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Syzygy functors for self-injective algebras

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Abstract: In this paper, homological properties of Ω-orbits and τ-orbits for self-injective algebras are studied. In addition, a class of self-injective algebras are proved to be satisfying the Auslander-Reiten conjecture. As a result, preprojective algebras of Dynkin type are showed to be satisfying the Auslander-Reiten conjecture.

Key words: self-injective algebra, Ω-period, Auslander-Reiten conjecture

1. Introduction

Throughout this paper, all algebras are Artin R-algebras, where R is a commutative Artin Ring. Recall that a ring Λ is called an Artin R-algebra if there is a ring homomorphism $\phi : R \to \Lambda$ with the image $\mathrm{Im}\phi$ in the center of Λ and Λ is a finitely generated right R-module. All modules are finitely generated right modules and $\mathbb{D} = \mathrm{Hom}(-, I_0(R/J(R)))$ is the ordinary duality, where $J(R)$ is the Jacobson radical of R and $I_0(R/J(R))$ is the injective envelope of $R/J(R)$.

An Artin R-algebra Λ is self-injective if Λ is injective as a right Λ-module. Self-injective algebras play an important role in the representation theory of Artin algebras. Many algebraists work on this topic and make the theory fruitful; (see [1, 8, 10, 12, 13, 18, 19] and so on). Among these results, Riedtmann [18, 19] and Asashiba [1] classified self-injective algebras of finite representation type in terms of derived equivalence. Duglas [8] built a connection between periodic algebras and self-injective algebras of finite representation type. Recall that an algebra Λ is called of finite representation type if there are finite number of indecomposable finitely generated right Λ-modules. We note that in these works homological properties of self-injective algebras were not studied in detail. In this paper we pay more attention to the homological properties of self-injective algebras.

On the other hand, homological conjectures [21] are important in the representation theory of Artin algebras. One of them is the Auslander-Reiten conjecure [3] which says the following:

(ARC) For an Artin algebra Λ and a finitely generated right Λ-module M, if $\mathrm{Ext}^i_\Lambda(M, M \bigoplus \Lambda) = 0$ holds for $i \geq 1$, then M is projective.

Auslander and Reiten [3, 4] proved **(ARC)** for self-injective algebras of finite representation type. Recall that an self-injective algebra Λ is called a symmetric algebra if $\Lambda \simeq \mathbb{D}\Lambda$ as a two-sided Λ-module. In 1980s, Hoshino [15] proved that **(ARC)** is true for symmetric algebras of radical cube zero. In 2012, Wei [20] proved that **(ARC)** is invariant under derived equivalence, that is, if Λ is derived equivalent to Γ, then Λ satisfies

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(ARC) if and only if Γ satisfies **(ARC)**. For more developments of **(ARC)**, we refer the reader to [6, 7, 16]. We consider the Auslander-Reiten conjecture for self-injective algebras in the present paper.

The paper is organized as follows:

In Section 2, we define Ω-orbit (resp. τ-orbit) and Ω-period (resp. τ-period). In addition, we give some preliminaries for later use.

In Section 3, we study the homological properties of Ω-periods and the τ-periods. We also determine when two indecomposable modules are in the same Ω-orbit (τ-orbit). Moreover, we show that the Ω-period and the τ-period for any right modules M over self-injective algebras of finite representation type are quite similar to the the order of elements in a finite group. We also give a connection between Ω-periods and the τ-periods for symmetric algebras of finite representation type. One of the main results is the following:

Theorem 1.1. *(Theorem 3.2) Let Λ be a self-injective algebra. Then the following are equivalent:*

(1) Two indecomposable modules M and N are in the same Ω-orbit (resp. τ-orbit).

(2) $\tau^i M$ and $\tau^i N$ (resp. $\Omega^i M$ and $\Omega^i N$) are in the same Ω-orbit (resp. τ-orbit) for any $i \in \mathbb{Z}$.

(3) There is a non-zero integer i_0 such that $\tau^{i_0} M$ and $\tau^{i_0} N$ (resp. $\Omega^{i_0} M$ and $\Omega^{i_0} N$) are in the same Ω-orbit (resp. τ-orbit).

In Section 4, we establish a new class of self-injective algebras over which the Auslander-Reiten conjecture holds and prove the following theorem.

Theorem 1.2. *(Theorem 4.1) Let Λ be a self-injective algebra with only finite number of indecomposable modules M satisfies $\mathrm{Ext}^k_\Lambda(M,M)=0$ for some $k \geq 1$. Then Λ satisfies the Auslander-Reiten conjecture.*

As a result of Theorem 1.2, a preprojective algebra Λ of Dynkin type (see Definition 2.5) satisfies **(ARC)**. But Λ is neither of finite representation type nor of radical cube zero in general.

In Section 5, we give some examples to illustrate our main results.

2. Preliminaries

In this section we recall some basic properties of self-injective algebras, syzygies and orbits. We begin this section with the definition of Artin R-algebras in [5, p26].

Definition 2.1. A ring Λ is called an *Artin R-algebra* if (1) there is a ring homomorphism $\phi: R \to \Lambda$ with the image of ϕ in the center of Λ, and (2) Λ is finitely generated as a right R-module.

For a right R-module M, we use $I_0(M)$ (resp. $P_0(M)$)to denote the injective envelope (resp. projective cover) of M. Denote by $\mathbb{D} = \mathrm{Hom}_R(-, I_0(R/J(R))$ and denote by $\mathrm{mod}\Lambda$ (resp. $\mathrm{mod}\Lambda^{op}$) the category of finitely generated right (resp. left) Λ-modules. Now we recall the following lemma [AuRS, p33, Theorem 4.2]

Lemma 2.1. *Let Λ be an Artin R-algebra. Then $\mathbb{D}: \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda^{op}$ is a duality.*

Denote by $\cdots \to P_i(M) \xrightarrow{f_i} \cdots \to P_1(M) \xrightarrow{f_1} P_0(M) \xrightarrow{f_0} M \to 0$ a minimal projective resolution of M. Denote by $\Omega^i M \simeq \mathrm{Ker} f_{i-1}$ the i-th syzygy of M. Dually, one can define the minimal injective resolution of M and co-syzygies $\Omega^{-i} M$ of M.

Applying the functor $(-)^* = \mathrm{Hom}_\Lambda(-,\Lambda)$ to a minimal projective resolution $\cdots \to P_1(M) \xrightarrow{f_1} P_0(M) \xrightarrow{f_0} M \to 0$ of M, one gets an exact sequence $0 \to M^* \to P_0(M)^* \xrightarrow{f_1^*} P_1(M)^* \to \mathrm{Coker} {f_1}^* \to 0$. Denote by

$\mathrm{Tr}M = \mathrm{Coker}f_1{}^*$ and denote by $\underline{\mathrm{mod}}\Lambda$ (resp. $\overline{\mathrm{mod}}\Lambda$) the stable subcategory of $\mathrm{mod}\Lambda$ modulo projective modules (resp. injective modules). Then we have the following lemma.

Lemma 2.2. *([5, Proposition 1.6]) (1)* $\mathrm{Tr}: \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda^{op}$ *is a duality.*

(2) $\tau = \mathbb{D}\mathrm{Tr}: \underline{\mathrm{mod}}\Lambda \to \overline{\mathrm{mod}}\Lambda$ *is an equivalence with the quasi-inverse* $\tau^- = \mathrm{Tr}\mathbb{D}$*.*

Denote by $id_\Lambda M$ (resp. $id_{\Lambda^{op}} N$ the injective dimension of a right Λ-module M (resp. left Λ-module N). Now it is time to recall the definition of self-injective algebras (resp. symmetric algebras).

Definition 2.2. ([5]) An algebra Λ is called *self-injective* if $id_\Lambda\Lambda = 0$. Moreover, a self-injective algebra Λ is called a *symmetric algebra* if $\mathbb{D}\Lambda \simeq \Lambda$ holds as a two-sided Λ-module.

In the following we recall some basic properties of syzygy functors for self-injective algebras.

Proposition 2.1. *([2, 4]) Let* Λ *be a self-injective algebra. Then*

(1) $\Omega: \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$ *is an equivalence with the quasi-inverse* Ω^{-1}

(2) $\mathrm{Ext}^1_\Lambda(M,N) \simeq \underline{\mathrm{Hom}}_\Lambda(\Omega^1 M, N)$ *holds for* $M, N \in \mathrm{mod}\Lambda$*.*

We also need the following definition of operations of a group on a set.

Definition 2.3. ([17]) Let $\mathcal{G}$ be a group with a unit e and let $\mathscr{S}$ be a non-empty set. We call a map $f: \mathcal{G} \times \mathscr{S}$ an action of $\mathcal{G}$ on $\mathscr{S}$ if the following are satisfied:

(1) $f(e,x) = x$ holds for any $x \in \mathscr{S}$.

(2) $f(g_1g_2, x) = f(g_1, f(g_2, x))$ for any $g_1, g_2 \in \mathcal{G}$ and $x \in \mathscr{S}$. In this case, we denote by $f(g,x) = g(x)$ for any $g \in \mathcal{G}$ and $x \in \mathscr{S}$.

For any $x, y \in \mathscr{S}$, we define a relation $x \sim y$ if $g(x) = y$ which is an equivalence relation on the set $\mathscr{S}$. We call the set $\mathcal{O}_x = \{g(x) | g \in \mathcal{G}\}$ the orbit of x.

We also need the following definitions of Ω-period and τ-period [8].

Definition 2.4. Let Λ be a self-injective algebra and let $M \in \mathrm{mod}\Lambda$ be non-projective. M is called of *Ω-period (resp. τ-period) n* if n is the smallest positive integer with $\Omega^n M \simeq M$ (resp. $\tau^n M \simeq M$). If there is no such a integer then we say that M is not Ω (resp. τ)-periodic. If M is an indecomposable projective module, then we say M is of Ω-period (resp. τ-period) 0.

The following definition of preprojective algebras is also needed in this paper.

Definition 2.5. Let $Q = (Q_0, Q_1)$ be a Dynkin quiver of $A_n (n \geq 2), D_n (n \geq 4), E_n (n = 6, 7, 8)$. Let $\overline{Q}$ be the double quiver of Q , which is obtained from Q by adding an arrow $a^*: j \to i$ if there is an arrow $a: i \to j$ in Q. The *preprojective algebra of Q* is defined as $\Lambda = \Lambda_Q = K\overline{Q}/(c)$, where K is an algebraic closed field and (c) is the ideal generated by the element $c = \sum_{a \in Q_1}(a^*a - aa^*)$.

We also need the following properties of preprojective algebras of Dynkin type [9].

Proposition 2.2. *Let* Λ *be a preprojective algebra of a Dynkin quiver* Q*. Then*

(1) Λ *is a finite dimensional self-injective algebra.*

(2) If $Q \neq A_2, A_3, A_4$*, then* Λ *is of infinite representation type, that is, there are infinite many indecomposable modules in* $\mathrm{mod}\Lambda$*.*

We end this section with the definition of Nakayama algebra [AuRS].

Definition 2.6. An algebra Λ is called a *Nakayama algebra* if every indecomposable projective modules and every indecomposable injective modules admit a unique composition series.

We remark that a Nakayama algebra Λ is of finite representation type, that is, there are finitely many indecomposable modules in $\mathrm{mod}\Lambda$.

3. Syzygy-orbits and τ-orbits for self-injective algebras

In this section we study the homological properties of syzygy-orbits and τ-orbits for a self-injective algebra Λ. We begin with the operations of integer group on sets of non-projective indecomposable Λ-modules.

Denote by $\mathscr{I}$ the set of indecomposable non-projective finitely generated right Λ-modules and denote by $\mathbb{Z}$ the additive group of integers. Now we can define two maps as follows:

$f_1 : \mathbb{Z} \times \mathscr{I} \to \mathscr{I}$ via $(i, M) \to \Omega^i M$ for $i \in \mathbb{Z}$ and $M \in \mathscr{I}$,

$f_2 : \mathbb{Z} \times \mathscr{I} \to \mathscr{I}$ via $(i, M) \to \tau^i M$ for $i \in \mathbb{Z}$ and $M \in \mathscr{I}$,

Then we have the following key proposition.

Proposition 3.1. *Let Λ be a self-injective algebra and let f_i be as above for $i = 1, 2$. Then*

(1) f_i is a group operation on a set.

(2) For $M \in \mathscr{I}$, $\mathcal{O}_M = \{\Omega^i M | i \in \mathbb{Z}\}$ (resp. $\mathcal{T}_M = \{\tau^i M | i \in \mathbb{Z}$) is a syzygy-orbit (resp. τ-orbit) which contains M.

Proof. (1) We show the case of f_1 since the case of f_2 is similar. By Proposition 2.1 (1), one gets $\Omega^i M \in \mathscr{I}$ if $M \in \mathscr{I}$. Then f_1 is a well-defined map. It is easy to get that $f_1(0, M) = \Omega^0 M = M$ and $f_1(i + j, M) = \Omega^{i+j} M = \Omega^i \Omega^j M = f_1(i, f_1(j, M))$. Then by Definition 2.6, f_1 is an operation of a group on a set.

(2) It is a straight result of (1) and Definition 2.3. □

In the following we show the relations between Ω-orbits and Ω-periods.

Proposition 3.2. *Let Λ be a self-injective algebra and let $\mathcal{O}_M$ and $\mathcal{T}_M$ be a syzygy-orbit and a τ-orbit of M, respectively.*

(1) If there is an N in $\mathcal{O}_M$ such that the Ω-period of N is n, then every module in $\mathcal{O}_M$ is of Ω-period n. And hence there are n indecomposable modules in $\mathcal{O}_M$.

(2) If there is an N in $\mathcal{T}_M$ such that the τ-period of N is n, then every module in $\mathcal{T}_M$ is of τ-period n. And hence there are n indecomposable modules in $\mathcal{T}_M$.

Proof. We only prove (1) since the proof of (2) is similar. By the assumption $N \in \mathcal{O}_M$, one gets $N \simeq \Omega^t M$, and hence $\Omega^{n+t} M \simeq \Omega^t M$ since N is of Ω-period n. By Proposition 2.1 (1), one can show $\Omega^n M \simeq M$ and hence $\Omega^{n+i} M \simeq \Omega^i M$ for any $i \in \mathbb{Z}$. This means that there are only n indecomposable modules in $\mathcal{O}_M$. Then for any $i \in \mathbb{Z}$, $\Omega^n(\Omega^i M) \simeq \Omega^i(\Omega^n M) \simeq \Omega^i M$. We only have to show n is the minimal positive integer desired. If there is $0 < m < n$ such that $\Omega^m(\Omega^i M) \simeq \Omega^i M$. Taking $i = t$, one gets an contradiction since $N = \Omega^t M$ is of Ω-period n. □

Now we can give a new statement of self-injective algebras of finite representation type.

Theorem 3.1. *Let Λ be a self-injective algebra. The following are equivalent.*

(1) Λ is of finite representation type.

(2) There are only finite number of Ω-orbits and each orbit admits a Ω-periodic element.

(3) There are only finite number of τ-orbits and each orbit admits a τ-periodic element.

Proof. We only show $(1) \Leftrightarrow (2)$. One can show $(1) \Leftrightarrow (3)$ similarly.

$(1) \Rightarrow (2)$ Since Λ is of finite representation type, then there are finite number of indecomposable modules in $\mod \Lambda$. For any indecomposable $M \in \mathrm{mod}\Lambda$, we get a Ω-orbit $\mathcal{O}_M$ by Proposition 3.1. Then there are finite number of indecomposable modules in $\mathcal{O}_M$. So we can find $0 < i < j \in \mathbb{Z}$ such that $\Omega^i M \simeq \Omega^j M$. Otherwise, one can get infinite number of indecomposable modules, a contradiction. Then M is of Ω-period $j - i$ by Proposition 2.1(1).

$(2) \Rightarrow (1)$ By Proposition 3.2(1) and the assumption, one gets that each Ω-orbit has finite indecomposable modules. Then one gets that Λ admits finite number of indecomposable modules since the number of orbits is finite. □

By Theorem 3.1 we get the following corollary immediately:

Corollary 3.1. *Let Λ be a self-injective algebra. If Λ is of finite representation type, then every indecomposable $M \in \mathrm{mod}\Lambda$ is Ω-periodic (τ-periodic).*

Proof. For any indecomposable projective $P \in \mod \Lambda$, by Definition 2.4 one gets P is of Ω-period (resp. τ-period) 0. Then by Theorem 3.1, every indecomposable non-projective module M is Ω-periodic (resp. τ-periodic). We are done. □

To study the Ω-period and τ-period of self-injective algebras of finite representation type, we need the following commutative properties of the functors Ω and τ.

Proposition 3.3. *Let Λ be a self-injective algebra and let M be an indecomposable right Λ-module. Then $\Omega^i \tau^j M \simeq \tau^j \Omega^i M$ for any $i, j \in \mathbb{Z}$.*

Proof. We only show the case of $i = \pm 1$ and $j = \pm 1$ since other cases follow by inductions on i and j, respectively. We show the assertion step by step.

(1) Firstly, we show $\Omega^1 \tau M \simeq \tau \Omega^1 M$.

If M is projective, then there is nothing to prove. Assume M is not projective. Take a minimal projective resolution of M: $\cdots \to P_2(M) \to P_1(M) \to P_0(M) \to M \to 0$. Applying the functor $(-)^* = \mathrm{Hom}_\Lambda(-, \Lambda)$, since Λ is self-injective, one gets the following exact sequences:

$$0 \to M^* \to (P_0(M))^* \to (P_1(M))^* \to (\Omega^2 M)^* \to 0 \quad (*1)$$

$$0 \to (\Omega^1 M)^* \to (P_1(M))^* \to (P_2(M))^* \to (\Omega^3 M)^* \to 0 \quad (*2)$$

$$0 \to (\Omega^2 M)^* \to (P_2(M))^* \to (\Omega^3 M)^* \to 0 \quad (*3)$$

Applying the functor $\mathbb{D}$ to the exact sequence $(*1)$, one gets the following exact sequence

$$0 \to \mathbb{D}(\Omega^2 M)^* \to \mathbb{D}(P_1(M))^* \to \mathbb{D}(P_0(M))^* \to \mathbb{D}M^* \to 0$$

Then one gets that $\tau M \simeq \mathbb{D}(\Omega^2 M)^*$. Similarly, one gets $\tau\Omega^1 M \simeq \mathbb{D}(\Omega^3 M)^*$. After applying the functor $\mathbb{D}$ to the sequence $(*3)$, one gets the assertion since Λ is self-injective.

(2) Hereby we prove $\Omega^{-1}\tau M \simeq \tau\Omega^{-1}M$.

Denote by $N = \Omega^{-1}M$. Then $M \simeq \Omega^1 N$ and $\tau\Omega^{-1}M = \tau N$ by proposition 2.5. One gets $\Omega^{-1}\tau M \simeq \Omega^{-1}\tau\Omega^1 N \simeq \Omega^{-1}\Omega^1\tau N \simeq \tau N$ by (1).

(3) The proof of $\Omega^1\tau^{-1}M \simeq \tau^{-1}\Omega^1 M$

Denote by $L = \tau^{-1}M$. Then $\Omega^1\tau^{-1}M \simeq \Omega^1 L$ and $M \simeq \tau L$ by proposition 2.1. One can show $\tau^{-1}\Omega^1 M \simeq \tau^{-1}\Omega^1\tau L \simeq \tau^{-1}\tau\Omega^1 L \simeq \Omega^1 L$ by (1). The assertion holds.

(4) We finally show $\Omega^{-1}\tau^{-1}M \simeq \tau^{-1}\Omega^{-1}M$.

Denote by $N = \Omega^{-1}M$. Then $M \simeq \Omega^1 N$ and $\tau^{-1}\Omega^{-1}M = \tau^{-1}N$ by proposition 2.5. By (3) one can show $\Omega^{-1}\tau^{-1}M \simeq \Omega^{-1}\tau^{-1}\Omega^1 N \simeq \Omega^{-1}\Omega^1\tau^{-1}N \simeq \tau^{-1}N$. The assertion holds. □

Now we are in a position to judge when two indecomposable modules are in the same τ-orbit or Ω-orbit.

Theorem 3.2. *Let Λ be a self-injective algebra. Then the following are equivalent:*

(1) Two indecomposable modules M and N are in the same Ω-orbit (resp. τ-orbit).

(2) $\tau^i M$ and $\tau^i N$ (resp. $\Omega^i M$ and $\Omega^i N$) are in the same Ω-orbit (resp. τ-orbit) for any $i \in \mathbb{Z}$.

(3) There is a non-zero positive integer i_0 such that $\tau^{i_0}M$ and $\tau^{i_0}N$ (resp. $\Omega^{i_0}M$ and $\Omega^{i_0}N$) are in the same Ω-orbit (resp. τ-orbit).

Proof. We only prove the case of Ω-orbits. One can show the case of τ-orbits similarly.

(1) $\Rightarrow$ (2) Since M and N are in the same Ω-orbit, one has $N \simeq \Omega^j M$ for some $j \in \mathbb{Z}$. Then $\tau^i N \simeq \tau^i\Omega^j M \simeq \Omega^j\tau^i M$ holds for any $i \in \mathbb{Z}$ by Proposition 3.3.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (1) Since $\tau^{i_0}N$ and $\tau^{i_0}M$ are in a same Ω-orbit, then there is an integer j such that $\Omega^j\tau^{i_0}M \simeq \tau^{i_0}N$. By Proposition 3.3, one gets that $\Omega^j\tau^{i_0}M \simeq \tau^{i_0}\Omega^j M \simeq \tau^{i_0}N$. By Proposition 2.2(2), $N \simeq \Omega^j M$, we are done. □

Although we do not know much about connections between Ω-orbits and τ-orbits, we have:

Proposition 3.4. *Let Λ be a self-injective algebra and M an indecomposable right Λ-module. Then*

(1) The Ω-orbit $\mathcal{O}_M$ is contained in the τ-orbit $\mathcal{T}_M$ if and only if $\Omega^1 M \simeq \tau^i M$ for some $i \in \mathbb{Z}$.

(2)The τ-orbit $\mathcal{T}_M$ is contained in the Ω-orbit $\mathcal{O}_M$ if and only if $\tau M \simeq \Omega^i M$ for some $i \in \mathbb{Z}$.

Proof. We only prove (1). One can prove (2) similarly. The necessity is trivial. Now we show the sufficiency. For any $N \in \mathcal{O}_M$, one gets $N \simeq \Omega^t M \simeq \Omega^{t-1}\Omega^1 M$. By the assumption and Proposition 3.3 one gets $N \simeq \Omega^{t-1}\tau^i M \simeq \tau^i\Omega^{t-1}M$. By induction on t, it is not difficult to show $N \simeq \tau^{it}M$, that is, $N \in \mathcal{T}_M$. □

In the following we focus on the Ω-periods and τ-periods for self-injective algebras. We get the following theorem which is very similar to the order of elements in a finite group.

Theorem 3.3. *Let Λ be a self-injective algebra and let M and N be indecomposable non-projective right Λ-modules.*

(1) If the Ω-period (resp. τ-period) of M and N is m and n, respectively. Then $M \oplus N$ is also Ω-periodic (resp. τ-periodic) and its Ω-period (resp. τ-period) is $[m, n]$, where $[m, n]$ is the least common multiple of m and n.

(2) If Λ is of finite representation type, then every $L \in \mathrm{mod}\Lambda$ without projective direct summands is Ω-periodic (resp. τ-periodic) and there is an upper bound of the Ω-period (resp. τ-period).

Proof. We only show the case of Ω-period since the case of τ-period is similar.

Denote by $t = [m, n]$, to show (1) we have to show (a) $\Omega^t(M \oplus N) \simeq M \oplus N$, and (b) If there is an $s \in \mathbb{Z}$ such that $\Omega^s(M \oplus N) \simeq M \oplus N$, then $t|s$.

It is clear that if M and N are in the same Ω-orbit, then by Proposition 3.2(1) and the assumption, $m = n$ and $t = n$ hold, and hence both (a) and (b) hold. In the following we can assume that M and N don't live in the same Ω-orbit.

(a) Since Ω is an additive functor, then $\Omega^t(M \oplus N) \simeq \Omega^t M \oplus \Omega^t N$. Notice that the Ω-period of M is m and $m|t$, then we have that $\Omega^t M \simeq \Omega^{mq} M \simeq \Omega^m M \simeq M$ for some positive integer q with $t = mq$. Similarly, one gets $\Omega^t N \simeq N$.

(b) Because of $\Omega^s(M \oplus N) \simeq \Omega^s M \oplus \Omega^s N$ and the assumption M and N don't live in the same orbit, one gets $\Omega^s M \simeq M$ and $\Omega^s N \simeq N$. Since the Ω-period of M and N is m and n, then $m|s$ and $n|s$, and hence $t|s$.

(2) Since Λ is of finite representation type, then $\mathrm{mod}\Lambda = \mathrm{add}T$, where T is a basic additive generator. Killing the indecomposable projective direct summands of T, we get a basic direct summand T' of T. Assume that $T' = \bigoplus_{i=1}^n T_i$. By Corollary 3.1 T_i is Ω-periodic, and we assume that the Ω-period of T_i is t_i. Then by (1), the Ω-period of T' is $[t_1, t_2, ..., t_n]$.

For any $L \in \mathrm{mod}\,\Lambda$ without projective direct summands, L can be written as $\bigoplus_{j=1}^n T_j^{i_j}$ for some non-negative integer i_j. By using (1), the Ω-period of ${T_i}^{i_j}$ is equal to that of T_i if $i_j > 0$. Using (1) again, one gets the Ω-period of L is a divisor of $[t_1, t_2, ..., t_n]$. □

As a corollary of Proposition 3.4 and Theorem 3.3, we have

Corollary 3.2. *Let Λ be a symmetric algebra and let $M \in \mathrm{mod}\,\Lambda$ be indecomposable non-projective. Then*

(1) $\mathcal{T}_M$ is contained in $\mathcal{O}_M$.

(2) If in addition Λ is of finite type, then the Ω-period of M is equal to the τ-period of M or two times of the τ-period of M.

Proof. (1) Since Λ is symmetric, then one gets $\tau M \simeq \Omega^2 M$. So one gets the assertion by Proposition3.4 (2).

(2) Since Λ is of finite representation type, then by Theorem 3.1 M is both Ω-periodic and τ-periodic. Let s, t be the Ω-period and τ-period of M, respectively. Then $\Omega^s M \simeq M \simeq \tau^t M$. We claim that $s = t$ or $s = 2t$.

Notice that Λ is symmetric, $\tau M \simeq \Omega^2 M$. By the definition of Ω-period, one can show $s|2t$ since $\tau^t M \simeq \Omega^{2t} M \simeq M$. On the other hand, $\Omega^s M \simeq M$ implies that $\Omega^{2s} M \simeq M$ by Proposition 3.2. That is, $\tau^s M \simeq M$. Since the τ-period of M is t, one can show $t|s$. Then one has $s = t$ or $s = 2t$. □

We end this section with the following question:

Question 3.1. Let Λ be an Artin algebra. If all indecomposable right Λ-modules are Ω-periodic, is Λ of finite representation type?

We should remark that if Λ is of radical cube zero, then there is a positive answer to the question above [11].

4. 4 Syzygies and Auslander-Reiten Conjecture

In this section, firstly we show some properties of syzygy functors then we show a class of self-injective algebras satisfying Auslander-Reiten conjecture. As a result, preprojective algebras of Dynkin type are showed to be satisfying Auslander-Reiten conjecture.

To show the main results of this section, we need the following lemma which is a generalization of Proposition 2.1 (2).

Lemma 4.1. *Let Λ be a self-injective algebra and $M, N \in \operatorname{mod} \Lambda$. Then $\operatorname{Ext}^i_\Lambda(M,N) \simeq \underline{\operatorname{Hom}}_\Lambda(\Omega^i M, N)$ holds for any $i \geq 1$.*

Proof. The case of $i = 1$ is clear form Proposition 2.5 (2). We only need to show the case of $i \geq 2$. Applying the functor $\operatorname{Hom}_\Lambda(-,N)$ to the following minimal projective resolution of M: $\cdots \to P_i(M) \to \cdots P_1(M) \to P_0(M) \to M \to 0$, one gets $\operatorname{Ext}^i_\Lambda(M,N) \simeq \operatorname{Ext}^{i-1}_\Lambda(\Omega^1 M, N) \simeq \cdots \simeq \operatorname{Ext}^1_\Lambda(\Omega^{i-1}M,N)$ for any $i \geq 2$. By Proposition 2.1 (2), one gets that $\operatorname{Ext}^1_\Lambda(\Omega^{i-1}M,N) \simeq \underline{\operatorname{Hom}}_\Lambda(\Omega^i M, N)$. We are done. □

Now we can state the following key proposition of this section.

Proposition 4.1. *Let Λ be a self-injective algebra and $M \in \operatorname{mod}\Lambda$. Then $\Omega^i M$ satisfies $\operatorname{Ext}^k_\Lambda(\Omega^i M, \Omega^i M) = 0$ in $\operatorname{mod}\Lambda$ for any $i \in \mathbb{Z}$ if and only if M satisfies $\operatorname{Ext}^k_\Lambda(M,M) = 0$ for some $k \geq 1$.*

Proof. If $i = 0$, then there is nothing to prove. If $i < 0$, then one gets that $M \simeq \Omega^{-i}\Omega^i M$. So it is enough to show the case of $i > 0$. By Lemma 4.1, one gets that $\operatorname{Ext}^k_\Lambda(\Omega^i M, \Omega^i M) \simeq \underline{\operatorname{Hom}}_\Lambda(\Omega^{k+i}M, \Omega^i M) \simeq \underline{\operatorname{Hom}}_\Lambda(\Omega^k M, M) \simeq \operatorname{Ext}^k_\Lambda(M,M)$. The assertion holds. □

As we mentioned above, Auslander and Reiten in [4] showed that if Λ is a self-injective algebra of finite representation type then Λ satisfies **(ARC)**. In the following we give a generalization of this result.

Theorem 4.1. *Let Λ be a self-injective algebra with only finite number of indecomposable modules M satisfies $\operatorname{Ext}^k_\Lambda(M,M) = 0$ for some $k \geq 1$. Then Λ satisfies Auslander-Reiten conjecture.*

Proof. Let N be an arbitrary right Λ-module satisfying $\operatorname{Ext}^i_\Lambda(N,N) = 0$ for any integer $i \geq 1$, by the definition of **(ARC)**, we need to show N is projective. On the other hand, since Λ is an Artin algebra, then N can be written into $\bigoplus_{j=1}^{s} N_j$ with N_j indecomposable and s a positive integer. Now it is enough to show N_j is projective for $1 \leq j \leq s$.

In the following we show that any indecomposable module M is projective if it satisfies $\operatorname{Ext}^i_\Lambda(M,M) = 0$ for any integer $i \geq 1$. By the assumption, there are only finite number of indecomposable modules M such that $\operatorname{Ext}^i_\Lambda(M,M) = 0$ for any integer $i \geq 1$.

We claim that M is Ω-periodic. Since M is indecomposable and Λ is self-injective, by Proposition 3.1 (2), one gets an Ω-orbit $\mathcal{O}_M = \{\Omega^j M | j \in \mathbb{Z}\}$. So there must be some $j_1 > j_2$ such that $\Omega^{j_1} M \simeq \Omega^{j_2} M$. Otherwise, we get an infinite Ω-orbit $\mathcal{O}_M$ by Proposition 2.1. By Lemma 4.1 $\Omega^j M$ satisfies $\mathrm{Ext}^i_\Lambda(\Omega^j M, \Omega^j M) = 0$ for any $i \geq 1$ and $j \in \mathbb{Z}$, which is a contradiction. So we get $\Omega^{j'} M \simeq M$, where $j' = j_1 - j_2$.

Now we show M is projective. Applying the functor $\mathrm{Hom}_\Lambda(-, M)$ to the following minimal projective resolution of M:

$$\cdots \to P_i(M) \to \cdots P_1(M) \to P_0(M) \to M \to 0,$$

one gets

$$0 = \mathrm{Ext}^{j'}_\Lambda(M, M) \simeq \mathrm{Ext}^{j'-1}_\Lambda(\Omega^1 M, M) \simeq \cdots \mathrm{Ext}^1_\Lambda(\Omega^{j'-1} M, M) \simeq \mathrm{Ext}^1_\Lambda(\Omega^{j'-1} M, \Omega^{j'} M).$$

This means that the short exact sequence $0 \to \Omega^{j'} M \to P_{j'-1}(M) \to \Omega^{j'-1}(M) \to 0$ splits, and hence $\Omega^{j'-1} M$ is projective. Notice that Λ is self-injective, then $\Omega^{j'-1} M$ is injective and hence the short exact sequence $0 \to \Omega^{j'-1} M \to P_{j'-2}(M) \to \Omega^{j'-2}(M) \to 0$ splits. Continue the same process, one gets that $0 \to \Omega^1 M \to P_0(M) \to M \to 0$ splits and hence M is projective. $\square$

In the following we give an application of Theorem 4.1 to preprojective algebras of Dynkin type. We should remark that in general preprojective algebras are neither of finite representation type nor of radical cube zero (comparing [15] and Proposition 2.2).

Corollary 4.1. *Let Λ be a preprojective algebra of Dynkin type. Then Λ satisfies the Auslander-Reiten conjecture.*

Proof. By Proposition 2.2 Λ is a finite dimensional algebra, and hence an Artin algebra. Moreover, Λ is self-injective. Then by [12], $\mathrm{Ext}^2_\Lambda(N, N) \neq 0$ holds for any non-projective N. That is, there are only finite indecomposable modules M such that $\mathrm{Ext}^2_\Lambda(M, M) = 0$. Then by Theorem 4.1, the assertion holds. $\square$

In the following we consider another class of self-injective algebras which also satisfies the condition of Theorem 4.1. We should remark that one can get the result by using a result in [5]. Here we give a completely short proof by using syzygy functors. Recall that an algebra Λ is called of radical square zero if $J^2(\Lambda) = 0$, where $J(\Lambda)$ is the Jacobson radical of Λ. For a Λ-module M, denote by $\mathrm{rad}M$ (resp. $\mathrm{soc}M$) the radical (resp. socle) of M, now we can state the following:

Proposition 4.2. *Let Λ be a self-injective algebra with radical square zero. Then Λ is a Nakayama algebra.*

Proof. By Definition 2.6, it suffices to show all indecomposable projective Λ-modules and all indecomposable injective Λ-modules admit a unique composition series. Since Λ is self-injective, then it is enough to show every indecomposable projective Λ-module admits a unique composition series.

For any indecomposable projective $P \in \mod \Lambda$, there is a simple module S such that $P_0(S) \simeq P$. Then we get the following exact sequence $0 \to \Omega^1 S \to P_0(S) \to S \to 0$. Since Λ is of radical square zero, then $\Omega^1 S \simeq \mathrm{rad} P_0(S)$ is semi-simple. On the other hand, the fact Λ is self-injective implies that $P_0(S)$ is indecomposable injective. Then $\mathrm{soc} P_0(S)$ is simple, and hence $\Omega^1 S \subset \mathrm{soc} P_0(S)$ is simple. So P admits a unique composition series. $\square$

Immediately, we have the following corollary.

Corollary 4.2. *Let Λ be a self-injective algebra with radical square zero. Then Λ satisfies Auslander-Reiten conjecture.*

Proof. By Proposition 4.2, Λ is a Nakayama algebra. Then by Definition 2.6, Λ is of finite representation type. Then by Theorem 4.1, the assertion holds. □

We also remark that radical square zero self-injective algebras are not necessarily symmetric algebras.

5. Examples

In this section we give examples to support our results in Section 3 and Section 4.

The following example implies that the number of indecomposable modules in a Ω-orbit is not necessary to be the divisor of the number of all indecomposable modules in general.

Example 5.1. *Let Λ be an algebra given by the quiver $\circlearrowleft^a$ with $a^3 = 0$. Then*

(1) Λ is a Nakayama local algebra with 3 indecomposable modules.

(2) The non-projective modules 1 and $\begin{smallmatrix}1\\1\end{smallmatrix}$ are of Ω-period 2 and the unique Ω-orbit is $\{1, \begin{smallmatrix}1\\1\end{smallmatrix}\}$.

Now we give an example to show Theorem 3.3.

Example 5.2. *Let Λ be the preprojective algebra of $1 \to 2 \to 3$. Then*

(1) Λ is a self-injective algebra with 12 indecomposable modules.

(2) The two Ω-orbits are

$\mathcal{O}_1 = \{2, {}^{1}{}_{2}{}^{3}, {}_{1}{}^{2}{}_{3}\}$

$\mathcal{O}_2 = \{1, 3, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\}$

(3) The two τ-orbits are

$\tau_1 = \{2, {}^{1}{}_{2}{}^{3}, {}_{1}{}^{2}{}_{3}\}$

$\tau_2 = \{1, 3, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\}$

(4) By (2) and (3) every indecomposable module M has the same τ-period and Ω-period, which is 3 or 6.

The following example (comparing Proposition 2.2 and Corollary 4.1) shows that Theorem 4.1 is far from trivial.

Example 5.3. *Let Λ be preprojective algebra of a quiver $Q = A_5 : 1 \to 2 \to 3 \to 4 \to 5$. Then*

(1) Λ is self-injective of infinite representation type.

(2) For any non-projective module $M \in \mathrm{mod}\Lambda$, one gets $\mathrm{Ext}^2_{\Lambda}(M, M) \neq 0$.

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