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The duals of $*$-operator frames for $End_{\mathcal{A}}^{*}(H)$

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Abstract: Frames play significant role in signal and image processing, which leads to many applications in differents fields. In this paper we define the dual of $*$-operator frames and we show their propreties obtained in Hilbert $\mathcal{A}$-modules and we establish some results.

Key words: $*$-frame, $*$-operator frame, C^*-algebra, Hilbert $\mathcal{A}$-modules.

1. Introduction

Frame theory is recently an active research area in mathematics, computer science, and engineering with many exciting applications in a variety of different fields. They are generalizations of bases in Hilbert spaces. Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [6] for study of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [5], and popularized from then on.
The aim of this papers is to study the dual of $*$-operator frames.

The paper is organized as follows:

In section 2, we briefly recall the definitions and basic properties of operator frame and $*$-operator frame in Hilbert C^*-modules.

In section 3, we introduce the dual $*$-operator frame, the $*$-operator frame transform and the $*$-frame operator.

In section 4, we investigate tensor product of Hilbert C^*-modules, we show that tensor product of dual $*$-operator frames for Hilbert C^*-modules $\mathcal{H}$ and $\mathcal{K}$, present a dual $*$-operator frames for $\mathcal{H}\otimes\mathcal{K}$.

2. Preliminaries

Let I and J be countable index sets. In this section we briefly recall the definitions and basic properties of C^*-algebra, Hilbert C^*-modules, operator frame and $*$-operator frame in Hilbert C^*-modules. For information about frames in Hilbert spaces we refer to [2]. Our reference for C^*-algebras is [3, 4]. For a C^*-algebra $\mathcal{A}$, an element $a\in\mathcal{A}$ is positive ($a\geq 0$) if $a=a^*$ and $sp(a)\subset\mathbf{R}^+$. $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 2.1. [8]. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product

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$\langle .,. \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x\rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z\rangle_{\mathcal{A}} = a\langle x, z\rangle_{\mathcal{A}} + \langle y, z\rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

(iii) $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x\rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x\rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Example 2.1. *[12] If $\{\mathcal{H}_k\}_{k\in\mathbf{N}}$ is a countable set of Hilbert $\mathcal{A}$-modules, then one can define their direct sum $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$. On the $\mathcal{A}$-module $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$ of all sequences $x = (x_k)_{k\in\mathbb{N}} : x_k \in \mathcal{H}_k$, such that the series $\sum_{k\in\mathbb{N}}\langle x_k, x_k\rangle_{\mathcal{A}}$ is norm-convergent in the C^*-algebra $\mathcal{A}$, we define the inner product by*

$$\langle x, y\rangle := \sum_{k\in\mathbb{N}}\langle x_k, y_k\rangle_{\mathcal{A}}$$

for $x, y \in \oplus_{k\in\mathbb{N}}\mathcal{H}_k$.

Then $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$ is a Hilbert $\mathcal{A}$-module.

The direct sum of a countable number of copies of a Hilbert C^*-module $\mathcal{H}$ is denoted by $l^2(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We also reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

Definition 2.2. [7]

A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$, if there exist two positive constants $A, B > 0$ such that

$$A\langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_i x, T_i x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}, \forall x \in \mathcal{H}. \tag{1}$$

The numbers A and B are called lower and upper bound of the operator frame, respectively. If $A = B = \lambda$, the operator frame is λ-tight. If $A = B = 1$, it is called a normalized tight operator frame or a Parseval operator frame.

Definition 2.3. [9]

A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an $*$-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$, if there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle x, x\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_i x, T_i x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^*, \forall x \in \mathcal{H}. \tag{2}$$

The elements A and B are called lower and upper bounds of the $*$-operator frame, respectively. If $A = B = \lambda$, the $*$-operator frame is λ-tight. If $A = B = 1_{\mathcal{A}}$, it is called a normalized tight $*$-operator frame or a Parseval $*$-operator frame. If only upper inequality of hold, then $\{T_i\}_{i\in i}$ is called an $*$-operator Bessel sequence for $End_{\mathcal{A}}^{*}(\mathcal{H})$.

If the sum in the middle of (2.1) is convergent in norm, the operator frame is called standard. If only upper inequality of (2.1) hold, then $\{T_i\}_{i\in I}$ is called an operator Bessel sequence for $End_{\mathcal{A}}^{*}(\mathcal{H})$.

Lemma 2.1. *[1]. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{K})$.*

(i) If T is injective and T has closed range, then the adjointable map T^*T is invertible and

$$\|(T^*T)^{-1}\|^{-1} I_{\mathcal{H}} \leq T^*T \leq \|T\|^2 I_{\mathcal{H}}.$$

(ii) If T is surjective, then the adjointable map TT^* is invertible and

$$\|(TT^*)^{-1}\|^{-1} I_{\mathcal{K}} \leq TT^* \leq \|T\|^2 I_{\mathcal{K}}.$$

3. Dual of $*$-operator Frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$

We begin this section with the following definition.

Definition 3.1. Let $\{T_i\}_{i\in I} \subset End_{\mathcal{A}}^{*}(\mathcal{H})$ be an $*$-operator frame for $\mathcal{H}$. If there exist an $*$-operator frame $\{\Lambda_i\}_{i\in I}$ such that $x = \sum_{i\in I} T_i^*\Lambda_i x$ for all $x \in \mathcal{H}$, then the $*$-operator frames $\{\Lambda_i\}_{i\in I}$ is called the duals $*$-operator frames of $\{T_i\}_{i\in I}$.

Example 3.1. *Let $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module over itself, let $\{f_j\}_{j\in J}$ be an $*$-frame for $\mathcal{A}$.*

We define the adjointable $\mathcal{A}$-module map $\Lambda_{f_j} : \mathcal{A} \to \mathcal{A}$ by $\Lambda_{f_j} f = \langle f, f_j\rangle$. Clearly, that $\{\Lambda_{f_j}\}_{j\in J}$ is an $*$-operator frame for $\mathcal{A}$.

Theorem 3.1. *Every $*$-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ has a dual $*$-operator frame.*

Proof.

Let $\{T_i\}_{i\in I} \subset End_{\mathcal{A}}^{*}(\mathcal{H})$ be an $*$-operator for $End_{\mathcal{A}}^{*}(\mathcal{H})$, with $*$-frame operator S.

We see that $\{T_iS^{-1}\}_{i\in I}$ is an $*$-operator frame.

Or, $\forall x \in \mathcal{H}$ we have :

$$Sx = \sum_{i\in I} T_i^*T_i x$$

then

$$x = \sum_{i\in I} T_i^*T_iS^{-1}x$$

hence $\{T_iS^{-1}\}_{i\in I}$ is a dual $*$-operator frame of $\{T_i\}_{i\in I}$.

It is called the canonique dual $*$-operator frame of $\{T_i\}_{i\in I}$. □

Theorem 3.2. *Let $\{\Lambda_i\}_{i\in I}$ be an $*$-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with $*$-pre-frame operator θ, the $*$-frame operator S and the canonical dual $*$-operator frames $\{\tilde{\Lambda}_i\}_{i\in \mathbb{J}}$.*

Let $\{\Omega_i\}_{i\in I}$ be an arbitrary dual $*$-operator frame of $\{\Lambda_i\}_{i\in I}$ with the $*$-pre-frame operator η; then the folowing statements are true:

(1) $\theta^*\eta = I$.

(2) $\Omega_i = \Pi_i\eta$ *for all* $i\in I$.

(3) *If* $\eta^{'} : \mathcal{H} \longmapsto l^2(\mathcal{H})$ *is any adjointable right inverse of* θ^* *then* $\{\Pi_i\eta^{'}\}_{i\in I}$ *is a dual $*$-operator frame of* $\{\Lambda_i\}_{i\in I}$ *with the $*$-pre-frame operator* $\eta^{'}$.

(4) *The $*$-frame operator* S_Ω *of* $\{\Omega_i\}_{i\in I}$ *is equal to* $S^{-1} + \eta^*(I - \theta S^{-1}\theta^*)\eta$.

(5) *Every adjointable right inverse* $\eta^{'}$ *of* θ^* *is the forme :*
$\eta^{'} = \theta S^{-1} + (I - \theta S^{-1}\theta^*)\psi$ *for some adjointable map* $\psi : \mathcal{H} \longmapsto l^2(\mathcal{H})$ *and vice versa.*

(6) *There exist a $*$-bessel operator* $\{\Delta_j\}_{j\in J}$ *in* $End_{\mathcal{A}}^*(\mathcal{H})$ *whose $*$-pre-frame operator is* η *and yields:*
$\Omega_j = \tilde{\Lambda}_j + \Delta_j - \sum_{k\in J}\tilde{\Lambda}_j\Lambda_k^*\Delta_k, \forall j\in J$

Proof.

(1) For $f, g \in \mathcal{H}$ we have :

$$\begin{aligned}\langle \theta^*\eta f, g\rangle &= \langle \eta f, \theta g\rangle \\ &= \langle \sum_{i\in I}\Omega_i f, \sum_{i\in I}\Lambda_i g\rangle \\ &= \sum_{i\in I}\langle \Omega_i f, \Lambda_i g\rangle = \sum_{i\in I}\langle \Lambda_i^*\Omega_i f, g\rangle \\ &= \langle \sum_{i\in I}\Lambda_i^*\Omega_i f, g\rangle = \langle f, g\rangle\end{aligned}$$

then $\theta^*\eta = I$.

(2) The proof is clear from the definition

(3) Since $\eta^{'}$ is adjointable, we have $\{\Pi_i\eta^{'}\}_{i\in I}$ is a $*$-bessel sequence in $\mathcal{H}$.
Also, since $(\eta^{'})^*\theta = I$; $(\eta^{'})^*$ is surjective, by lemme 1.5, for $f\in\mathcal{H}$ we have:

$$||(\eta^{'})^*\eta^{'})^{-1}||^{-1}\langle f, f\rangle \leq \langle \eta^{'} f, \eta' f\rangle = \sum_{i\in I}\langle \pi_i\eta' f, \pi_i\eta' f\rangle$$

clearly, $\eta^{'}$ is the $*$-pre-frame operator of $\{\Pi_i\eta^{'}\}_{i\in I}$

(4)

$$\begin{aligned} S_\Omega &= \eta^*\eta \\ &= \eta^*\theta S^{-1} + \eta^*\eta - \eta^*\theta S^{-1} \\ &= \eta^*\theta S^{-1} + \eta^*\eta - \eta^*\theta S^{-1}\theta^*\eta \\ &= \eta^*\theta S^{-1} + \eta^*(I - \theta S^{-1}\theta^*)\eta \end{aligned}$$

(5) If $\eta^{'}$ is such a right inverse of θ, then

$$\theta S^{-1} + (I - \theta S^{-1}\theta^*)\eta^{'} = \theta S^{-1} + \eta' - \theta S^{-1}\theta^*\eta^{'} = \theta S^{-1} + \eta' - \theta S^{-1} I = \eta'$$

(6) Let $\{\Delta_i\}_{i\in I}$ be an $*$-operator bessel sequence for $End^*_{\mathcal{A}}(\mathcal{H})$ with the preframe operator η. For $i \in I$, let $\Omega_i = \tilde{\Lambda}_i + \Delta_i - \sum_{k\in I}\tilde{\Lambda}_i\Lambda_k^*\Delta_k$ Let S and θ be the $*$-frame operator and the preframe operator of $\{\Delta_i\}_{i\in I}$, resp. we define the linear operator $\psi : \mathcal{H} \longmapsto l^2(\mathcal{H})$ by $\psi f = (\Omega_i f)_{i\in I}$. clearly, ψ is adjointable, for every $i \in I$, we have

$$\begin{aligned} \pi_i\psi &= \Omega_i \\ &= \Lambda_i S^{-1} + \Delta_i - \Lambda_i S^{-1}\sum_{k\in I}\Lambda_k^*\Delta_k \\ &= \Lambda_i S^{-1} + \pi_i\eta - \sum_{k\in I}\Lambda_k^*\Delta_k \\ &= \pi_i\theta S^{-1} + \pi_i\eta - \pi_i\theta S^{-1}\theta^*\eta \\ &= \pi_i(\theta S^{-1} + \eta - \theta S^{-1}\theta^*\eta) \end{aligned}$$

then

$$\psi = \theta S^{-1} + \eta - \theta S^{-1}\theta^*\eta$$

by parts (3) and (5) of the theorem; $\{\Omega_i\}_{i\in I}$ becomes a dual of $*$-operator $\{\Lambda_i\}_{i\in I}$

□

Example 3.2. *Let $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module over itself, let $\{f_j\}_{j\in J} \subset \mathcal{A}$.*

We define the adjointable $\mathcal{A}$-module map $\Lambda_{f_j} : \mathcal{A} \to \mathcal{A}$ with $\Lambda_{f_j}.f = \langle f, f_j\rangle$, clearly $\{f_j\}_{j\in J}$ is a $*$-frame in $\mathcal{A}$ if and only if $\{\Lambda_{f_j}\}_{j\in J}$ is a $*$-operator frame in $\mathcal{A}$.

In the folowing, we study the duals of such $*$-operator frame.

(a) Let $\{g_j\}_{j\in J} \subset \mathcal{A}$ for all $f \in \mathcal{A}$:

$$\sum_{j\in J}\Lambda_{g_j}^*\Lambda_{f_j} f = \sum_{j\in J}\langle f, f_j\rangle g_j = \sum_{j\in J}\langle f, g_j\rangle f_j = \sum_{j\in J}\Lambda_{f_j}^*\Lambda_{g_j} f.$$

Therefore, $\{g_j\}_{j\in J}$ is a dual $*$-frame of $\{f_j\}_{j\in J}$ if and only if $\{\Lambda_{g_j}\}_{j\in J}$; is a dual $*$-operator of $\{\Lambda_{f_j}\}_{j\in J}$

(b) Let S and S_Λ be the $*$-frame operators of $\{f_j\}_{j\in J}$ and $\{\Lambda_{f_j}, \mathcal{A}\}_{j\in J}$ respectively.

For all $f \in \mathcal{A}$ *we have:*

$$\sum_{j\in J}\langle f, f_j\rangle f_j = \sum_{j\in J} f f_j^* f_j = \sum_{j\in J}\langle\langle f, f_j\rangle, f_j^*\rangle = \sum_{j\in J}\Lambda_{f_j}^*\Lambda_{f_j} f.$$

It follows that $S = S_\Lambda$

(c) It is clearly to see that $\{h_j\}_{j\in J} \subset \mathcal{A}$ *is an* $*$*-bessel sequence if and only if* $\{\Lambda_{h_j}, \mathcal{A}\}_{j\in J}$ *is an* $*$*-bessel operator.*

(d) for a $*$*-bessel sequence* $\{h_j\}_{j\in J}$ *we define*

$$g_j = S^{-1}f_j + h_j - \sum_{k\in J}\langle S^{-1}f_j, f_k\rangle h_k$$

then the sequence $\{g_j\}_{j\in J}$ *is a dual* $*$*-frame of* $\{f_j\}_{j\in J}$.

By the last theorem, the sequence $\{\Gamma_j\}_{j\in J}$ *is a dual* $*$*-operator frame of* $\{\Lambda_{f_j}\}_{j\in J}$*, where*

$$\Gamma_j = \tilde{\Lambda}_{f_j} + \Lambda_{h_j} + \sum_{k\in J}\tilde{\Lambda}_{f_j}\Lambda_{f_k}^*\Lambda_{h_k}, \forall j \in J$$

now we clain that $\Gamma_j = \Lambda_{g_j}$

In fact, $\forall f \in \mathcal{A}$ *we have*

$$\begin{aligned}
\Gamma_j f &= \tilde{\Lambda}_{f_j} f + \Lambda_{h_j} f - \sum_{k\in J}\tilde{\Lambda}_{f_j}\Lambda_{f_k}^*\Lambda_{h_k} f \\
&= \Lambda_{f_j}S^{-1}f + \Lambda_{h_j}f - \sum_{k\in J}\Lambda_{f_j}S^{-1}\Lambda_{f_k}^*\langle f, h_k\rangle \\
&= \langle S^{-1}f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J}\langle S^{-1}\Lambda_{f_k}^*\langle f, h_k\rangle, f_j\rangle \\
&= \langle S^{-1}f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J}\langle S^{-1}\Lambda_{f_k}^*\Lambda_{h_k} f, f_j\rangle \\
&= \langle S^{-1}f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J}\langle S^{-1}\Lambda_{h_k} f f_k, f_j\rangle \\
&= \langle S^{-1}f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J}\langle S^{-1} f h_k^* f_k, f_j\rangle \\
&= \langle f, S^{-1}f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J}\langle f h_k^* f_k, S^{-1}f_j\rangle \\
&= \langle f, S^{-1}f_j + h_j\rangle - \sum_{k\in J}\langle f, f_k^* h_k S^{-1}f_j\rangle \\
&= \langle f, S^{-1}f_j + h_j - \sum_{k\in J}\langle S^{-1}f_j, f_k\rangle h_k\rangle \\
&= \langle f, g_j\rangle = \Lambda_{g_j} f.
\end{aligned}$$

therefore, every $*$*-operator frame of* $\{\Lambda_{f_j}\}_{j\in J}$ *has the form :*

$$\tilde{\Lambda}_{f_j} + \Lambda_{h_j} - \sum_{k\in J}\tilde{\Lambda}_{f_j}\Lambda_{f_k}^*\Lambda_{h_k}$$

where $\{h_j\}_{j\in J}$ *is a* $*$*-bessel sequence in* $\mathcal{A}$.

4. Tensor product

In this section, we study the tensor product of the duals $*$-operator frames.

Theorem 4.1. *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *are two Hilbert* C^**-modules over unitary* C^**-Algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively, let* $\{\Lambda_i\}_{i\in I}\subset End^*_{\mathcal{A}}(\mathcal{H})$ *and* $\{\Gamma_j\}_{j\in J}\subset End^*_{\mathcal{B}}(\mathcal{K})$ *are an* $*$*-operators frames. If* $\{\tilde{\Lambda}_i\}_{i\in I}$ *is a dual of* $\{\Lambda_i\}_{i\in I}$ *and* $\{\tilde{\Gamma_j}\}_{j\in J}$ *is a dual of* $\{\Gamma_j\}_{j\in J}$ *then* $\{\tilde{\Lambda_i}\otimes\tilde{\Gamma_j}\}_{i\in I,j\in J}$ *is a dual* $*$*-operator frame of* $\{\Lambda_i\otimes\Gamma_j\}_{i\in I,j\in J}$.

Proof. Let $x\in\mathcal{H}$ and $y\in\mathcal{K}$, we have :

$$\begin{aligned}\sum_{i\in I,j\in J}(\Lambda_i\otimes\Gamma_j)^*(\tilde{\Lambda_i}\otimes\tilde{\Gamma_j})(x\otimes y)&=\sum_{i\in I,j\in J}(\Lambda_i^*\otimes\Gamma_j^*)(\tilde{\Lambda_i}x\otimes\tilde{\Gamma_j}y)\\&=\sum_{i\in I,j\in J}(\Lambda_i^*\tilde{\Lambda_i}x\otimes\Gamma_j^*\tilde{\Gamma_j}y)\\&=\sum_{i\in I}\Lambda_i^*\tilde{\Lambda_i}x\otimes\sum_{j\in J}\Gamma_j^*\tilde{\Gamma_j}y\\&=x\otimes y\end{aligned}$$

then

$$\sum_{i\in I,j\in J}(\Lambda_i\otimes\Gamma_j)^*(\tilde{\Lambda_i}\otimes\tilde{\Gamma_j})=I$$

hence $\{\tilde{\Lambda_i}\otimes\tilde{\Gamma_j}\}_{i\in I,j\in J}$ is a dual $*$-operator frames of $\{\Lambda_i\otimes\Gamma_j\}_{i\in I,j\in J}$. □

Corollary 4.1. *Let* $(\Lambda_{ij})_{0\leq i\leq n;j\in J}$ *be a family of* $*$*-operator and* $(\tilde{\Lambda}_{ij})_{0\leq i\leq n;j\in J}$ *its their dual, then* $(\tilde{\Lambda}_{0j}\otimes\tilde{\Lambda}_{1j}\otimes...\otimes\tilde{\Lambda}_{nj})_{j\in J}$ *is a dual of* $(\Lambda_{0j}\otimes\Lambda_{1j}\otimes...\otimes\Lambda_{nj})_{j\in J}$.

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References

[1] A.Alijani, M.Dehghan, $*$-frames in Hilbert $\mathcal{C}^*$ modules, U. P. B. Sci. Bull. Series A 2011.

[2] O. Christensen, An Introduction to Frames and Riesz bases, Brikhouser,2016.

[3] J.B.Conway, A Course In Operator Theory, AMS, V.21, 2000.

[4] F. R. Davidson, $\mathcal{C}^*$-algebra by example, Fields Ins. Monog. 1996.

[5] I.Daubechies, A.Grossmann, and Y.Meyer, Painless nonorthogonal expansions, J.Math. Phys.27 (1986), 1271-1283

[6] R. J. Duffin, A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952), 341-366.

[7] M. Rossafi, S. Kabbaj, $*$*-operator frame for* $End^*_{\mathcal{A}}(\mathcal{H})$, Wavelet and Linear Algebra, Vol 5, No. 02, 2018, 1-13.

[8] A. Khosravi, B. Khosravi, Frames and bases in tensor products of Hilbert spaces and Hilbert $\mathcal{C}^*$-modules, *Proc. Indian Acad. Sci. Math. Sci.* **117** (2007), 1-12.

[9] M. Rossafi, S. Kabbaj, Operator Frame for $End^*_{\mathcal{A}}(\mathcal{H})$, J.Linear.Topological.Algebra. Vol.08, No.02, 2019, 85-95.

[10] M. Rossafi, S. Kabbaj, Frames and Operator Frames fo $\mathcal{B}(H)$, Asia Mathematika, Volume 2, Issue 3, December 2018, Pages: 19-23.

[11] M. Rossafi, S. Kabbaj, K-operator Frames for $End^*_{\mathcal{A}}(\mathcal{H})$, Asia Mathematika, Volume 2, Issue 2, August 2018, Pages: 52-60.

[12] V. M. Manuilov, E. V. Troitsky, Hilbert $\mathcal{C}^*$-modules, AMS, V. 226, 2005.