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Generalized Hyers-Ulam type stability of the additive functional equation inequalities with $2n$-variables on an approximate group and ring homomorphism

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Abstract:
In this paper we study to solve additive functional inequality with $2n-$ variables and their Hyers-Ulam type stability. First are investigated results with a direction method of group homomorphism and last are investigated in ring homomorphism. Then I will show that the solutions of inequality are additive mapping. These are the main results of this paper .

Key words: stability, functional equation Banach space; generalized Hyers-Ulam stability. Jordan-homomorphism, Lie-homomorphism, equation functional inequality
Mathematics Subject Classification: 39B52, 46S10, 47S10, 12J25

1. Introduction

The study of the functional equation stability originated from a question of S.M. Ulam [23], concerning the stability of group homomorphisms. Let $\left(\mathbb{G}, *\right)$ be a group and let $\left(\mathbb{G}', \circ, d\right)$ be a metric group with metric $d\left(\cdot,\cdot\right)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : \mathbb{G} \to \mathbb{G}'$ satisfies

$$d\left(f\left(x * y\right), f\left(x\right) \circ f\left(y\right)\right) < \delta$$

for all $x, y \in \mathbb{G}$ then there is a homomorphism $h : \mathbb{G} \to \mathbb{G}'$ with

$$d\left(f\left(x\right), h\left(x\right)\right) < \epsilon$$

for all $x \in \mathbb{G}$?, if the answer, is affirmative, we would say that equation of homomorphism $h\left(x * y\right) = h\left(y\right) \circ h\left(y\right)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation? Hyers[13] gave a first affirmative answes the question of Ulam as follows:

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Theorem 1.1. *(D. H. Hyers) Let* $\epsilon \geq 0$ *and let* f *be a function defined on an Abelian group* $\left(\mathbb{G}, +\right)$ *with values in Banach spaces* $\left(\mathbb{Y}, +\right)$ *satisfying*

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\| \leq \epsilon,$$

for all $x, y \in \mathbb{G}$ *and some* $\epsilon \geq 0$. *Then there exists a unique additive mapping* $T : \mathbb{G} \to \mathbb{Y}$, *such that*

$$\left\|f\left(x\right)-T\left(x\right)\right\| \leq \epsilon, \forall x \in \mathbb{G}.$$

Next Th. M. Rassias [20] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded:

Theorem 1.2. *(Th. M. Rassias.) Consider* $\mathbb{E}, \mathbb{E}'$ *to be two Banach spaces, and let* $f : \mathbb{E} \to \mathbb{E}'$ *be a mapping such that* $f\left(tx\right)$ *is continous in* t *for each fixed* x. *Assume that there exist* $\theta > 0$ *and* $p \in [0, 1]$ *such that*

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\| \leq \epsilon\left(\left\|x\right\|^p+\left\|y\right\|^p\right), \forall x, y \in \mathbb{E}.$$

then there exists a unique linear $L : \mathbb{E} \to \mathbb{E}'$ *satifies*

$$\left\|f\left(x\right)-L\left(x\right)\right\| \leq \frac{2\theta}{2-2^p}\|x\|^p, x \in \mathbb{E}.$$

Next Badora in [6] provided the following result concering the stability of a ring homomorphism:

Theorem 1.3. *Let* $\Re$ *be a ring and* $\mathbb{Y}$ *be Banach algebra and* $\epsilon, \delta \geq 0$. *Assume that* $f : \Re \to \mathbb{Y}$ *satisfies*

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\| \leq \epsilon$$

and

$$\left\|f\left(x \cdot y\right)-f\left(x\right)f\left(y\right)\right\| \leq \delta,$$

$\forall x, y \in \Re$. *Then there exists a unique ring homomorphism* $T : \Re \to \mathbb{Y}$ *such that*

$$\left\|f(x)-T(x)\right\| \leq \epsilon, \forall x \in \Re.$$

The stability problems for several functional equations have been extensively investigated by a number of authors and and there are many interesting results concerning this problem. Such as in 1978, Rassias in [20] prove a generalization of Hyers' theorem the Cauchy difference to be unbounded and R. Badora in [6] prove generalization the result on a ring homomorphisms next in [24] proved the generalized Hyers' theorem and

Badora' theorem. Recently, in [6, 13, 20, 24] the authors studied the Hyers-Ulam stability for the following functional inequalities

$$\left\| f\left(\sum_{k=1}^{r} x_k\right) - \sum_{k=1}^{r} f\left(x_k\right)\right\|_{\mathbb{Y}} \leq \epsilon, \forall \epsilon \geq 0.$$

and

$$\left\| f\left(\prod_{j=1}^{r} x_j\right) - \prod_{j=1}^{r} f\left(x_j\right)\right\|_{\mathbb{Y}} \leq \delta, \forall \delta \geq 0$$

in group and ring homomorphisms. So that we solve and proved the Hyers-Ulam type stability for functional inequalities

$$\left\| f\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) - \sum_{j=1}^{n} f\left(x_j\right) - \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \epsilon, \forall \epsilon \geq 0 \tag{1}$$

and

$$\left\| f\left(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) - \prod_{j=1}^{n} f\left(x_j\right) - \prod_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \delta, \forall \delta \geq 0 \tag{2}$$

ie the functional inequalities with $2n$-variables. Under suitable assumptions on spaces $\mathbb{X}$ and $\mathbb{Y}$, we will prove that the mappings satisfying the functional inequalities (1) and (2). Thus, the results in this paper are generalization of those in [6, 13, 20, 24] for functional inequalities with $2n$-variables.
The paper is organized as followns: In section preliminaries we remind some basic notations in [3, 4, 14] such as solutions of the inequalities.

Section 3 is devoted to prove the Hyers-Ulam type stability of the additive functional inequalities (1) when $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be a Banach space.

Section 4 is devoted to prove the Hyers-Ulam type stability of the additive functional inequalitie (1) and (2) when $\mathbb{X}$ be a ring and $\mathbb{Y}$ be a Banach algebra, $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be a Banach space

2. preliminaries

2.1. Banach spaces.

Definition 2.1. Let $\left\{x_n\right\}$ be a sequence in a normed space $\mathbb{X}$.

1. A sequence $\left\{x_n\right\}_{n=1}^{\infty}$ in a space $\mathbb{X}$ is a Cauchy sequence iff the sequence $\left\{x_{n+1} - x_n\right\}_{n=1}^{\infty}$ converges to zero;
2. The sequence $\left\{x_n\right\}_{n=1}^{\infty}$ is said to be convergent if, there exists $x \in \mathbb{X}$ such that, for any $\epsilon > 0$, there is a positive integer N such that

$$\Big\|x_n - x\Big\| \leq \epsilon.\forall n \geq N.$$

Then the point x$\in X$ is called the limit of sequence x_n and denoted by $\lim_{n\to\infty} x_n = x$;

3. If every sequence Cauchy in $\mathbb{X}$ converger, then the normed space X is called a Banach space.

2.2. Solutions of the inequalities.

The functional equation

$$f\Big(x+y\Big) = f\Big(x\Big) + f\Big(y\Big)$$

is called the Cauchuy equation. In particular, every solution of the Cauchuy equation is said to be an *additive mapping*.

3. Stability of Approximate Group Homomorphisms

Now, we first study the solutions of (1). Note that for this inequalitie, $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ is a Banach spaces. Under this setting, we can show that the mapping satisfying (1) is additive. These results are give in the following.

Theorem 3.1. *Let $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be Banach space. If $\epsilon \geq 0$, $n \in \mathbb{N}$, $n \geq 2$ and $f : \mathbb{X} \to \mathbb{Y}$ such that*

$$\left\|f\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) - \sum_{j=1}^{n} f\Big(x_j\Big) - \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \epsilon \tag{3}$$

for all $x_1, x_2, ..., x_{2n} \in X$, then there exists a unique additive mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\Big\|f(x) - H(x)\Big\|_{\mathbb{Y}} \leq \frac{1}{2n-1}\epsilon, \forall x \in \mathbb{X}. \tag{4}$$

Proof. We will show that

$$\left\|\frac{f\Big((2n)^k x\Big)}{(2n)^k} - f\Big(x\Big)\right\|_{\mathbb{Y}} \leq \epsilon\sum_{m=1}^{k}(2n)^{-m}, \forall x \in \mathbb{X}. \tag{5}$$

for any positive integer k and for all $x \in \mathbb{X}$. The proof of (5) follows by induction on k. With $k = 1$ and letting $x_j = x, x_{n+j} = nx$ for all $j = 1, 2, ..., n$ by the hypothesis (3), we have

$$\left\|\frac{f\Big(2nx\Big)}{2n} - f\Big(x\Big)\right\|_{Y} = \frac{1}{2n}\left\|f\Big(2nx\Big) - 2nf\Big(x\Big)\right\|_{\mathbb{Y}} \leq \frac{1}{2n}\epsilon, \forall x \in \mathbb{X}.$$

Assume now that (5) holds for k and we want to prove it for the case $k+1$. Replacing x by $2nx$ in (5) we obtain

$$\left\|\frac{f\Big((2n)^k \cdot 2nx\Big)}{(2n)^k} - f\Big(2nx\Big)\right\|_{\mathbb{Y}} \leq \epsilon\sum_{m=1}^{k}(2n)^{-m}, \forall x \in \mathbb{X}.$$

therefore

$$\left\|\frac{f\Big((2n)^{k+1}x\Big)}{(2n)^{k+1}}-\frac{1}{2n}f\Big(2nx\Big)\right\|_{\mathbb{Y}}\le\epsilon\sum_{m=2}^{k+1}(2n)^{-m},\forall x\in\mathbb{X}.$$

Now, using the triangle inequality we deduce

$$\begin{aligned}\left\|\frac{f\Big((2n)^{k+1}x\Big)}{(2n)^{k+1}}-f\Big(x\Big)\right\|_{\mathbb{Y}}&\le\left\|\frac{f\Big((2n)^{k+1}x\Big)}{(2n)^{k+1}}-\frac{1}{2n}f\Big(2nx\Big)\right\|_{\mathbb{Y}}+\left\|\frac{1}{2n}f\Big(2nx\Big)-f\Big(x\Big)\right\|_{\mathbb{Y}}\\&\le\frac{\epsilon}{2n}+\epsilon\sum_{m=2}^{k+1}(2n)^{-m}\\&\le\epsilon\sum_{m=1}^{k+1}(2n)^{-m}.\end{aligned}$$

Thus, (5) is valid for all $k\in\mathbb{N}$. Since $\sum_{m=1}^{k}(2n)^{-m}$ is increasingly convergent to $\frac{1}{2n-1}$, we get from (5) that

$$\left\|\frac{f\Big((2n)^{k+1}x\Big)}{(2n)^{k+1}}-f\Big(x\Big)\right\|_{Y}\le\frac{1}{2n-1}\epsilon,\forall x\in X.\tag{6}$$

Fixing an $x\in\mathbb{X}$, for all $h,k\in\mathbb{N}$ with $h>k$, we have, from (6) that

$$\begin{aligned}\left\|\frac{f\Big((2n)^{h}x\Big)}{(2n)^{h}}-\frac{1}{(2n)^{k}}f\Big((2n)^{k}x\Big)\right\|_{\mathbb{Y}}&=\frac{1}{(2n)^{h}}\left\|\frac{1}{(2n)^{h-k}}f\Big((2n)^{h}x\Big)-f\Big((2n)^{k}x\Big)\right\|_{\mathbb{Y}}\\&\le\frac{1}{(2n)^{k}}\cdot\frac{1}{2n-1}\epsilon.\end{aligned}$$

Therefore

$$\lim_{h,k\to\infty}\left\|\frac{f\Big((2n)^{h}x\Big)}{(2n)^{h}}-\frac{1}{(2n)^{k}}f\Big((2n)^{k}x\Big)\right\|_{\mathbb{Y}}=0.$$

Since Y is Banach space, the sequence $\left\{\frac{f\big((2n)^{k}x\big)}{(2n)^{k}}\right\}$ converges. Set

$$H\Big(x\Big)=\lim_{k\to\infty}\frac{f\Big((2n)^{k}x\Big)}{(2n)^{k}},\forall x\in\mathbb{X}.\tag{7}$$

Then we obtain a mapping $H:\mathbb{X}\to\mathbb{Y}$. From (10), for all $x_1,x_2,...,x_{2n}\in X$ and for all $k\in\mathbb{N}$, We compute that

$$\left\|f\left((2n)^k\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}f\Big((2n)^kx_j\Big)-\sum_{j=1}^{n}f\left((2n)^k\frac{x_{n+j}}{n}\right)\right\|_Y\le\epsilon,$$

and so

$$\frac{1}{(2n)^k}\left\|f\left((2n)^k\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)\right)-\sum_{j=1}^{n}f\Big((2n)^kx_j\Big)-\sum_{j=1}^{n}f\left((2n)^k\frac{x_{n+j}}{n}\right)\right\|_Y\le\frac{1}{(2n)^k}\epsilon.$$

We will prove that H is additive.
Consequently,

$$\begin{aligned}&\left\|H\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}H\Big(x_j\Big)-\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)\right\|_Y\\&=\lim_{k\to\infty}\left\|\frac{1}{(2n)^k}f\left((2n)^k\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)\right)\right.\\&\left.-\sum_{j=1}^{n}\frac{1}{(2n)^k}f\Big((2n)^kx_j\Big)-\sum_{j=1}^{n}\frac{1}{(2n)^k}f\left((2n)^k\frac{x_{n+j}}{n}\right)\right\|_Y\\&\le\lim_{k\to\infty}\left\|\frac{1}{(2n)^k}\epsilon\right\|=0.\end{aligned}$$

It follows from (7) that

$$\left\|H\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}H\Big(x_j\Big)-\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)\right\|_Y=0.$$

Hence

$$H\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)=\sum_{j=1}^{n}H\Big(x_j\Big)+\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right),$$

for all $x_1,x_2,...,x_{2n}\in X$.

Clearly, $H\Big(0\Big)=0$ and so H is an additive mapping. From (6) and (7) we see that (4) is valid. Now we prove the uniqueness of H. Assume that $H_1:X\to Y$ is an additive mapping such that

$$\left\|f\Big(x\Big)-H_1\Big(x\Big)\right\|\le\frac{1}{2n-1}\epsilon,\forall x\in X.$$

Since both H and H_1 are additive, we deduce that, for each $\forall x\in X$ and for all $n\in\mathbb{N}$,

$$2n\left\|H(x)-H_1\Big(x\Big)\right\|_Y = \left\|H(2nx)+H_1(2nx)\right\|_Y$$
$$\leq \left\|H(2nx)-f\Big(2nx\Big)\right\|_Y + \left\|f(2nx)+H_1(2nx)\right\|_Y$$
$$\leq \frac{2\epsilon}{2n-1},$$

so that

$$\left\|H(x)-H_1\Big(x\Big)\right\|_{\mathbb{Y}} \leq \frac{2\epsilon}{n\Big(2n-1\Big)}$$

for all $x\in\mathbb{X}$ and hence $H\Big(x\Big)=H_1\Big(x\Big)$ for all $x\in\mathbb{X}$. This completes the proof. □

Corollary 3.1. *Let $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be Banach space. If $\epsilon\geq 0$, $n\in\mathbb{N}$, $n\geq 2$, $f\Big(0\Big)=0$ and $f:\mathbb{X}\to\mathbb{Y}$ such that*

$$\left\|f\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}f\Big(x_j\Big)-\sum_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\leq\epsilon, \tag{8}$$

for all $x_1,x_2,...,x_{2n}\in X$, then there exists a unique additive group homomorphism $H:\mathbb{X}\to\mathbb{Y}$ such that

$$\left\|f(x)-H(x)\right\|_{\mathbb{Y}}\leq\frac{1}{2n-1}\epsilon,\forall x\in\mathbb{X}. \tag{9}$$

4. Stability of a Ring Homomorphism

Now, we first study the solutions of (2). Note that for this inequalitie, $\mathbb{X}$ be a ring and $\mathbb{Y}$ is a Banach algebra and $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ is a Banach spaces. Under this setting, we can show that the mapping satisfying (2) is additive. These results are give in the following.

Theorem 4.1. *Let $\Re$ be a ring and $\mathbb{Y}$ be Banach algebra and $\epsilon,\delta\geq 0$ and $n\in\mathbb{N}$, $n\geq 2$. If a mapping $f:\Re\to\mathbb{Y}$ satisfies*

$$\left\|f\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}f\Big(x_j\Big)-\sum_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\leq\epsilon \tag{10}$$

and

$$\left\|f\left(\prod_{j=1}^{n}x_j+\frac{1}{n}\prod_{j=1}^{n}x_{n+j}\right)-\prod_{j=1}^{n}f\Big(x_j\Big)-\prod_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\leq\delta \tag{11}$$

for all $x_1,x_2,...,x_{2n}\in\Re$, then there exists a unique additive mapping $H:\Re\to\mathbb{Y}$ such that

$$H\left(\prod_{j=1}^{n}x_j+\frac{1}{n}\prod_{j=1}^{n}x_{n+j}\right)=\prod_{j=1}^{n}H\Big(x_j\Big)+\prod_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right) \tag{12}$$

for all $x_1, x_2, ..., x_{2n} \in \Re$ *and*

$$\Big\|f\Big(x\Big) - H\Big(x\Big)\Big\|_{\mathbb{Y}} \leq \frac{1}{2n-1}\epsilon, \forall x \in \Re. \tag{13}$$

Proof. Theorem 3.1 show that there exists a unique additive mapping $H : \Re \to \mathbb{Y}$ satisfies (13). By the proof of Theorem 3.1, we see that the mapping H is give by

$$H\Big(x\Big) = \lim_{k\to\infty} \frac{1}{\big(2n\big)^k} f\Big(\big(2n\big)^k x\Big), \forall x \in \Re \tag{14}$$

for all $x_1, x_2, ..., x_{2n} \in \Re$, let

$$h\Big(x_1, x_2, ..., x_{2n}\Big) = f\left(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) - \prod_{j=1}^{n} f\Big(x_j\Big) - \prod_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right).$$

The using inequality (10), we get

$$\lim_{k\to\infty} \frac{1}{\big(2n\big)^k} h\Big(\big(2n\big)^k x_1, x_2, ..., x_{2n}\Big) = 0.$$

Therefore

$$\begin{aligned}
H\left(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) &= H\Big(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\Big) \\
&= \lim_{k\to\infty} \frac{1}{\big(2n\big)^k} f\Big(\big(2n\big)^k \Big(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{+2} \cdots x_{2n}\Big) \\
&= \lim_{k\to\infty} \frac{1}{\big(2n\big)^k} h\Big(\Big(\big(2n\big)^k x_1\Big) x_2 \cdots x_n + \frac{1}{n}\Big(\big(2n\big)^k x_{n+1}\Big) x_{n+2} \cdots x_{2n}\Big)\Big) \\
&= \lim_{k\to\infty} \frac{1}{\big(2n\big)^k} \Bigg[h\Big((2n)^k x_1, x_2, ..., x_{2n}\Big) + f\Big((2n)^k x_1\Big) f\Big(x_2\Big) \cdots f\Big(x_n\Big) \\
&+ f\left(\big(2n\big)^k \cdot \frac{x_{n+1}}{n}\right) f\left(\frac{x_{n+2}}{n}\right) \cdots f\left(\frac{x_{2n}}{n}\right)\Bigg] \\
&= \prod_{j=1}^{n} H\Big(x_j\Big) + \prod_{j=1}^{n} H\left(\frac{x_{n+j}}{n}\right),
\end{aligned}$$

$\forall x_1, x_2, ..., x_{2n} \in \Re$.
From the last equation and the additivity of H we see that, for all $k \in \mathbb{N}$

$$H\Big(x_1\Big)f\Big((2n)^k x_2\Big)\cdots f\Big(x_n\Big)+H\left(\frac{x_{n+1}}{n}\right)f\left(\Big(2n\Big)^k\cdot\frac{x_{n+2}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right)$$
$$=H\left(x_1\cdot(2n)^k x_2\cdots x_n+\frac{1}{n}x_{n+1}\cdot(2n)^k x_{n+2}\cdots x_{2n}\right)$$
$$=H\left((2n)^k\cdot x_1\cdot x_2\cdots x_n+\frac{1}{n}(2k)^k x_{n+1}\cdot x_{n+2}\cdots x_{2n}\right)$$
$$=(2n)^k H\Big(x_1\Big)f\Big(x_2\Big)\cdots f\Big(x_n\Big)$$
$$+(2n)^k H\left(\frac{x_{n+1}}{n}\right)f\left(\frac{x_{n+2}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right)$$

and so

$$H\Big(x_1\Big)\frac{f\Big((2n)^k x_2\Big)}{(2n)^k}\cdots f\Big(x_n\Big)+H\left(\frac{x_{n+1}}{n}\right)\frac{f\left(\Big(2n\Big)^k\cdot\frac{x_{n+2}}{n}\right)}{(2n)^k}\cdots f\left(\frac{x_{2n}}{n}\right)$$
$$=H\Big(x_1\Big)f\Big(x_2\Big)\cdots f\Big(x_n\Big)+H\left(\frac{x_{n+1}}{n}\right)f\left(\frac{x_{n+2}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right).$$

Sending k to to infinty, we see that

$$H\Big(x_1\Big)H\Big(x_2\Big)H\Big(x_3\Big)\cdots f\Big(x_n\Big)+H\left(\frac{x_{n+1}}{n}\right)H\left(\frac{x_{n+2}}{n}\right)H\left(\frac{x_{n+3}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right)$$
$$=H\left(x_1x_2\cdots x_n+\frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right),\tag{15}$$

$\forall x_1,x_2,...,x_{2n}\in\Re$.
Suppose that

$$H\Big(x_1\Big)H\Big(x_2\Big)H\Big(x_3\Big)\cdots H\Big(x_{n-1}\Big)f\Big(x_n\Big)+H\left(\frac{x_{n+1}}{n}\right)H\left(\frac{x_{n+2}}{n}\right)H\left(\frac{x_{n+3}}{n}\right)$$
$$\cdots H\left(\frac{x_{2n-1}}{n}\right)f\left(\frac{x_{2n}}{n}\right)$$
$$=H\left(x_1x_2\cdots x_n+\frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right),\tag{16}$$

$\forall x_1, x_1, ..., x_{2n} \in \Re$. Then, from (16), we get that, for all $k \in \mathbb{N}$.

$$\begin{aligned}
&\frac{1}{(2n)^k} H\Big(x_1\Big) H\Big(x_2\Big) H\Big(x_3\Big) \cdots H\Big(x_{n-1}\Big) f\Big((2n)^k x_n\Big) \\
&\quad + \frac{1}{(2n)^k} H\left(\frac{x_{n+1}}{n}\right) H\left(\frac{x_{n+2}}{n}\right) H\left(\frac{x_{n+3}}{n}\right) \cdots H\left(\frac{x_{2n-1}}{n}\right) f\left((2n)^k \cdot \frac{x_{2n}}{n}\right) \\
&\quad = \frac{1}{(2n)^k} H\left((2n)^k \left(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\right)\right) \\
&\quad = H\left(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\right).
\end{aligned} \tag{17}$$

By letting $k \to \infty$ we see that

$$\begin{aligned}
&H\Big(x_1\Big) H\Big(x_2\Big) H\Big(x_3\Big) \cdots H\Big(x_n\Big) + H\left(\frac{x_{n+1}}{n}\right) H\left(\frac{x_{n+2}}{n}\right) H\left(\frac{x_{n+3}}{n}\right) \cdots H\left(\frac{x_{2n}}{n}\right) \\
&\qquad = H\left(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\right).
\end{aligned} \tag{18}$$

$\forall x_1, x_2, ..., x_{2n} \in \Re$ which is the desired identity (12) □

Theorem 4.2. *Let $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be Banach space. If $\epsilon \geq 0$, $\forall k \in \mathbb{N}$, and $\varphi : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ such that*

$$2^k \varphi\Big(x, y, z\Big) = \varphi\Big(2^k x, y, z\Big) = \varphi\Big(x, 2^k y, z\Big) = \varphi\Big(x, y, 2^k z\Big)$$

$\forall k \in \mathbb{N}, x, y, z \in \mathbb{X}$
and $\psi : \mathbb{Y} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ be a continuous mapping such that

$$2^k \psi\Big(x, y, z\Big) = \psi\Big(2^k x, y, z\Big) = \psi\Big(x, 2^k y, z\Big) = \psi\Big(x, y, 2^k z\Big)$$

$\forall k \in \mathbb{N}, x, y, z \in \mathbb{Y}$ *and* $\epsilon, \delta \geq 0$.
If $f : \mathbb{X} \to \mathbb{Y}$ satisfies

$$\left\| f\left(\frac{x+y}{2} + z\right) - f\Big(\frac{x+y}{2}\Big) - f\Big(z\Big) \right\|_{\mathbb{Y}} \leq \epsilon \tag{19}$$

for all $x, y, z \in \mathbb{X}$ and

$$\left\| f\left(\varphi\Big(x, y, z\Big)\right) - \psi\left(f\Big(x\Big), f\Big(y\Big), f\Big(z\Big)\right) \right\|_{\mathbb{Y}} \leq \delta \tag{20}$$

for all $x, y, z \in \mathbb{X}$
. Then there exists a unique additive mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$H\left(\varphi\Big(x, y, z\Big)\right) = \psi\left(H\Big(x\Big), H\Big(y\Big), H\Big(z\Big)\right) \tag{21}$$

for all $x, y, z \in \mathbb{X}$

$$\Big\|f\big(x\big) - H\big(x\big)\Big\|_{\mathbb{Y}} \le \frac{1}{2}\epsilon, \forall x \in \mathbb{X}. \tag{22}$$

Proof. We will show that

$$\left\|\frac{f\Big(2^k x\Big)}{2^k} - f\Big(x\Big)\right\|_{\mathbb{Y}} \le \epsilon \sum_{m=1}^{k} 2^{-m}, \forall x \in \mathbb{X}. \tag{23}$$

for any positive integer k and for all $x \in \mathbb{X}$. The proof of (23) follows by induction on k. With $k = 1$ and letting $x = y = z$ by the hypothesis (19), we have

$$\left\|\frac{f\Big(2x\Big)}{2} - f\Big(x\Big)\right\|_{Y} = \frac{1}{2}\left\|f\Big(2x\Big) - 2f\Big(x\Big)\right\|_{\mathbb{Y}} \le \frac{1}{2}\epsilon, \forall x \in \mathbb{X}.$$

Assume now that (23) holds for k and we want to prove it for the case $k+1$. Replacing x by $2x$ in (23) we obtain

$$\left\|\frac{f\Big(2^k \cdot 2x\Big)}{2^k} - f\Big(2x\Big)\right\|_{\mathbb{Y}} \le \epsilon \sum_{m=1}^{k} 2^{-m}, \forall x \in \mathbb{X}.$$

therefore

$$\left\|\frac{f\Big(2^{k+1} x\Big)}{2^{k+1}} - \frac{1}{2}f\Big(2x\Big)\right\|_{\mathbb{Y}} \le \epsilon \sum_{m=2}^{k+1} 2^{-m}, \forall x \in \mathbb{X}.$$

Now, using the triangle inequality we deduce

$$\begin{aligned}\left\|\frac{f\Big(2^{k+1} x\Big)}{2^{k+1}} - f\Big(x\Big)\right\|_{\mathbb{Y}} &\le \left\|\frac{f\Big(2^{k+1} x\Big)}{2^{k+1}} - \frac{1}{2}f\Big(2x\Big)\right\|_{\mathbb{Y}} + \left\|\frac{1}{2}f\Big(2x\Big) - f\Big(x\Big)\right\|_{\mathbb{Y}} \\ &\le \frac{\epsilon}{2} + \epsilon \sum_{m=2}^{k+1} 2^{-m} \\ &\le \epsilon \sum_{m=1}^{k+1} 2^{-m}.\end{aligned}$$

Thus, (23) is valid for all $k \in \mathbb{N}$. Since $\sum_{m=1}^{k} 2^{-m}$ is increasingly convergent to $\frac{1}{2}$, we get from (23) that

$$\left\|\frac{f\Big(2^{k+1} x\Big)}{2^{k+1}} - f\Big(x\Big)\right\|_{\mathbb{Y}} \le \frac{1}{2}\epsilon, \forall x \in X. \tag{24}$$

Fixing an $x \in \mathbb{X}$, for all $h, k \in \mathbb{N}$ with $h > k$, we have, from (24) that

$$\left\|\frac{f\left(2^h x\right)}{2^h} - \frac{1}{2^k} f\left(2^k x\right)\right\|_{\mathbb{Y}} = \frac{1}{2^h}\left\|\frac{1}{2^{h-k}} f\left(2^h x\right) - f\left(2^k x\right)\right\|_{\mathbb{Y}}$$
$$\leq \frac{1}{2^k} \cdot \frac{1}{2}\epsilon.$$

Therefore

$$\lim_{h,k\to\infty}\left\|\frac{f\left(2^h x\right)}{2^h} - \frac{1}{2^k} f\left(2^k x\right)\right\|_{\mathbb{Y}} = 0.$$

Since Y is Banach space, the sequence $\left\{\frac{f\left(2^k x\right)}{2^k}\right\}$ converges. Set

$$H\left(x\right) = \lim_{k\to\infty}\frac{f\left(2^k x\right)}{2^k}, \forall x \in \mathbb{X}. \tag{25}$$

Then we obtain a mapping $H : \mathbb{X} \to \mathbb{Y}$. From (19), for all $x, y, z \in X$ and for all $k \in \mathbb{N}$, We compute that

$$\left\|f\left(2^k\left(\frac{x+y}{2} + z\right)\right) - f\left(2^k \frac{x+y}{2}\right) - f\left(2^k z\right)\right\|_{\mathbb{Y}} \leq \epsilon,$$

and so

$$\frac{1}{2^k}\left\|f\left(2^k\left(\frac{x+y}{2} + z\right)\right) - f\left(2^k \frac{x+y}{2}\right) - f\left(2^k z\right)\right\|_{\mathbb{Y}} \leq \frac{1}{2^k}\epsilon.$$

We will prove that H is additive.
Consequently,

$$\left\|H\left(\frac{x+y}{2} + z\right) - H\left(\frac{x+y}{2}\right) - H\left(z\right)\right\|_{\mathbb{Y}}$$
$$= \lim_{k\to\infty}\left\|\frac{1}{2^k} f\left(2^k\left(\frac{x+y}{2} + z\right)\right) - \frac{1}{2^k} f\left(2^k \frac{x+y}{2}\right) - \frac{1}{2^k} f\left(2^k z\right)\right\|_{\mathbb{Y}}$$
$$\leq \lim_{k\to\infty}\left\|\frac{1}{2^k}\epsilon\right\| = 0.$$

It follows from (25) that

$$\left\|H\left(\frac{x+y}{2} + z\right) - H\left(\frac{x+y}{2}\right) - H\left(z\right)\right\|_{\mathbb{Y}} = 0.$$

Hence

$$H\left(\frac{x+y}{2} + z\right) = H\left(\frac{x+y}{2}\right) + H\left(z\right)$$

for all $x, y, z \in X$.

Clearly, $H\Big(0\Big) = 0$ and so H is an additive mapping. From (24) and (25) we see that (22) is valid. Now we prove the uniqueness of H. Assume that $H_1 : X \to Y$ is an additive mapping such that

$$\left\|f\Big(x\Big) - H_1\Big(x\Big)\right\|_{\mathbb{Y}} \leq \frac{1}{2}\epsilon, \forall x \in X.$$

Since both H and H_1 are additive, we deduce that, for each $\forall x \in X$ and for all $k \in \mathbb{N}$,

$$\begin{aligned} k\left\|H(x) - H_1\Big(x\Big)\right\|_Y &= \left\|H(kx) + H_1(kx)\right\|_Y \\ &\leq \left\|H(kx) - f\Big(kx\Big)\right\|_Y + \left\|f\Big(kx\Big) + H_1\Big(kx\Big)\right\|_Y \\ &\leq \epsilon, \end{aligned}$$

so that

$$\left\|H(x) - H_1\Big(x\Big)\right\|_{\mathbb{Y}} \leq \frac{\epsilon}{k}$$

for all $x \in \mathbb{X}$ and hence $H\Big(x\Big) = H_1\Big(x\Big)$ for all $x \in \mathbb{X}$.

Next to show that the mapping H satisfies (20), us define

$$Q\Big(x, y, z\Big) = f\left(\varphi\Big(x, y, z\Big)\right) - \psi\left(f\Big(x\Big), f\Big(y\Big), f\Big(z\Big)\right), \forall x, y, z \in \mathbb{X}.$$

Then from condition (20), we see that

$$\lim_{k\to\infty} \frac{1}{2^k} Q\Big(2^k x, y, z\Big) = 0, \forall x, y, z \in \mathbb{X}.$$

Thus, by (25) we have for all $x, y, z \in \mathbb{X}$

$$\begin{aligned} H\left(\varphi\Big(x, y, z\Big)\right) &= \lim_{k\to\infty} \frac{1}{2^k} f\left(\varphi\Big(2^k x, y, z\Big)\right) \\ &= \lim_{k\to\infty} \frac{1}{2^k}\left(\psi\left(\frac{f\Big(2^k x\Big)}{2^k}, f(y), f(z)\right) + \frac{Q\Big(2^k x, y, z\Big)}{2^k}\right) \\ &= \psi\Big(H(x), f(y), f(z)\Big) \end{aligned}$$

From the last equation and the additivity of H, we obtain that

$$H\left(\varphi\Big(x, y, z\Big)\right) = \frac{1}{2^k} H\left(\varphi\Big(x, y, 2^k z\Big)\right) = \psi\left(H\Big(x\Big), f(y), \frac{1}{2^k} f(2^k z)\right).$$

Letting $k \to \infty$ yields (21). This completes the proof. □

From proving the theorems we have corollarys:

Corollary 4.1. *Let* $\Re$ *be a ring with a unit* 1 *and* $\mathbb{Y}$ *be Banach algebra with a unit* e *and* $\epsilon, \delta \geq 0$ *and* $n \in \mathbb{N}$, $n \geq 2$. *If a mapping* $f : \Re \to \mathbb{Y}$ *satisfies*

$$\left\| f\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) - \sum_{j=1}^{n} f\left(x_j\right) - \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \epsilon \tag{26}$$

and

$$\left\| f\left(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) - \prod_{j=1}^{n} f\left(x_j\right) - \prod_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \delta \tag{27}$$

for all $x_1, x_2, ..., x_{2n} \in \Re$ *and* $f\left(1\right) = e$, *then there exists a unique ring homomorphism* $H : \Re \to \mathbb{Y}$ *such that*

$$\left\| f(x) - H(x)\right\|_{\mathbb{Y}} \leq \frac{1}{2n-1}\epsilon, \forall x \in \Re. \tag{28}$$

Corollary 4.2. *Let* $\mathbb{X}$ *be an algebra,* $\mathbb{Y}$ *be Banach algebra and* $\epsilon, \delta \geq 0$. *If a mapping* $f : \mathbb{X} \to \mathbb{Y}$ *satisfies*

$$\left\| f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x+y}{2}\right) - f\left(z\right)\right\|_{\mathbb{Y}} \leq \epsilon \tag{29}$$

and for all $x, y, z \in \mathbb{X}$,

$$\left\| f\left(\left[x, y, z\right]\right) - \left[f\left(x\right), f\left(y\right), f\left(z\right)\right]\right\|_{\mathbb{Y}} \left(resp. \left\| f\left(x \circ y \circ z\right) - f\left(x\right) \circ f\left(y\right) \circ f\left(z\right)\right\|_{\mathbb{Y}}\right) \leq \delta, \tag{30}$$

then there exists a unique additive mapping $H : \mathbb{X} \to \mathbb{Y}$ *such that*

$$H\left(\left[x, y, z\right]\right) = \left[H\left(x\right), H\left(y\right), H\left(z\right)\right], \left(resp. f\left(x \circ y \circ z\right) = f\left(x\right) \circ f\left(y\right) \circ f\left(z\right)\right), \tag{31}$$

for all $x, y, z \in \mathbb{X}$ *and*

$$\left\| f(x) - H(x)\right\|_{\mathbb{Y}} \leq \epsilon, \forall x \in \mathbb{X}. \tag{32}$$

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