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Nonexistence results for semi-linear structurally damped wave equation and system of derivative type

Ali HAKEM *
Laboratory ACEDP, Djillali Liabes university,
22000 Sidi Bel Abbes, Algeria. Orchid iD: 0000-0001-6145-4514

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Abstract: Main purpose of this paper is to study the nonexistence of global weak solution for the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + g(t)(-\Delta)^{\frac{\alpha}{2}} u_{tt} = |u_t|^p, & x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & x \in \mathbb{R}^n, \end{cases}$$

where $\alpha \in (0,2)$, $p > 1, n \geq 1, g(t) = t^\lambda, \lambda \geq 1$ and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator of order $\frac{\alpha}{2}$. Then, this result is extended to the case of 2×2-system of the same type. The results obtained in this paper extend several contributions in this field.

Key words: Weak solution, test functions, nonexistence.

1. Introduction

In this paper, we are first concerned with the nonexistence of global weak solutions for the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + g(t)(-\Delta)^{\frac{\alpha}{2}} u_{tt} = |u_t|^p, & x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $p > 1, n \geq 1$, $\alpha \in (0,2), g(t) = t^\lambda, \lambda \geq 1$ and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator of order $\frac{\alpha}{2}$. Then we extend our analysis to the 2×2 system of the same type, namely

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + g(t)(-\Delta)^{\frac{\alpha}{2}} u_{tt} = |v_t|^p, & x \in \mathbb{R}^n, \\ v_{tt} - \Delta v_t - \Delta v + f(t)(-\Delta)^{\frac{\beta}{2}} v_{tt} = |u_t|^q, & x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x), v_t(0,x) = v_1(x) & x \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $p, q > 1, n \geq 1$, $\alpha, \beta \in (0,2), g(t) = t^\lambda, f(t) = t^\gamma, \lambda, \gamma \geq 1$ and $(-\Delta)^{\frac{\alpha}{2}}, (-\Delta)^{\frac{\beta}{2}}$ are the fractional Laplacian operators of order $\frac{\alpha}{2}$ and $\frac{\beta}{2}$. We mention below some motivations for studying the considered

*Correspondence: hakemali@yahoo.com

problems.

Recently, Tuan Anh Dao in [11] investigated the nonexistence of global(in time) solutions to the following system

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{\delta_1} u_t = |v|^p, \quad x \in \mathbb{R}^n, t > 0, \\ v_{tt} - \Delta v + (-\Delta)^{\delta_2} v_t = |u|^q, \quad x \in \mathbb{R}^n, t > 0 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x), v_t(0,x) = v_1(x) \quad x \in \mathbb{R}^n. \end{cases} \tag{3}$$

It was shown that if $\delta_1, \delta_2 \in \left[0, \frac{1}{2}\right], u_0 = u_1 = 0$ and $u_1, v_1 \in \mathbb{L}^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} u_1(x)dx > \varepsilon_1, \quad \int_{\mathbb{R}^n} u_1(x)dx > \varepsilon_2,$$

and

$$\frac{n}{2} \leq \frac{1 + q\frac{1-\delta_2}{1-\delta_1} + (pq-1)\delta_2}{(q-1)\frac{\delta_1-\delta_2}{1-\delta_2} + (pq-1)} \quad \text{if} \quad \delta_1 \geq \delta_2,$$

$$\frac{n}{2} \leq \frac{1 + p\frac{1-\delta_1}{1-\delta_2} + (pq-1)\delta_2}{(p-1)\frac{\delta_2-\delta_1}{1-\delta_1} + (pq-1)} \quad \text{if} \quad \delta_2 \geq \delta_1.$$

Then, there is no global (in time) Sobolev solution $(u, v) \in \mathcal{C}\left([0, \infty) \times \mathbb{L}^2(\mathbb{R}^n)\right) \times \mathcal{C}\left([0, \infty) \times \mathbb{L}^2(\mathbb{R}^n)\right)$ to (3). Very recently, the critical exponent to the following structurally damped wave equation with the power nonlinearity $|u_t|^p$:

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{\alpha}{2}} u_t = |u_t|^p, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \quad x \in \mathbb{R}^n, \end{cases} \tag{4}$$

has been studied by Tuan Anh Dao and Ahmad Z. Fino [12]. It was shown in [12] that if

$$1 < p \leq 1 + \frac{\tilde{\alpha}}{n} \quad \text{where} \quad \tilde{\alpha} = min\{1, \alpha\},$$

then, there is no global (in time) weak solution to (4). As far as we know that one of the most typical important methods to verify critical exponent is well-known test function method. Concretely, this method is used to prove the nonexistence of global solutions by a contradiction argument. However, standard test function method seems difficult to directly apply to (1) containing pseudo-differential operators $(-\Delta)^{\frac{\alpha}{2}}$ for any $\alpha \in (0, 2)$, well-known non-local operators. Nonlocal operators have been receiving increased attention in recent years due to their usefulness in physics. To overcome the difficulty caused by the nonlocal property of the fractional Laplacian operator, D' Abbicco and Reissig [6] investigated the structurally damped wave equation with the power nonlinearity $|u|^p$. The critical exponent has been studied and they proposed to distinguish between (parabolic like models) in the case $\sigma \in (0, 1]$, the so-called effective damping, and (hyperbolic like models) in the remaining case $\sigma \in (1, 2]$, the so-called noneffective damping according to expected decay estimates (see more [2]). In the former case, they proved the existence of global (in time) solutions when

$$p > p_c = 1 + \frac{2}{(n-\sigma)_+} \quad \text{where} \quad (n-\sigma)_+ = \max(n-\sigma, 0),$$

for the small initial data and low space dimensions $2 \leq n \leq 4$ by using the energy estimates.

For other contributions related to the structurally damped wave equation with the power nonlinearity of derivative type, see ([7],[8], [5]), for example and the references therein. Motivated by the above contributions, in particular by [12], our goal in this paper is to investigate problems (1) and (2) for nonexistence of global weak solutions by using the method of the test function which has been introduced by Mitidieri and Pohozaev [1]. The paper is organized as follows. In the next section, we give some auxiliary results and formulate our main results. In Section 3, we prove our main results.

2. Mathematical statements and theorems

Definition 2.1. ([9],[3]). Let $s \in (0,1)$ and X be a suitable set of functions defined on $\mathbb{R}^n$. Then, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ is a non-local operator given by

$$(-\Delta)^s : f \in X \to (-\Delta)^s f(x) = C_{n,s}\ P.V \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\ dy,$$

as long as the right-hand side exists, where $P.V$ stands for the Cauchy's principal value and $C_{n,s} = \dfrac{4^s\Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}\Gamma(-s)}$ is the normalization constant and Γ denotes the Gamma function.

Definition 2.2. (Weak solution for (1)). Let $T>0, p>1$, and $(u_0,u_1) \in \mathbb{L}^1(\mathbb{R}^n) \times \mathbb{L}^2(\mathbb{R}^n)$. We say that $u \in \mathbb{L}^1_{loc}\left((0,\infty), \mathbb{L}^2(\mathbb{R}^n)\right)$ satisfying $u_t \in \mathbb{L}^p_{loc}\left((0,\infty), \mathbb{L}^{2p}(\mathbb{R}^n)\right) \cap \mathbb{L}^1_{loc}\left((0,\infty), \mathbb{L}^2(\mathbb{R}^n)\right)$ is a local weak solution to (1) if

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} |u_t(x,t)|^p\varphi(t,x)dxdt + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx = -\int_0^T\int_{\mathbb{R}^n} u_t(x,t)\varphi_t(t,x)dxdt \\
&-\int_0^T\int_{\mathbb{R}^n} u(x,t)\Delta\varphi(t,x)dxdt - \int_0^T\int_{\mathbb{R}^n} u_t(x,t)\left(g(t)(-\Delta)^{\frac{\alpha}{2}}\varphi(x,t)\right)_t dxdt \\
&-\int_0^T\int_{\mathbb{R}^n} u_t(x,t)\Delta\varphi(x,t)dxdt,
\end{aligned} \tag{5}$$

for any test function $\varphi \in \mathcal{C}\left([0,\infty); H^2(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,\infty); \mathbb{L}^2(\mathbb{R}^n)\right)$ such that its support in time is compact. If $T=\infty$, we say that u is a global weak solution to (1).

Definition 2.3. (Weak solution for (2)). Let $p,q>1$ and $T>0$. We say that (u,v) is a local weak solution to the problem (2) if $(u,v) \in \mathbb{L}^q_{loc}([0,T) \times \mathbb{R}^n) \times \mathbb{L}^p_{loc}([0,T) \times \mathbb{R}^n)$ and satisfies the equations

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} |v_t(x,t)|^p\varphi(t,x)dxdt + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx = -\int_0^T\int_{\mathbb{R}^n} u_t(x,t)\varphi_t(t,x)dxdt \\
&-\int_0^T\int_{\mathbb{R}^n} u(x,t)\Delta\varphi(t,x)dxdt - \int_0^T\int_{\mathbb{R}^n} u_t(x,t)\left(g(t)(-\Delta)^{\frac{\alpha}{2}}\varphi(x,t)\right)_t dxdt \\
&-\int_0^T\int_{\mathbb{R}^n} u_t(x,t)\Delta\varphi(x,t)dxdt,
\end{aligned} \tag{6}$$

and

$$\int_0^T \int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi(t,x)dxdt + \int_{\mathbb{R}^n} v_1(x)\varphi(0,x)dx = -\int_0^T \int_{\mathbb{R}^n} v_t(x,t)\varphi_t(t,x)dxdt$$
$$-\int_0^T \int_{\mathbb{R}^n} v(x,t)\Delta\varphi(t,x)dxdt - \int_0^T \int_{\mathbb{R}^n} v_t(x,t)\left(f(t)(-\Delta)^{\frac{\beta}{2}}\varphi(x,t)\right)_t dxdt \tag{7}$$
$$-\int_0^T \int_{\mathbb{R}^n} v_t(x,t)\Delta\varphi(x,t)dxdt,$$

for any test function $\varphi \in \mathcal{C}_0^\infty([0,T)\times\mathbb{R}^n)$. If $T=\infty$, we say that (u,v) is a global weak solution to (2).

Now, we are ready to state the main results of this paper.

Theorem 2.1. *Let $\alpha \in (0,2]$ and $\tilde{\alpha} = min\{1,\alpha\}$. We assume that $(u_0,u_1) \in \mathbb{L}^1(\mathbb{R}^n)\times\mathbb{L}^2(\mathbb{R}^n)$ satisfying the following condition:*

$$\int_{\mathbb{R}^n} u_1(x)dx \geq 0. \tag{8}$$

If

$$1 < p \leq \frac{n+\tilde{\alpha}}{n+\lambda-\alpha+\tilde{\alpha}}, \tag{9}$$

then, there is no global (in time) weak solution to problem (1).

Theorem 2.2. *We assume that $(u_0,u_1) \in \mathbb{L}^1(\mathbb{R}^n)\times\mathbb{L}^2(\mathbb{R}^n)$ and $(v_0,v_1) \in \mathbb{L}^1(\mathbb{R}^n)\times\mathbb{L}^2(\mathbb{R}^n)$ satisfying the following conditions:*

$$\int_{\mathbb{R}^n} u_1(x)dx > 0 \quad and \quad \int_{\mathbb{R}^n} v_1(x)dx > 0. \tag{10}$$

If

$$n \leq \frac{1}{pq-1}\max\Big\{p(\alpha-\gamma)-\lambda pq+\beta, q(\beta-\lambda)-\gamma pq+\alpha\Big\}, \tag{11}$$

then, there is no global (in time) weak solution to (2).

3. Proofs

In this section, we give the proofs of Theorems 2.1 and 2.2. We shall use the nonlinear capacity method combined with the following pointwise estimate (see Fujiwara [2] and Dao and Reissig [13]).

Lemma 3.1. *([13]) Let $\langle x\rangle = \big(1+(|x|-1)^4\big)^{\frac{1}{4}}$. Let $s\in(0,1)$ and $\phi:\mathbb{R}^n\to\mathbb{R}$ be the function defined by*

$$\phi(x) = \begin{cases} \langle x\rangle^{-n-2s} & if \quad |x|\geq 1, \\ 1 & if \quad |x|\leq 1. \end{cases} \tag{12}$$

Then $\phi\in\mathcal{C}^2(\mathbb{R}^n)$, and the following estimate holds

$$|(-\Delta)^s\phi(x)| \leq C\phi(x), x\in\mathbb{R}^n, \tag{13}$$

where C is a constant independent of x.

Lemma 3.2. *([13]) Let $s \in (0,1)$. Let ψ be a smooth function satisfying $\partial_x^2 \psi \in \mathbb{L}^\infty(\mathbb{R}^n)$. For any $R > 0$, let ψ_R be a function defined by*

$$\psi_R(x) = \psi\left(\frac{x}{R}\right), \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Then, $(-\Delta)^s \psi_R$ satisfies the following scaling properties:

$$(-\Delta)^s(\psi_R)(x) = R^{-2s}(-\Delta)^s \psi\left(\frac{x}{R}\right) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Remark 3.1. *Throughout, C denotes a positive constant, whose value may change from line to line.*

3.1. Proof of Theorem 2.1

Let u be a global weak solution to (1), then for all $\varphi \in \mathcal{C}\left([0,\infty); H^2(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,\infty); \mathbb{L}^2(\mathbb{R}^n)\right)$, one has

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi(t,x)dxdt + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx = -\int_0^{+\infty}\int_{\mathbb{R}^n} u_t(x,t)\varphi_t(t,x)dxdt \\
&- \int_0^{+\infty}\int_{\mathbb{R}^n} u(x,t)\Delta\varphi(t,x)dxdt - \int_0^{+\infty}\int_{\mathbb{R}^n} u_t(x,t)\left(g(t)(-\Delta)^{\frac{\alpha}{2}}\varphi(x,t)\right)_t dxdt \qquad (14)\\
&- \int_0^{+\infty}\int_{\mathbb{R}^n} u_t(x,t)\Delta\varphi(x,t)dxdt.
\end{aligned}$$

First, we introduce the function $\phi = \phi(x)$ as defined in (12) with $s = \frac{\alpha}{2}$ and the function $\eta = \eta(t)$ having the following properties:

1. $\eta \in \mathcal{C}_0^\infty([0,\infty))$ and $\begin{cases} 1 & \text{if} \quad 0 \le t \le \frac{1}{2}, \\ \text{decreasing} & \text{if} \quad \frac{1}{2} \le t \le 1, \\ 0 & \text{if} \quad t \ge 1. \end{cases}$

2. $\eta^{-\frac{1}{p}}(t)|\eta'(t)| + |\eta(t)| \le C$ for any $t \in [\frac{1}{2}, 1]$.

Let R be a large parameter in $[0,\infty)$. We define the following test function:

$$\varphi_R(x,t) = \eta_R(t)\phi_R(x),$$

where $\eta_R(t) = \eta(R^{-\tilde{\alpha}}t))$ and $\phi_R(x) = \phi(R^{-1}K^{-1}x)$ for some $K \ge 1$ which will be fixed later. Moreover, we check easily that *supp*$(\eta) \subset [0, R^{\tilde{\alpha}}]$. Let

$$\Psi_R(t) = \int_t^{R^{\tilde{\alpha}}} \eta_R(t) = R^{\tilde{\alpha}} - t.$$

We define the functionals

$$I_1 = \int_0^{+\infty}\int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi_R(t,x)dxdt = \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi_R(t,x)dxdt,$$

and

$$I_2=\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}|u_t(x,t)|^p\varphi_R(t,x)dxdt,\quad I_3=\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}}|u_t(x,t)|^p\varphi_R(t,x)dxdt.$$

From (14), one obtains

$$I_1+\int_{\mathbb{R}^n}u_1(x)\phi_R(x)dx=-\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}u_t(x,t)\eta_R'(t)\phi_R(x)dxdt-\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}}u(x,t)\eta_R(t)\Delta\phi_R(x)dxdt$$
$$-\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}}u_t(x,t)\eta_R(t)\Delta\phi_R(x)dxdt-\int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}u_t(x,t)\left(g(t)\eta_R(t)(-\Delta)^{\frac{\alpha}{2}}\phi_R(x)\right)_t dxdt.$$

Using integrating by parts, one has

$$\begin{aligned}
&I_1+\int_{\mathbb{R}^n}u_1(x)\phi_R(x)dx+\int_{\mathbb{R}^n}u_0(x)\Psi_R(0)\Delta\phi_R(x)dx=-\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}u_t(x,t)\eta_R'(t)\phi_R(x)dxdt\\
&-\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}}u_t(x,t)\Psi_R(t)\Delta\phi_R(x)dxdt-\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}}u_t(x,t)\eta_R(t)\Delta\phi_R(x)dxdt\\
&-\int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}u_t(x,t)g'(t)\eta_R(t)(-\Delta)^{\frac{\alpha}{2}}\phi_R(x)dxdt-\int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}u_t(x,t)g(t)\eta_R'(t)(-\Delta)^{\frac{\alpha}{2}}\phi_R(x)dxdt\\
&=-J_1-J_2-J_3-J_4-J_5.
\end{aligned}\tag{15}$$

Applying Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$, we can proceed the estimate for J_1 as follows:

$$\begin{aligned}
|J_1|\preceq\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}|u_t(x,t)||\eta_R'(t)|\phi_R(x)dxdt&\leq\left(\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}\left(|u_t(x,t)|\varphi_R^{\frac{1}{p}}(t,x))\right)^p\right)^{\frac{1}{p}}\\
&\times\left(\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}\left(|\eta_R'(t)|\phi_R(x)\varphi_R^{-\frac{1}{p}}(t,x)\right)^{p'}\right)^{\frac{1}{p'}}\\
&\preceq I_2^{\frac{1}{p}}\left(\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}\eta_R^{-\frac{p'}{p}}(t)|\eta_R'(t)|^{p'}\phi_R(x)dxdt\right)^{\frac{1}{p'}}.
\end{aligned}$$

Using change of variables $\tilde{t}=R^{-\tilde{\alpha}}t$ and $\tilde{x}=R^{-1}K^{-1}x$, we get

$$|J_1|\preceq I_2^{\frac{1}{p}}R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}}K^{\frac{n}{p'}}\left(\int_{\mathbb{R}^n}\langle\tilde{x}\rangle^{-n-\alpha}\right)^{\frac{1}{p'}}\preceq I_2^{\frac{1}{p}}R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}}K^{\frac{n}{p'}}.\tag{16}$$

Now let us turn to estimate $J_2,J_3,J_4,$ and J_5. Applying Hölder's inequality again as we estimated J_1 leads to

$$|J_2|\preceq I_3^{\frac{1}{p}}\left(\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}}\Psi_R^{p'}(t)\eta_R^{-\frac{p'}{p}}(t)\phi_R^{-\frac{p'}{p}}(x)|\Delta\phi_R(x)|^{p'}dxdt\right)^{\frac{1}{p'}}\preceq I_3^{\frac{1}{p}}R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}}K^{-2+\frac{n}{p'}},\tag{17}$$

$$|J_3|\preceq I_3^{\frac{1}{p}}\left(\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}}\eta_R(t)\phi_R^{-\frac{p'}{p}}(x)|\Delta\phi_R(x)|^{p'}dxdt\right)^{\frac{1}{p'}}\preceq I_3^{\frac{1}{p}}R^{-2+\frac{n+\tilde{\alpha}}{p'}}K^{-2+\frac{n}{p'}},\tag{18}$$

and

$$|J_4| \preceq I_1^{\frac{1}{p}} \left(\int_0^{R^{\tilde{\alpha}}} \int_{\mathbb{R}^n} g'^{p'}(t) \eta_R(t) \phi_R^{-\frac{p'}{p}}(x) |(-\Delta)^{\frac{\alpha}{2}} \phi_R(x)|^{p'} dx dt \right)^{\frac{1}{p'}} \preceq I_1^{\frac{1}{p}} R^{\lambda-1-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}}, \tag{19}$$

$$|J_5| \preceq I_1^{\frac{1}{p}} \left(\int_0^{R^{\tilde{\alpha}}} \int_{\mathbb{R}^n} g^{p'}(t) {\eta'_R}^{p'}(t) \eta_R^{-\frac{p'}{p}}(t) \phi_R^{-\frac{p'}{p}}(x) |(-\Delta)^{\frac{\alpha}{2}} \phi_R(x)|^{p'} dx dt \right)^{\frac{1}{p'}} \preceq I_1^{\frac{1}{p}} R^{\lambda-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}}. \tag{20}$$

Combining the estimates from (16) to (20) we may arrive at

$$\begin{aligned} I_1 + \int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx \leq \int_{\mathbb{R}^n} |u_0(x)||\Psi_R(0)||\Delta\phi_R(x)|dx + C\Big(I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} \\ + I_3^{\frac{1}{p}} R^{-2+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} + I_1^{\frac{1}{p}} R^{\lambda-1-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} + I_1^{\frac{1}{p}} R^{\lambda-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} \Big). \end{aligned}$$

Moreover, it is clear that

$$\Psi_R(t) = \int_t^{R^{\tilde{\alpha}}} \eta_R(t) = R^{\tilde{\alpha}} - t \quad \text{then} \quad \Psi_R(0) = R^{\tilde{\alpha}}.$$

We can easily check that $|\Delta\phi_R(x)| \leq R^{-2}\phi_R(x)$. Therefore, this implies that

$$\begin{aligned} I_1 + \int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx \leq R^{\tilde{\alpha}-2} \int_{\mathbb{R}^n} |u_0(x)|\phi_R(x)dx + C\Big(I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} \\ + I_3^{\frac{1}{p}} R^{-2+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} + I_1^{\frac{1}{p}} R^{\lambda-1-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} + I_1^{\frac{1}{p}} R^{\lambda-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} \Big). \end{aligned} \tag{21}$$

Since $u_0 \in \mathbb{L}^1(\mathbb{R}^n)$, it implies immediately that

$$\lim_{R\to\infty} \left[R^{\tilde{\alpha}-2} \int_{\mathbb{R}^n} |u_0(x)|\phi_R(x)dx \right] = 0.$$

Invoking the assumption (8), one obtains

$$R^{\tilde{\alpha}-2} \int_{\mathbb{R}^n} |u_0(x)|\phi_R(x)dx < \frac{1}{2} \int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx.$$

From (21), we easily see that

$$\begin{aligned} I_1 + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx \leq C\Big(I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} \\ + I_1^{\frac{1}{p}} R^{\lambda-1-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} + I_1^{\frac{1}{p}} R^{\lambda-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} \Big). \end{aligned} \tag{22}$$

By choosing $K = 1$ and noticing the relations $I_2 \leq I_1$ and $I_3 \leq I_1$ we may arrive, particularly, at

$$\begin{aligned} I_1 + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx \leq C\Big(I_1^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} + I_1^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} + I_1^{\frac{1}{p}} R^{-2+\frac{n+\tilde{\alpha}}{p'}} \\ + I_1^{\frac{1}{p}} R^{\lambda-1-\alpha+\frac{n+\tilde{\alpha}}{p'}} + I_1^{\frac{1}{p}} R^{\lambda-\alpha+\frac{n+\tilde{\alpha}}{p'}} \Big) \leq C I_1^{\frac{1}{p}} R^{\lambda-\alpha+\frac{n+\tilde{\alpha}}{p'}}. \end{aligned} \tag{23}$$

Thanks to the following ε-Young's inequality:

$$ab \leq \varepsilon a^p + C(\varepsilon) b^{p'}, \quad \text{for all} \quad a, b > 0 \quad \text{and} \quad \text{for any} \quad \varepsilon > 0,$$

we conclude

$$C I_1^{\frac{1}{p}} R^{\lambda - \alpha + \frac{n+\tilde{\alpha}}{p'}} \leq \varepsilon I_1 + C(\varepsilon) R^{(\lambda-\alpha)p' + n + \tilde{\alpha}}.$$

Consequently, from (23) we derive

$$(1-\varepsilon) I_1 + \frac{1}{2} \int_{\mathbb{R}^n} u_1(x) \phi_R(x) dx \leq C(\varepsilon) R^{(\lambda-\alpha)p' + n + \tilde{\alpha}},$$

which follows that

$$I_1 \leq C R^{(\lambda-\alpha)p' + n + \tilde{\alpha}}, \tag{24}$$

$$\int_{\mathbb{R}^n} u_1(x) \phi_R(x) dx \leq C R^{(\lambda-\alpha)p' + n + \tilde{\alpha}}. \tag{25}$$

It is clear that the assumption (9) is equivalent to $(\lambda - \alpha)p' + n + \tilde{\alpha} \leq 0$. For this reason, we will split our consideration into two cases.

Case 1:In the subcritical case $(\lambda - \alpha)p' + n + \tilde{\alpha} < 0$, letting $R \to \infty$ in (25) we easily deduce

$$\int_{\mathbb{R}^n} u_1(x) dx \leq 0,$$

which contradicts the assumption (8).

Case 2: For the critical case $(\lambda - \alpha)p' + n + \tilde{\alpha} = 0$, from (24) we can see that $I_1 \leq C$. Using Beppo Levi's theorem on monotone convergence, one obtains

$$\int_0^\infty \int_{\mathbb{R}^n} |u_t(x,t)|^p dx dt = \lim_{R\to\infty} \int_0^{R^{\tilde{\alpha}}} \int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi_R(x,t) dx dt = \lim_{R\to\infty} I_1 \leq C.$$

We conclude that $u_t \in \mathbb{L}^p((0,\infty) \times \mathbb{R}^n)$. By the absolute continuity of the Lebesgue integral, it follows that $I_2 \to 0$ and $I_3 \to 0$ as $R \to \infty$. Using again the fact that $\alpha - \lambda = \frac{n+\tilde{\alpha}}{p'}$, we obtain from (22) the following estimate:

$$\begin{aligned} I_1 + \frac{1}{2} \int_{\mathbb{R}^n} u_1(x) \phi_R(x) dx \leq C \Big(& I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\alpha-\lambda} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\alpha-\lambda} K^{-2+\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\alpha-\lambda} K^{-2+\frac{n}{p'}} \\ & + I_1^{\frac{1}{p}} R^{-1} K^{-\alpha+\frac{n}{p'}} + I_1^{\frac{1}{p}} K^{-\alpha+\frac{n}{p'}} \Big), \end{aligned} \tag{26}$$

for all $K \geq 1$.

1. If $\alpha \in (0,1]$, then $\alpha = \tilde{\alpha}$. Consequently, from (26) we have

$$\begin{aligned} I_1 + \frac{1}{2} \int_{\mathbb{R}^n} u_1(x) \phi_R(x) dx \leq C \Big(& I_2^{\frac{1}{p}} R^{-\lambda} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2(1-\alpha)-\lambda} K^{-2+\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\alpha-\lambda} K^{-2+\frac{n}{p'}} \\ & + I_1^{\frac{1}{p}} R^{-1} K^{-\alpha+\frac{n}{p'}} + I_1^{\frac{1}{p}} K^{-\alpha+\frac{n}{p'}} \Big). \end{aligned} \tag{27}$$

Letting $R \to \infty$ in (27) we get

$$\int_{\mathbb{R}^n} u_1(x))dx \preceq K^{-\alpha+\frac{n}{p'}} \quad \text{for all} \quad K \geq 1. \tag{28}$$

It is obvious that $-\alpha + \frac{n}{p'} < 0$. We can fix a sufficiently large constant $K \geq 1$ in (28) to gain a contradiction to (8).

2. If $\alpha \in (1,2]$, then $\tilde{\alpha} = 1$. As a result, choosing $K = 1$ we may conclude from (26) that

$$I_1 + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx \leq C\Big(I_2^{\frac{1}{p}} R^{-1+\alpha-\lambda} + I_3^{\frac{1}{p}} R^{-1+\alpha-\lambda} + I_3^{\frac{1}{p}} R^{-2+\alpha-\lambda} + I_1^{\frac{1}{p}} R^{-1} + I_1^{\frac{1}{p}}\Big). \tag{29}$$

Since $\alpha > 1$, letting $R \to \infty$ in (29) we obtain a contradiction to (8) again.

Summarizing, the proof of the Theorem 2.1 is completed.

3.2. Proof of Theorem 2.2

Let $\alpha, \beta \in (0,2]$ and $\tilde{\alpha} = min\{1,\alpha\}, \tilde{\beta} = min\{1,\beta\}$. First, we introduce the same test function as in Theorem 1.1. Let us assume that (u,v) is the global solution to (2). We define the functionals

$$J_1 = \int_0^{+\infty}\int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi_R(t,x)dxdt = \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi_R(t,x)dxdt,$$

and

$$J_2 = \int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi_R(t,x)dxdt, \quad J_3 = \int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} |u_t(x,t)|^q \varphi_R(t,x)dxdt,$$

$$I_1 = \int_0^{+\infty}\int_{\mathbb{R}^n} |v_t(x,t)|^p \varphi_R(t,x)dxdt = \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |v_t(x,t)|^p \varphi_R(t,x)dxdt,$$

and

$$I_2 = \int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |v_t(x,t)|^p \varphi_R(t,x)dxdt, \quad I_3 = \int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} |v_t(x,t)|^p \varphi_R(t,x)dxdt.$$

From (6) and (7), on has

$$I_1 + \int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx = -\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} u_t(x,t)\eta_R'(t)\phi_R(x)dxdt - \int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} u(x,t)\eta_R(t)\Delta\phi_R(x)dxdt$$

$$-\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} u_t(x,t)\eta_R(t)\Delta\phi_R(x)dxdt - \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} u_t(x,t)\left(g(t)\eta_R(t)(-\Delta)^{\frac{\alpha}{2}}\phi_R(x)\right)_t dxdt,$$

and

$$J_1 + \int_{\mathbb{R}^n} v_1(x)\phi_R(x)dx = -\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} v_t(x,t)\eta_R'(t)\phi_R(x)dxdt - \int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} v(x,t)\eta_R(t)\Delta\phi_R(x)dxdt$$

$$-\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} v_t(x,t)\eta_R(t)\Delta\phi_R(x)dxdt - \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} v_t(x,t)\left(f(t)\eta_R(t)(-\Delta)^{\frac{\alpha}{2}}\phi_R(x)\right)_t dxdt.$$

Repeating the steps of the proof from (16) to (23) we may conclude the following estimates

$$I_1 \leq J_1^{\frac{1}{q}} R^{(\lambda-\alpha)+\frac{n+\tilde{\alpha}}{q'}}. \tag{30}$$

In the analogous way, one obtains

$$J_1 \leq I_1^{\frac{1}{p}} R^{(\gamma-\beta)+\frac{n+\tilde{\beta}}{p'}}. \tag{31}$$

From (30) and (31) we obtain

$$I_1^{\frac{pq-1}{pq}} \leq R^{((\gamma-\beta)+\frac{n+\tilde{\beta}}{p'})\frac{1}{q}+(\lambda-\alpha)+\frac{n+\tilde{\alpha}}{q'}} = R^{\delta_1}, \tag{32}$$

$$J_1^{\frac{pq-1}{pq}} \leq R^{((\lambda-\alpha)+\frac{n+\tilde{\alpha}}{q'})\frac{1}{p}+(\gamma-\beta)+\frac{n+\tilde{\beta}}{p'}} = R^{\delta_2}. \tag{33}$$

It is clear that the assumption (11) is equivalent to $\max\{\delta_1,\delta_2\} \leq 0$. For this reason, we will split our consideration into two cases.
Case 1:In the subcritical case $\max\{\delta_1,\delta_2\} < 0$, letting $R \to \infty$ in (32)and (33) we easily deduce

$$\int_{\mathbb{R}^n} v_1(x)\phi_R(x)dx \leq 0 \quad \text{and} \quad \int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx \leq 0,$$

which contradicts the assumption (10).
Case 2: For the critical case $\delta_2 = 0$, from (24) we can see that $J_1 \leq C$. Using Beppo Levi's theorem on monotone convergence, one obtains

$$\int_0^\infty \int_{\mathbb{R}^n} |u_t(x,t)|^q dxdt = \lim_{R\to\infty} \int_0^{R^{\tilde{\alpha}}} \int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi_R(x,t)dxdt = \lim_{R\to\infty} J_1 \leq C.$$

We conclude that $u_t \in \mathbb{L}^q((0,\infty)\times\mathbb{R}^n)$. By the absolute continuity of the Lebesgue integral, it follows that $J_2 \to 0$ and $J_3 \to 0$ as $R \to \infty$. Using again the fact that $\delta_2 = 0$, we obtain from (22) the following estimate:

$$\begin{aligned} J_1 + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx \leq C\Big(& I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\alpha-\lambda} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\alpha-\lambda} K^{-2+\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\alpha-\lambda} K^{-2+\frac{n}{p'}} \\ & + I_1^{\frac{1}{p}} R^{-1} K^{-\alpha+\frac{n}{p'}} + I_1^{\frac{1}{p}} K^{-\alpha+\frac{n}{p'}}\Big), \end{aligned} \tag{34}$$

for all $K \geq 1$.

1. If $\alpha \in (0,1]$, then $\alpha = \tilde{\alpha}$. Consequently, from (34) we have

$$\begin{aligned} J_1 + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)dx \leq C\Big(& I_2^{\frac{1}{p}} R^{-\lambda} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2(1-\alpha)-\lambda} K^{-2+\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\alpha-\lambda} K^{-2+\frac{n}{p'}} \\ & + I_1^{\frac{1}{p}} R^{-1} K^{-\alpha+\frac{n}{p'}} + I_1^{\frac{1}{p}} K^{-\alpha+\frac{n}{p'}}\Big). \end{aligned} \tag{35}$$

   Letting $R \to \infty$ in (35) we get

$$\int_{\mathbb{R}^n} u_1(x))dx \preceq K^{-\alpha+\frac{n}{p'}} \quad \text{for all} \quad K \geq 1. \tag{36}$$

It is obvious that $-\alpha+\frac{n}{p'}<0$. We can fix a sufficiently large constant $K\geq 1$ in (36) to gain a contradiction to (10).

2. If $\alpha\in(1,2]$, then $\tilde{\alpha}=1$. As a result, choosing $K=1$ we may conclude from (34) that

$$J_1+\frac{1}{2}\int_{\mathbb{R}^n}u_1(x)\phi_R(x)dx\leq C\Big(I_2^{\frac{1}{p}}R^{-1+\alpha-\lambda}+I_3^{\frac{1}{p}}R^{-1+\alpha-\lambda}+I_3^{\frac{1}{p}}R^{-2+\alpha-\lambda}+I_1^{\frac{1}{p}}R^{-1}+I_1^{\frac{1}{p}}\Big). \quad (37)$$

Since $\alpha>1$, letting $R\to\infty$ in (37) we obtain a contradiction to (10) again.

In the case $\delta_1=0$ we repeat the same arguments as in $\delta_2=0$. Summarizing, the proof of the Theorem 2.2 is completed.

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