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Existence and stability results for hybrid fractional q- differential pantograph equations

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Abstract: In this manuscript, we discuss the existence, uniqueness, estimate and stability of solutions for nonlinear hybrid fractional $q-$ differential pantograph equations. The existence and uniqueness of solutions are derived from Banach contraction principle, while the estimate of solutions is proved via generalization of Gronwall's inequality. Stability of solutions is also established. An illustrative example is presented.

Key words: Caputo $q-$derivative, hybrid differential equations, pantograph equations, Ulam-Hyers stability, existence

1. Introduction

Fractional differential equations involving fractional calculus operators and fractional $q-$integral calculus operators have attracted the attentions of many scholars working in a variety of disciplines, due to the development and applications of these equations in many fields such as engineering, physics, mathematics, etc, see [2, 5, 13, 15, 20] and references therein, one of the motivating topics in this area is the study of the existence and uniqueness of solutions, see for example [2, 9, 12, 18, 22]. Pantograph type equations have attracted the attention of many authors working in a variety of disciplines due to the development and applications of these equations in many fields such engineering, electro-dynamic, biology, control, see [11, 23, 24]. Moreover, fractional-order pantograph differential equations have been considered by many researchers, for instance, see [1, 3, 8, 16] and the references cited therein. The Ulam stability of differential equations with fractional derivative has been investigated by different authors, we refer the reader to the papers [6, 12, 13]. Recently, the stability of fractional pantograph differential equations has been investigated by many researchers, we refer the reader to [1, 3, 4, 17]. More recently, fractional hybrid pantograph differential equations have also been studied by several authors, for instance, see [10, 19]. In [10], the authors studied the existence solutions of the following hybrid fractional pantograph equation

$$\begin{cases} D_{0^+}^{\alpha}\left[\frac{u(t)}{\varphi(t,u(t),u(\lambda t))}\right] = \psi\left(t,u\left(t\right),u\left(\eta t\right)\right), 0<t<1, \\ u\left(0\right)=0, \end{cases}$$

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where $\alpha,\lambda,\eta\in(0,1)$ and D_{0+}^{α} denotes the Riemann-Liouville fractional derivative. In [19], the authors studied the following hybrid generalized fractional pantograph equation

$$\begin{cases} D_{0+}^{\alpha}\left[\frac{u(t)}{\varphi(t,u(t),u(\theta(t)))}\right]=\psi\left(t,u\left(t\right),u\left(\rho\left(t\right)\right)\right),0<t<1, \\ u\left(0\right)=0, \end{cases}$$

where $\alpha\in(0,1)$, D_{0+}^{α} denotes the Riemann-Liouville fractional derivative and $\theta,\rho:[0,1]\to[0,1]$ are given functions. In this work, we study the existence, uniqueness, Ulam-Hyers-stability and Ulam-Hyers-Rassias stability of solutions for the following hybrid Caputo fractional $q-$differential pantograph equations

$$\begin{cases} D_{q}^{\alpha}\left[\dfrac{x\left(t\right)}{\sum_{i=1}^{k}\phi_{i}\left(t,x\left(t\right),x\left(\lambda t\right)\right)}\right] \\ =\sum_{i=1}^{k}\varphi_{i}\left(t,x\left(t\right),x\left(\eta t\right),D_{q}^{\alpha}\left[\dfrac{x\left(t\right)}{\sum_{i=1}^{k}\phi_{i}\left(t,x\left(t\right),x\left(\lambda t\right)\right)}\right]\right),0<\lambda,\eta<1, \\ x\left(0\right)+\psi\left(x\right)=x_{0},x_{0}\in\mathbb{R}, \end{cases} \tag{1}$$

where $0<q,\alpha<1, D_{q}^{\alpha}$ is the Caputo fractional $q-$derivative, $t\in J=[0,T]$, $\phi_{i}\in C\left(J\times\mathbb{R}^{2},\mathbb{R}-\{0\}\right)$ and $\varphi_{i}\in C\left(J\times\mathbb{R}^{3},\mathbb{R}\right), i=1,...,k,\ k\in\mathbb{N}^{*}$. The operator D_{q}^{α} is the fractional $q-$derivative in the sense of Caputo [7, 25], defined by

$$\begin{cases} D_{q}^{\alpha}f\left(t\right)=I_{q}^{n-\alpha}D_{q}^{n}f\left(t\right),\ \alpha>0, \\ D_{q}^{0}f\left(t\right)=f\left(t\right), \end{cases}$$

where n is the smallest integer greater than or equal to α. The fractional $q-$integral of the Riemann-Liouville type [7, 25] is given by

$$\begin{cases} I_{q}^{\alpha}f\left(t\right)=\frac{1}{\Gamma_{q}(\alpha)}\int_{0}^{t}\left(t-qs\right)^{(\alpha-1)}f\left(s\right)d_{q}s,\alpha>0, \\ I_{q}^{0}f\left(t\right)=f\left(t\right), \end{cases}$$

where the $q-$gamma function is defined by $\Gamma_{q}\left(\alpha\right)=\frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}},\alpha\in\mathbb{R}\backslash\left\{0,-1,-2,...\right\}$ and satisfies

$$\Gamma_{q}\left(\alpha+1\right)=[\alpha]_{q}\Gamma_{q}\left(\alpha\right),\ [a]_{q}=\frac{1-q^{a}}{1-q},a\in\mathbb{R}.$$

We need the the following lemmas [25].

Lemma 1.1. *Let $\alpha,\beta\geq0$ and f be a function defined in $[0,1]$. Then the following formulas hold*

$$I_{q}^{\alpha}I_{q}^{\beta}f\left(t\right)=I_{q}^{\alpha+\beta}f\left(t\right)\ \text{ and }\ D_{q}^{\alpha}I_{q}^{\alpha}f\left(t\right)=f\left(t\right).$$

Lemma 1.2. *Let $\alpha>0$ and σ be a positive integer. Then the following equality holds*

$$I_{q}^{\alpha}D_{q}^{\sigma}f\left(t\right)=D_{q}^{\sigma}I_{q}^{\alpha}f\left(t\right)-\sum_{j=0}^{\sigma-1}\frac{t^{\alpha-\sigma+j}}{\Gamma_{q}\left(\alpha+j-\sigma+1\right)}D_{q}^{j}f\left(0\right).$$

Lemma 1.3. *For $\alpha \in \mathbb{R}_+$ and $\beta > -1$, we have*

$$I_q^{\alpha}\left[(t-x)^{(\beta)}\right] = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(t-x)^{(\alpha+\beta)}.$$

In particular, for $x = 0$ and $\beta = 0$, we have

$$I_q^{\alpha}[1] = \frac{1}{\Gamma_q(\alpha+1)}t^{(\alpha)}.$$

We give the following generalization of Gronwall's lemma for singular kernels [21] which will be used in the following sections.

Lemma 1.4. *Let $x : J \to [0,+\infty)$ be a real function and $F(.)$ is a nonnegative, locally integrable function on J and there are constants $a > 0$ and $0 < \alpha, q < 1$ such that*

$$x(t) \leq F(t) + a\int_0^t (t-qs)^{(\alpha-1)} x(s)\, d_q s,$$

then, there exists a constant $\theta = \theta(\alpha)$ such that

$$x(t) \leq F(t) + \theta a\int_0^t (t-qs)^{(\alpha-1)} F(t)\, d_q s,\ t \in J.$$

Also, we present the equivalence of the problem (1). The proof of the following Lemma can be as similar to the proof of Lemma 6.2 given in [14].

Lemma 1.5. *Let $i = 1, ..., k$ and $0 < \alpha, q < 1$. If $\psi_i \in C\left(J \times \mathbb{R}^2, \mathbb{R} - \{0\}\right)$ and $\varphi_i \in C\left(J \times \mathbb{R}^3, \mathbb{R}\right)$. Then the fractional problem (1) is equivalent to nonlinear fractional Volterra integro-differential equation*

$$\begin{aligned} x(t) &= x_0 - \psi(x) + \sum_{i=1}^{k}\phi_i(t, x(t), x(\lambda t)) \qquad (2)\\ &\times \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \sum_{i=1}^{k}\varphi_i\left(s, x(s), x(\eta s), D_q^{\alpha}\left[\frac{x(s)}{\sum_{i=1}^{k}\phi_i(s, x(s), x(\lambda s))}\right]\right) d_q s, \end{aligned}$$

where $t \in J$ and $k \in \mathbb{N}^*$.

Let $C(J, \mathbb{R})$ denote the Banach space of continuous functions from J into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,T]}|x(t)|$. For $\delta > 0$ and $g : J \to \mathbb{R}^+$, we consider the following inequalities

$$\begin{aligned} &D_q^{\alpha}\left[\frac{y(t)}{\sum_{i=1}^{k}\phi_i(t, y(t), y(\lambda t))}\right] \qquad (3)\\ &- \sum_{i=1}^{k}\varphi_i\left(t, y(t), y(\eta t), D_q^{\alpha}\left[\frac{y(t)}{\sum_{i=1}^{k}\phi_i(t, y(t), y(\lambda t))}\right]\right) \leq \delta, \end{aligned}$$

and

$$D_q^{\alpha}\left[\frac{y(t)}{\sum_{i=1}^{k}\phi_i(t,y(t),y(\lambda t))}\right] \tag{4}$$

$$-\sum_{i=1}^{k}\varphi_i\left(t,y(t),y(\eta t),D_q^{\alpha}\left[\frac{y(t)}{\sum_{i=1}^{k}\phi_i(t,y(t),y(\lambda t))}\right]\right)\leq \delta g(t).$$

Definition 1.1. The problem (1) is Ulam-Hyers stable if there exists a real number $\beta_{\phi,\varphi}>0$ such that for all $\delta>0$ and for each solution y of the inequality (3), there exists a solution x of the problem (1) with

$$|y(t)-x(t)|\leq \beta_{\phi,\varphi}\delta,\ t\in J.$$

Definition 1.2. The problem (1) is generalized Ulam-Hyers stable if there exists $\Psi_{\phi,\varphi}\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\Psi_{\phi,\varphi}(0)=0$, such that for all solution y of the inequality (3), there exists a solution x of the the problem (1) with

$$|y(t)-x(t)|\leq \Psi_{\phi,\varphi}(\delta),\ t\in J.$$

Definition 1.3. The problem (1) is Ulam-Hyers-Rassias stable with respect to $g\in C(J,\mathbb{R}_+)$ if there exists a real number $\beta_{\phi,\varphi,g}>0$ such that for all $\delta>0$ and for each solution y of the inequality (4), there exists a solution x of the problem (1) with

$$|y(t)-x(t)|\leq \beta_{\phi,\varphi,g}\delta g(t),\ t\in J.$$

Remark 1.1. *A function $y\in C(J,\mathbb{R})$ is a solution of the inequality (3) if and only if there exists a function $\mu:J\to\mathbb{R}$ such that*

$$|\mu(t)|\leq\delta, t\in J.$$

and

$$\begin{aligned} & D_q^{\alpha}\left[\frac{y(t)}{\sum_{i=1}^{k}\phi_i(t,y(t),y(\lambda t))}\right] \\ = & \sum_{i=1}^{k}\varphi_i\left(t,y(t),y(\eta t),D_q^{\alpha}\left[\frac{y(t)}{\sum_{i=1}^{k}\phi_i(t,y(t),y(\lambda t))}\right]\right)+\mu(t), t\in J.\end{aligned}$$

2. Existence and uniqueness results

In this section, we establish the existence and uniqueness of solutions to the fractional problem (1) and prove the estimate of solution of (1). For the sake of convenience, we impose the following conditions.

(C_1) : There exist constants ϖ_i such that for all $t\in J$ and $u_j,v_j\in\mathbb{R}$ $(j=1,2)$, we have

$$|\phi_i(t,u_1,u_2)-\phi_i(t,v_1,v_2)|\leq\varpi_i(|u_1-v_1|+|u_2-v_2|), i=1,...,k.$$

(C_2) : There exist constants ω_i such that for all $t\in J$ and $u_l,u_l\in\mathbb{R}$ $(l=1,2,3)$, we have

$$\begin{aligned} & |\varphi_i(t,u_1,u_2,u_3)-\varphi_i(t,v_1,v_2,v_3)| \\ \leq & \ \omega_i(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|), i=1,...,k,\end{aligned}$$

with $\sum_{i=1}^{k}\omega_i<1.$

(C_3) : There exist $A_i,B_i\in\mathbb{R}^+$ such that

$$\left|\phi_i\left(t,u,v\right)\right|\leq A_i \text{ and } \left|\varphi_i\left(t,u,v,w\right)\right|\leq B_i, i=1,...,k,$$

for each $t\in J$ and $u,v,w\in\mathbb{R}.$

(C_4) : $\psi:C\left(\left[0,T\right],\mathbb{R}\right)\rightarrow\mathbb{R}$ is a continuous function satisfying the condition

$$\left|\psi\left(u\right)-\psi\left(v\right)\right|\leq\vartheta\left\|u-v\right\|, \text{ fo rall } u,v\in C\left(J,\mathbb{R}\right),\vartheta<1.$$

Now, we discuss the existence and uniqueness of solutions for the problem (1) by means of Banach's contraction mapping principle.

Theorem 2.1. *Assume that the continuous functions* $\phi_i:J\times\mathbb{R}^2\rightarrow\mathbb{R}-\{0\},\varphi_i:J\times\mathbb{R}^3\rightarrow\mathbb{R},i=1,...,k,$ *satisfy* $(C_1)-(C_3)$ *and suppose that* (C_4) *holds. If the inequality*

$$\sum_{i=1}^{k}\frac{\left|B_i\right|T^{(\alpha)}}{\Gamma_q\left(\alpha+1\right)}\sum_{i=1}^{k}\varpi_i+2\sum_{i=1}^{k}\frac{\left|A_i\right|T^{(\alpha)}}{\Gamma_q\left(\alpha+1\right)}\sum_{i=1}^{k}\frac{\omega_i}{1-\sum_{i=1}^{k}\omega_i}<\frac{1-\vartheta}{2}, \tag{5}$$

is valid, then the problem (1) has a unique solution.

Proof. We have

$$D_q^{\alpha}\left[\frac{x\left(t\right)}{\sum_{i=1}^{k}\phi_i\left(t,x\left(t\right),x\left(\lambda t\right)\right)}\right]=z_x\left(t\right),u\left(0\right)+\psi\left(x\right)=x_0,k\in\mathbb{N}^*,$$

by Lemma 1.5, we can write

$$x\left(t\right)=x_0-\psi\left(u\right)+\sum_{i=1}^{k}\phi_i\left(t,x\left(t\right),x\left(\lambda t\right)\right)I_q^{\alpha}\left[z_x\left(t\right)\right],t\in J,k\in\mathbb{N}^*, \tag{6}$$

where

$$z_x\left(t\right)=\sum_{i=1}^{k}\varphi_i\left(t,x_0-\psi\left(x\right)+\sum_{i=1}^{k}\phi_i\left(t,x\left(t\right),x\left(\lambda t\right)\right)I_q^{\alpha}\left[z_x\left(t\right)\right],x\left(\eta t\right),z_x\left(t\right)\right). \tag{7}$$

Now, we define a fractional integral operator $O:C\left(J,\mathbb{R}\right)\rightarrow C\left(J,\mathbb{R}\right)$ by

$$Ox\left(t\right)=x_0-\psi\left(x\right)+\sum_{i=1}^{k}\phi_i\left(t,x\left(t\right),x\left(\lambda t\right)\right)\int_0^t\frac{\left(t-qs\right)^{(\alpha-1)}}{\Gamma_q\left(\alpha\right)}z_x\left(s\right)d_qs. \tag{8}$$

For $x,y\in C\left(J,\mathbb{R}\right)$, we have

$$\begin{aligned}
&\left|Ox\left(t\right)-Oy\left(t\right)\right|\\
\leq\ &\left|\psi\left(x\right)-\psi\left(y\right)\right|\\
&+\sum_{i=1}^{k}\left|\phi_i\left(t,x\left(t\right),x\left(\lambda t\right)\right)-\phi_i\left(t,y\left(t\right),y\left(\lambda t\right)\right)\right|\int_0^t\frac{\left(t-qs\right)^{(\alpha-1)}}{\Gamma_q\left(\alpha\right)}\left|z_x\left(s\right)\right|d_qs\\
&+\sum_{i=1}^{k}\left|\phi_i\left(t,x\left(t\right),x\left(\lambda t\right)\right)\right|\int_0^t\frac{\left(t-qs\right)^{(\alpha-1)}}{\Gamma_q\left(\alpha\right)}\left|z_x\left(s\right)-z_y\left(s\right)\right|d_qs.
\end{aligned}$$

Using $(C_j)_{j=1,2,3,4}$, we get

$$\begin{aligned}|Ox(t)-Oy(t)| &\leq \vartheta\|x-y\|+2\sum_{i=1}^{k}\int_0^t\frac{B_i(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\sum_{i=1}^{k}\varpi_i\|x-y\| \\ &+2\sum_{i=1}^{k}A_i\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|z_x(s)-z_y(s)|\,d_qs. \qquad (9)\end{aligned}$$

Also, we have

$$\begin{aligned}&|z_x(t)-z_y(t)| \\ =&\left|\sum_{i=1}^{k}\varphi_i(t,x(t),x(\eta t),z_x(t))-\sum_{i=1}^{k}\varphi_i(t,y(t),y(\eta t),z_y(t))\right| \\ \leq& 2\sum_{i=1}^{m}\omega_i\|x-y\|+\sum_{i=1}^{m}\omega_i\|z_x-z_y\|,\end{aligned}$$

which implies that

$$|z_x(t)-z_y(t)|\leq\frac{2\sum_{i=1}^{k}\omega_i}{1-\sum_{i=1}^{k}\omega_i}\|x-y\|. \qquad (10)$$

By (9) and (10), we have

$$\begin{aligned}&|Ox(t)-Oy(t)| \\ \leq& \vartheta\|x-y\|+2\sum_{i=1}^{k}\int_0^t\frac{B_i(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\sum_{i=1}^{k}\varpi_i\|x-y\| \\ &+2\sum_{i=1}^{k}A_i\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\frac{2\sum_{i=1}^{k}\omega_i}{1-\sum_{i=1}^{k}\omega_i}\|x-y\|\,d_qs \\ \leq& \vartheta\|x-y\|+2\sum_{i=1}^{k}B_i\frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}\sum_{i=1}^{k}\varpi_i\|x-y\| \\ &+2\sum_{i=1}^{k}A_i\frac{(T)^{(\alpha)}}{\Gamma_q(\alpha+1)}\frac{2\sum_{i=1}^{k}\omega_i}{1-\sum_{i=1}^{k}\omega_i}\|x-y\|.\end{aligned}$$

Hence

$$\begin{aligned}\|Ox-Oy\| \leq& \left[\vartheta+2\sum_{i=1}^{k}\frac{B_iT^{(\alpha)}}{\Gamma_q(\alpha+1)}\sum_{i=1}^{k}\varpi_i\right. \\ &\left.+4\sum_{i=1}^{k}\frac{A_iT^{(\alpha)}}{\Gamma_q(\alpha+1)}\sum_{i=1}^{k}\frac{\omega_i}{1-\sum_{i=1}^{k}\omega_i}\right]\|x-y\|.\end{aligned}$$

By (5), we conclude that O is contractive. Consequently, by Banach fixed point theorem, O has a unique fixed point which is a solution of (1). □

Next, we prove the estimate of solution of the fractional problem (1).

Theorem 2.2. *Suppose that* $\phi_i : J \times \mathbb{R}^2 \to \mathbb{R} - \{0\}, \varphi_i : J \times \mathbb{R}^3 \to \mathbb{R}, i = 1, ..., k,$ *are continuous functions satisfying the conditions* $(C_1) - (C_3)$ *and assume that* (C_4) *and* (5) *hold. If* x *is a solution of problem (1), then*

$$\begin{aligned} \|x\| \leq & \frac{|x_0| + M}{(1-\vartheta)} + \sum_{i=1}^{k} A_i \frac{\sum_{i=1}^{k} N_i T^{(\alpha)}}{\left(1 - \sum_{i=1}^{k} \omega_i\right) \Gamma_q(\alpha+1)(1-\vartheta)} \\ & + \theta \sum_{i=1}^{k} A_i \frac{2 \sum_{i=1}^{k} \omega_i T^{(\alpha)}}{\left(1 - \sum_{i=1}^{k} \omega_i\right)(1-\vartheta)^2 \Gamma_q(\alpha+1)} \\ & \times \left((|x_0| + M) + \sum_{i=1}^{k} A_i \frac{\sum_{i=1}^{k} N_i T^{(\alpha)}}{\left(1 - \sum_{i=1}^{k} \omega_i\right) \Gamma_q(\alpha+1)} \right), \end{aligned} \tag{11}$$

where $M = |\psi(0)|$ *and* $N_i = \sup_{t \in J} |\varphi_i(t, 0, 0, 0)|, i = 1, ..., k, k \in \mathbb{N}^*$.

Proof. By Lemma 1.5, we can write

$$x(t) = x_0 - \psi(x) + \sum_{i=1}^{k} \phi_i(t, x(t), x(\lambda t)) I_q^\alpha [z_x(t)], t \in J, k \in \mathbb{N}^*.$$

Then, for all $t \in J$, we have

$$\begin{aligned} |x(t)| & \leq |x_0| + |\psi(x)| + \sum_{i=1}^{k} |\phi_i(t, x(t), x(\lambda t))| \left|I_q^\alpha [z_x(t)]\right| \\ & \leq |x_0| + |\psi(x) - \psi(0)| + |\psi(0)| + \sum_{i=1}^{k} A_i \left|I_q^\alpha [z_x(t)]\right| \\ & \leq |x_0| + \vartheta \|x\| + M + \sum_{i=1}^{k} A_i \left|I_q^\alpha [z_x(t)]\right|. \end{aligned}$$

On the other hand, for all $t \in J$, we get

$$\begin{aligned}
|z_x(t)| &= \sum_{i=1}^{k} |\varphi_i(t, x(t), x(\eta t), z_x(t))| \\
&\leq \sum_{i=1}^{k} |\varphi_i(t, x(t), x(\eta t), z_x(t)) - \varphi_i(t, 0, 0, 0)| + \sum_{i=1}^{k} |\varphi_i(t, 0, 0, 0)| \\
&\leq 2\sum_{i=1}^{k} \omega_i \|x\| + \sum_{i=1}^{k} \omega_i \|z_x\| + \sum_{i=1}^{k} N_i \\
&\leq \frac{2\sum_{i=1}^{k} \omega_i}{1 - \sum_{i=1}^{k} \omega_i} \|x\| + \frac{\sum_{i=1}^{k} N_i}{1 - \sum_{i=1}^{k} \omega_i}.
\end{aligned}$$

Consequently

$$\begin{aligned}
|x(t)| &\leq |x_0| + \vartheta \|x\| + M \\
&\quad + \sum_{i=1}^{k} A_i I_q^{\alpha} \left[\frac{2\sum_{i=1}^{k} \omega_i}{1 - \sum_{i=1}^{k} \omega_i} \|x\| + \frac{\sum_{i=1}^{k} N_i}{1 - \sum_{i=1}^{k} \omega_i} \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
(1-\vartheta)|x(t)| &\leq |x_0| + M + \sum_{i=1}^{k} A_i \frac{\sum_{i=1}^{k} N_i T^{(\alpha)}}{\left(1 - \sum_{i=1}^{k} \omega_i\right) \Gamma_q(\alpha+1)} \\
&\quad + \sum_{i=1}^{k} A_i \frac{2\sum_{i=1}^{k} \omega_i}{\left(1 - \sum_{i=1}^{k} \omega_i\right)(1-\vartheta)} I_q^{\alpha} [(1-\vartheta)|x(t)|].
\end{aligned}$$

Then Lemma 1.4 implies that for each $t \in J$

$$\begin{aligned}
&(1-\vartheta)|x(t)| \\
\leq\ & |x_0| + M + \sum_{i=1}^{k} A_i \frac{\sum_{i=1}^{k} N_i T^{(\alpha)}}{\left(1 - \sum_{i=1}^{k} \omega_i\right) \Gamma_q(\alpha+1)} \\
&+\theta \sum_{i=1}^{k} A_i \frac{2\sum_{i=1}^{k} \omega_i T^{(\alpha)}}{\left(1 - \sum_{i=1}^{k} \omega_i\right)(1-\vartheta) \Gamma_q(\alpha+1)} \\
&\times \left((|x_0| + M) + \sum_{i=1}^{m} A_i \frac{\sum_{i=1}^{k} N_i T^{(\alpha)}}{\left(1 - \sum_{i=1}^{k} \omega_i\right) \Gamma_q(\alpha+1)} \right),
\end{aligned}$$

where $\theta = \theta(\alpha)$ is a constant.

So,

$$\begin{aligned}\|x\| &\leq \frac{|x_0|+M}{(1-\vartheta)}+\sum_{i=1}^{k}A_i\frac{\sum_{i=1}^{k}N_iT(\alpha)}{\left(1-\sum_{i=1}^{k}\omega_i\right)\Gamma_q(\alpha+1)(1-\vartheta)}\\ &+\theta\sum_{i=1}^{k}A_i\frac{2\sum_{i=1}^{k}\omega_iT^{(\alpha)}}{\left(1-\sum_{i=1}^{k}\omega_i\right)(1-\vartheta)^2\Gamma_q(\alpha+1)}\\ &\times\left((|x_0|+M)+\sum_{i=1}^{k}A_i\frac{\sum_{i=1}^{k}N_iT^{(\alpha)}}{\left(1-\sum_{i=1}^{k}\omega_i\right)\Gamma_q(\alpha+1)}\right).\end{aligned}$$

This completes the proof. □

3. Ulam-Hyers-Rassias-stability

This section is devoted to the investigation of the Hyers-Ulam-Rassias stability of our proposed problem.

Lemma 3.1. *Let $\phi_i : J\times\mathbb{R}^2\rightarrow\mathbb{R}-\{0\}$ and $\varphi_i : J\times\mathbb{R}^3\rightarrow\mathbb{R}, i=1,...,k,$ are continuous functions satisfying (C_3). If $y\in C(J,\mathbb{R})$ is a solution of the inequality (3), then y is a solution of the following inequality*

$$|y(t)-Oy(t)|\leq\sum_{i=1}^{k}A_i\frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}\delta, \tag{12}$$

where $\delta>0$.

Proof. Let $y\in C(J,\mathbb{R})$ be a solution of the inequality (3). Then using Remark 1.1, we obtain

$$D_q^{\alpha}\left[\frac{y(t)}{\sum_{i=1}^{k}\phi_i(t,y(t),y(\lambda t))}\right]$$

$$-\sum_{i=1}^{k}\varphi_i\left(t,y(t),y(\eta t),D_q^{\alpha}\left[\frac{y(t)}{\sum_{i=1}^{k}\phi_i(t,y(t),y(\lambda t))}\right]\right)=\mu(t),$$

where $|\mu(t)|\leq\delta,\ t\in J.$

Thanks to Lemma 1.5, we can write

$$x(t)-x_0+\psi(x)-\sum_{i=1}^{k}\phi_i(t,x(t),x(\lambda t))\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}z_x(s)\,d_qs$$

$$=\sum_{i=1}^{k}\phi_i(t,x(t),x(\lambda t))\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\mu(s)\,d_qs,$$

where z_x is given by (7).

By (C_3), we obtain

$$\begin{aligned} |y(t)-Oy(t)| &= \left| \sum_{i=1}^{k} \phi_i(t, x(t), x(\lambda t)) \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |\mu(s)| d_q s \right|. \\ &\leq \sum_{i=1}^{k} A_i \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)} \delta, \end{aligned}$$

which is satisfied inequality (12). □

Theorem 3.1. *Let $\phi_i : J \times \mathbb{R}^2 \to \mathbb{R} - \{0\}, \varphi_i : J \times \mathbb{R}^3 \to \mathbb{R}, i = 1, ..., k,$ are continuous functions satisfying the conditions $(C_1) - (C_3)$ and assume that (C_4) and (5) hold. Then the problem (1) is Ulam-Hyers stable and consequently, generalized Ulam-Hyers stable.*

Proof. Let $y \in C(J, \mathbb{R})$ be a solution of the inequality (3). Then for all $t \in J$, we have

$$\begin{aligned} & |y(t)-x(t)| \\ = & \Bigg| y(t) - x(t) - x_0 + \psi(x) \\ & - \sum_{i=1}^{k} \phi_i(t, x(t), x(\lambda t)) \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} z_x(s) d_q s \Bigg| \\ = & |y(t) - Oy(t) + Oy(t) - Ox(t)| \\ \leq & |y(t) - Oy(t)| + |Oy(t) - Ox(t)|. \end{aligned}$$

Thus by $(C_1), (C_2)$ and Lemma 3.1, we get

$$\begin{aligned} |y(t)-x(t)| \leq & \sum_{i=1}^{k} A_i \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)} \delta + \Bigg[\vartheta + 2 \sum_{i=1}^{k} \frac{B_i T^{(\alpha)}}{\Gamma_q(\alpha+1)} \sum_{i=1}^{k} \varpi_i \\ & + 4 \sum_{i=1}^{k} A_i \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)} \sum_{i=1}^{k} \frac{\omega_i}{1 - \sum_{i=1}^{m} \omega_i} \Bigg] \|x - y\|, \end{aligned}$$

We obtain by (5)

$$\|x-y\| \leq \frac{\sum_{i=1}^{k} A_i \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}}{1-\Lambda} \delta,$$

where

$$\Lambda = \vartheta + 2 \sum_{i=1}^{k} \frac{B_i T^{(\alpha)}}{\Gamma_q(\alpha+1)} \sum_{i=1}^{k} \varpi_i + 4 \sum_{i=1}^{k} A_i \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)} \sum_{i=1}^{k} \frac{\omega_i}{1 - \sum_{i=1}^{k} \omega_i}. \tag{13}$$

Then, for each $t \in J$, we have

$$|x(t)-y(t)| \leq \frac{\sum_{i=1}^{k} A_i \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}}{1-\Lambda} \delta = \beta_{\phi,\varphi} \vartheta.$$

Hence, the problem (1) is Ulam-Hyers stable. By putting $\Psi_{\phi,\varphi}(\delta) = \beta_{\phi,\varphi}\delta$, $\Psi_{\phi,\varphi}(0) = 0$ yields that the problem (1) generalized Ulam-Hyers stable. □

For the next stability results, we introduce the following condition.

(C_5) : There exists an function $g \in C(J, \mathbb{R}_+)$ and there exists $\gamma_g > 0$ such that for any $t \in J$

$$\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) ds \leq \gamma_g g(t). \tag{14}$$

Lemma 3.2. *Let $\phi_i : J \times \mathbb{R}^2 \to \mathbb{R} - \{0\}, \varphi_i : J \times \mathbb{R}^3 \to \mathbb{R}, i = 1, ..., k,$ are continuous functions satisfying the conditions (C_3) and suppose that (C_5) holds. If $y \in C(J, \mathbb{R})$ is a solution of the inequality (4), then, y is a solution of the following inequality*

$$|y(t) - Oy(t)| \leq \sum_{i=1}^{k} A_i \delta \gamma_g g(t). \tag{15}$$

Proof. Using Remark 1.1, we can write

$$D_q^\alpha \left[\frac{y(t)}{\sum_{i=1}^k \phi_i(t, y(t), y(\lambda t))} \right]$$
$$- \sum_{i=1}^{k} \varphi_i \left(t, y(t), y(\eta t), D_q^\alpha \left[\frac{y(t)}{\sum_{i=1}^k \phi_i(t, y(t), y(\lambda t))} \right] \right) = \mu(t),$$

where $y \in C(J, \mathbb{R})$ be a solution of the inequality (4) and $|\mu(t)| \leq \delta g(t),\ t \in J$. Applying Lemma 1.5, we obtain

$$y(t) - Oy(t) = \sum_{i=1}^{k} \phi_i(t, x(t), x(\lambda t)) \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mu(s) d_q s.$$

Since $|\mu(t)| \leq \delta g(t),\ t \in J,$ then we have

$$\begin{aligned} |y(t) - Oy(t)| &\leq \sum_{i=1}^{k} |\phi_i(t, x(t), x(\lambda t))| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |\mu(s)| d_q s \\ &\leq \sum_{i=1}^{k} |\phi_i(t, x(t), x(\lambda t))| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \delta g(s) d_q s. \end{aligned}$$

Thanks to (C_3), we obtain

$$|y(t) - Oy(t)| \leq \delta \sum_{i=1}^{k} A_i \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s.$$

So, by (C_5), we have

$$|y(t) - Oy(t)| \leq \delta \sum_{i=1}^{k} A_i \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s \leq \sum_{i=1}^{k} A_i \delta \gamma_g g(t),$$

which is satisfied inequality (15). □

Theorem 3.2. *Suppose that* $\phi_i : J \times \mathbb{R}^2 \to \mathbb{R} - \{0\}$ *and* $\varphi_i : J \times \mathbb{R}^3 \to \mathbb{R}, i = 1, ..., k,$ *are continuous functions satisfy* $(C_1) - (C_3)$. *Furthermore, assume that* $(C_4), (C_5)$ *and (5) hold. Then the problem (1) is Ulam-Hyers-Rassias stable.*

Proof. For $y \in C(J, \mathbb{R})$ and $t \in J,$ we have

$$\begin{aligned} & \left|y(t) - x(t)\right| \\ = & \left|y(t) - x(t) - x_0 + \psi(x) \right. \\ & \left. - \sum_{i=1}^{k} \phi_i(t, x(t), x(\lambda t)) \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} w_u(s)\, d_q s \right| \\ = & \left|y(t) - Oy(t) + Oy(t) - Ox(t)\right| \\ \leq & \left|y(t) - Oy(t)\right| + \left|Oy(t) - Ox(t)\right|. \end{aligned}$$

Thanks to (15) and (5), we obtain

$$\begin{aligned} \left|y(t) - x(t)\right| \leq & \sum_{i=1}^{k} A_i \delta \gamma_g g(t) + \left[\vartheta + 2 \sum_{i=1}^{k} \frac{B_i T^{(\alpha)}}{\Gamma_q(\alpha+1)} \sum_{i=1}^{k} \varpi_i \right. \\ & \left. + 4 \sum_{i=1}^{k} A_i \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)} \sum_{i=1}^{k} \frac{\omega_i}{1 - \sum_{i=1}^{m} \omega_i} \right] \|x - y\|, \end{aligned}$$

which implies that

$$\|x - y\| (1 - \Lambda) \leq \sum_{i=1}^{k} A_i \delta \gamma_g g(t).$$

Thus

$$\left|y(t) - x(t)\right| \leq \frac{\sum_{i=1}^{k} A_i \gamma_g}{1 - \Lambda} \delta g(t) = \beta_{\phi,\varphi,g} \delta g(t), \ t \in J.$$

Hence, the problem (1) is Ulam-Hyers-Rassias stable. □

4. Example

Consider the following hybrid problem given by

$$\begin{cases} D_q^{\alpha}\left[\dfrac{x(t)}{\frac{t}{37}\cos(x(t))+\frac{t}{37}\cos\left(x\left(\frac{2}{5}t\right)\right)+\frac{e^{-t}}{11}+\frac{3}{25}x(t)+\frac{3}{25}x\left(\frac{3}{4}t\right)}\right] \\ \quad =\frac{t}{36}\left(\sin x(t)+\sin x\left(\frac{5}{6}t\right)+\frac{1}{2}\right) \\ \quad +\dfrac{\cos(t)}{6+D_q^{\alpha}\left[\dfrac{x(t)}{\frac{1}{37}\cos(t)x(t)+\frac{1}{37}\cos(t)x\left(\frac{2}{5}t\right)+\frac{e^{t}}{11}+\frac{3}{25}x(t)+\frac{3}{25}x\left(\frac{3}{4}t\right)}\right]} \\ \quad +\frac{e^{-t}}{25+t^2}+\frac{1}{(25+t)}x(t)+\frac{e^{-5t}}{25}\sin x\left(\frac{2}{3}t\right) \\ \quad +\dfrac{1}{5+\left|D_q^{\alpha}\left[\dfrac{x(t)}{\frac{1}{37}\cos(t)x(t)+\frac{1}{37}\cos(t)x\left(\frac{2}{5}t\right)+\frac{e^{t}}{11}+\frac{3}{25}x(t)+\frac{3}{25}x\left(\frac{3}{4}t\right)+\frac{1}{11}}\right]\right|} \\ x(0)+\sum_{l=1}^{n}d_l x(t_l)=\frac{\sqrt{5}}{2}, \end{cases} \tag{16}$$

where $1<t_1<t_2<...<t_n<1$ and $d_l, l=1,2,...,n$ are given positive constants with $\sum_{l=1}^{n}d_l<\frac{1}{6}$.

Consider the hybrid fractional problem with $\alpha=\frac{1}{8}, q=\frac{1}{2}$, and for $x_1,x_2,x_3\in\mathbb{R},\ t\in[0,1]$, we have

$$\begin{aligned} \phi_1(t,x_1,x_2) &= \frac{\cos(t)}{37}x_1+\frac{\cos(t)}{37}x_2. \\ \phi_2(t,x_1,x_2) &= \frac{e^{-t}}{11}+\frac{3}{25}x_1+\frac{3}{25}x_2, \end{aligned}$$

$$\begin{aligned} \varphi_1(t,x_1,x_2,x_3) &= \frac{t}{36}\left(\sin x_1+\sin x_2+\frac{1}{2}\right)+\frac{\cos(t)}{6+|x_3|}, \\ \varphi_2(t,x_1,x_2,x_3) &= \frac{e^{-t^2}}{25}+\frac{1}{(25+t)}x_1+\frac{e^{-5t}}{25}\sin x_2+\frac{1}{5+|x_3|}, \end{aligned}$$

and

$$\psi(x_1)=\sum_{l=1}^{n}d_l x_1(t_l).$$

So, for $t\in[0,1]$ and $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$, we have

$$\begin{aligned} |\phi_1(t,x_1,x_2)-\phi_1(t,y_1,y_2)| &\leq \frac{1}{37}(|x_1-y_1|+|x_2-y_2|), \\ |\phi_1(t,x_1,x_2)-\phi_1(t,y_1,y_2)| &\leq \frac{3}{25}(|x_1-y_1|+|x_2-y_2|), \end{aligned}$$

and for $t \in [0,1]$ and $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$, we have

$$\begin{aligned} |\varphi_1(t, x_1, x_2, x_3) - \varphi_1(t, y_1, y_2, y_3)| &\leq \frac{1}{36}(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|), \\ |\varphi_1(t, x_1, x_2, x_3) - \varphi_1(t, y_1, y_2, y_3)| &\leq \frac{1}{25}(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|), \end{aligned}$$

Hence, the conditions (C_1) and (C_2) hold with $\omega_1 = \frac{1}{37}, \omega_2 = \frac{3}{25}, \varpi_1 = \frac{1}{36}$ and $\varpi_2 = \frac{1}{25}$ respectively. Also, for all $x_1, x_2 \in C([0,1])$, we have

$$\psi(x_1) - \psi(x_2) = \left| \sum_{i=1}^{n} d_i x_1(t_i) - \sum_{i=1}^{n} d_i x_2(t_i) \right| \leq \sum_{i=1}^{n} d_i |x_1 - x_2|.$$

Thus, the condition (C_4) is satisfied with $\vartheta = \sum_{i=1}^{n} d_i < \frac{1}{6}$. For $t \in [0,1]$ and $x_1, x_2, x_3 \in \mathbb{R}$, we have

$$|\phi_1(t, x_1, x_2)| \leq \frac{2}{37}, \quad |\phi_2(t, x_1, x_2)| \leq \frac{91}{275},$$

and

$$|\varphi_1(t, x_1, x_2, x_3)| \leq \frac{51}{216}, \quad |\varphi_2(t, x_1, x_2, x_3)| \leq \frac{8}{25}.$$

Now, using the given data, we find that

$$\begin{aligned} & \sum_{i=1}^{2} \frac{|B_i| T^{(\alpha)}}{\Gamma_q(\alpha+1)} \sum_{i=1}^{2} \varpi_i + 2 \sum_{i=1}^{2} \frac{|A_i| T^{(\alpha)}}{\Gamma_q(\alpha+1)} \sum_{i=1}^{2} \frac{\omega_i}{1 - \sum_{i=1}^{2} \omega_i} \\ \simeq \; & 0.177\,89 < \frac{1-\vartheta}{2} = \frac{1-\frac{1}{6}}{2} \simeq 0.416\,67. \end{aligned}$$

By Theorem 2.1, we conclude that the problem (16) has a unique solution on $[0,1]$ and by Theorem 3.1, we deduce that (16) is Ulam-Hyers stable.

Now, if we take $g(t) = \rho, \ \rho > 0$, we have

$$\frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} g(s) ds = \frac{\rho t^{(\alpha)}}{\Gamma_{\frac{1}{2}}\left(\frac{9}{8}\right)} \leq \frac{1}{\Gamma_{\frac{1}{2}}\left(\frac{9}{8}\right)} \rho = \gamma_g g(t).$$

Thus condition (C_5) is satisfied with $g(t) = \rho$ and $\gamma_g = \frac{1}{\Gamma_{\frac{1}{2}}\left(\frac{9}{8}\right)}$, it follows from Theorem 3.2 that the problem (16) is Ulam-Hyers-Rassias stable.

5. Conclusion

In this work, we have discussed the existence, estimate and Ulam-type stability of solutions for nonlinear hybrid fractional $q-$differential pantograph equations with fractional Caputo $q-$derivative. We have establish the existence and uniqueness results applying the Banach fixed point theorem. Also the estimate of solutions have been proved via generalization of Gronwall's inequality. Moreover, the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability have been discussed. To illustrate our theoretical results we have given an example.

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