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# On the chaotic response of the $\Lambda$-fractional Van der Pol oscillators

**Konstantinos A. LAZOPOULOS**[1], **Dimitrios KARAOULANIS**[2],
**Anastassios K. LAZOPOULOS**[3*] **and Athanassios P. PIRENTIS**[4]
[1]Theatrou 14, Rafina, 19009 Greece
[2]National Technical University of Athens External Science Collaborator,
Korai 21, Chalandri, Athens, Greece
[3]Mathematical Sciences Department, Hellenic Army Academy,
Vari, 16673 Greece
[4]Institute of Applied and Computational Mathematics, Foundation for Research and Technology - Hellas,
Heraklion, Crete, Greece

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**Abstract:** For a recently introduced fractional $\Lambda$-derivative, corresponding to a differential, according to Differential Topology, a modified version of the Van der Pol oscillator is studied. In the proposed fractional $\Lambda$-space the oscillator behaves in the conventional way. Nevertheless, the fractional order response of the Van der Pol equation is revealed when the the Van der Pol response is transferred to the initial space. Finally, this article concludes with the response of the forced Van der Pol oscillator.

**Key words:** fractional $\Lambda$-derivative, differential, Van der Pol oscillator, fractional $\Lambda$-space, initial space, chaos

## 1. Introduction

Fractional derivatives are mainly used in non-local models which account for long-range (non-local) dependence of phenomena. Therefore, these derivatives describe the non-local phenomena in a more precise way. In fact the various fractional derivatives have been introduced just to underline the non-local action of the derivatives. Models for describing viscoelastic interaction have been presented through fractional time derivatives [2, 3]. Lazopoulos has proposed a model for non-local deformations defining the $\Lambda$-strain [4], where Noll's axiom of local-action [5] has been lifted. Fractional derivatives, suggested by Leibniz [6], and discussed by Liouville [7], Riemann [8] and many other famous mathematicians [9], have recently applied to modern advances in almost all applied science fields, due to their importance. However, all the well known fractional derivatives have mainly an operative character, instead of a derivative's one, since they fail to satisfy the conditions demanded by Differential Topology for having the character of the derivative. In fact, there are three prerequisites for defining a derivative corresponding to a differential [14]; namely

$(i)$ *Linearity*: $D(a\,f(x)+b\,g(x)) = a\,Df(x)+b\,Dg(x)$

$(ii)$ *Leibniz rule*: $D(f(x)\cdot g(x)) = Df(x)\cdot g(x)+f(x)\cdot Dg(x)$

$(iii)$ *Chain rule.* $D(g(f))(x) = Dg(f(x))\cdot Df(x)$

These conditions are necessary for defining a differential corresponding to the derivative. Since no differential geometry may be generated using fractional derivative and no mechanics or physics may mathematically be

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*Correspondence: Orfeakos74@gmail.com

established without a mathematically defined derivative, the use of fractional derivatives in mathematics and physics is questionable.

Hence their use was not mathematically established, but it has an ad-hoc character. Lazopoulos trying to fill that gap, proposed initially the fractional L-derivative [15]. Recently, Lazopoulos proposed the fractional Λ-derivative [1], that is a modification of the fractional L-derivative, along with the fractional Λ-space where the fractional Λ-derivative behaves according to conventional derivative rules.

Dynamical systems have mainly been developed for the study of non-linear oscillators. The second order Van der Pol (VdP) oscillator [16] may correspond to the description of the behavior of a system composed by a mass, a spring and a damper. The damping coefficient is not constant, but it depends upon a non-linear coefficient. That oscillator has been used to describe models in many applied areas such as acoustics, electronics, biomechanics, biology, control, etc. Since the fractional Λ-space is similar to the conventional one, it is assumed that the Van der Pol oscillator is described by the equation

$$\frac{d^2Y}{dT^2} - \mu(1-Y^2)\frac{dY}{dT} + Y = 0,$$

where the displacement Y and time T in Λ-space will be defined in the next section.

The fractional Λ-derivative along with the fractional Λ-space are described in the following section, where the Λ-derivative, in the Λ-space, behaves in the conventional way. Developing the conventional solution of the VdP oscillator in the fractional Λ-space, the results are transferred into the initial space. The solutions of the VdP oscillator along with the corresponding phase portraits will be derived in the initial space, through the solution of the fractional Λ-space. Further, the fractional VdP equation will be discussed along the same lines of thought. The fractional VdP equation has also been discussed by Barbosa and Machado [17]. Further and recent applications of Fractional Calculus may be found in the provided references [18–23].

The present work does not include more specific topics, such as bifurcations, etc., because its purpose is to show how to deal with the solution of the fractional VdP equation.

## 2. The Λ-Fractional Derivative

A very brief outline of fractional calculus will be presented in the present section, while the interested reader is referred to literature [10–12] for more information.

The left and right fractional integrals for a real fractional dimension $0 < \gamma \leq 1$ are defined as

$$ {}_aI_x^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(s)}{(x-s)^{1-\gamma}}\, ds, \tag{1}$$

$$ {}_xI_b^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_x^b \frac{f(s)}{(s-x)^{1-\gamma}}\, ds, \tag{2}$$

respectively, where $\gamma$ is the order of fractional integrals and $\Gamma(\gamma)$ Euler's Gamma function. Further, the left and right Riemann-Liouville (RL) fractional derivatives are defined as

$$ {}_a^{RL}D_x^\gamma f(x) = \frac{d}{dx}\left({}_aI_x^{1-\gamma} f(x)\right) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_a^x \frac{f(s)}{(x-s)^\gamma}\, ds, \tag{3}$$

$$ {}_{x}^{RL}D_{b}^{\gamma}f(x) = \frac{d}{dx}({}_{x}I_{b}^{1-\gamma}f(x)) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int\limits_{x}^{b}\frac{f(s)}{(s-x)^{\gamma}}\,ds, \tag{4} $$

respectively. For the left fractional integral and derivative holds that

$$ {}_{a}^{RL}D_{x}^{\gamma}({}_{a}I_{x}^{\gamma}f(x)) = f(x). \tag{5} $$

A similar relation is true for the right fractional Riemann-Liouville derivative and integral.

The $\Lambda$-fractional derivative ($\Lambda$-FD) is defined as

$$ {}_{a}^{\Lambda}D_{x}^{\gamma}f(x) = \frac{{}_{a}^{RL}D_{x}^{\gamma}f(x)}{{}_{a}^{RL}D_{x}^{\gamma}x}. \tag{6} $$

Recalling the definition of the RL fractional derivative, Eq. (3), $\Lambda$-FD is expressed by

$$ {}_{a}^{\Lambda}D_{x}^{\gamma}f(x) = \frac{\frac{d\ {}_{a}I_{x}^{1-\gamma}f(x)}{dx}}{\frac{d\ {}_{a}I_{x}^{1-\gamma}x}{dx}} = \frac{d\ {}_{a}I_{x}^{1-\gamma}f(x)}{d\ {}_{a}I_{x}^{1-\gamma}x}. \tag{7} $$

Considering

$$ \begin{aligned} X &= {}_{a}I_{x}^{1-\gamma}x, \\ F(X) &= {}_{a}I_{x}^{1-\gamma}f(x(X)), \end{aligned} \tag{8} $$

the $\Lambda$-FD appears to behave as a conventional derivative in the fractional $\Lambda$-space $(X, F(X))$ with local properties. In fact, the Fractional Differential Geometry may be formulated as a conventional differential geometry in the fractional $\Lambda$-space $(X, F(X))$. Then the results may be transferred to the initial space invoking Eq. (5). Indeed, we may transfer the results from the $\Lambda$-fractional space to the initial one, using the relation

$$ f(x) = {}_{a}^{RL}D_{x}^{1-\gamma}F(X(x)) = {}_{a}^{RL}D_{x}^{1-\gamma}I^{1-\gamma}f(x). \tag{9} $$

In case the contribution of the right side fractional derivative should be taken into consideration, the $\Lambda$-fractional space may be defined with

$$ I^{1-\gamma}f(x) = \frac{1}{2}({}_{a}I_{x}^{1-\gamma}f(x) + {}_{x}I_{b}^{1-\gamma}f(x)) = \frac{1}{2}({}_{a}I_{x}^{1-\gamma}f(x) + {}_{a}I_{x}^{1-\gamma}f(b-x)). \tag{10} $$

Further, for that case, Eqs. (8) become

$$ \begin{aligned} X &= {}_{a}I_{x}^{1-\gamma}x, \\ F(X) &= I^{1-\gamma}f(x(X)), \end{aligned} \tag{11} $$

and $\Lambda$-FD, Eq. (6), is defined by

$$ {}_{a}^{\Lambda}D_{x}^{\gamma}f(x) = \frac{\frac{d({}_{a}I_{x}^{1-\gamma}f(x)\ +\ {}_{a}I_{x}^{1-\gamma}f(b-x))}{dx}}{\frac{d\ {}_{a}I_{x}^{1-\gamma}x}{dx}} = \frac{d\ I^{1-\gamma}f(x)}{d_{a}I_{x}^{1-\gamma}x}. \tag{12} $$

In order to clarify the ideas, let

$$f(x) = x^2. \tag{13}$$

Then, the $\Lambda$-fractional plane $(X, F(X))$ is defined by

$$X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)}, \tag{14}$$

$$F(X) \;=\; {}_0I_x^{1-\gamma} f(x(X)) = \frac{1}{\Gamma(1-\gamma)} \int\limits_0^x \frac{s^2}{(x-s)^{1-\gamma}}\, ds = \frac{2x^{3-\gamma}}{\Gamma(4-\gamma)}. \tag{15}$$

Solving Eq. (14) for $x$ yields

$$x = (\Gamma(3-\gamma)\ X)^{1/(2-\gamma)}, \tag{16}$$

and replacing to Eq. (15) results to

$$F(X) = \frac{2(\Gamma(3-\gamma)\ X)^{(3-\gamma)/(2-\gamma)}}{\Gamma(4-\gamma)}. \tag{17}$$

Therefore, the curve in the original plane $(x, f(x))$, shown in Figure 1, corresponds to the fractional plane

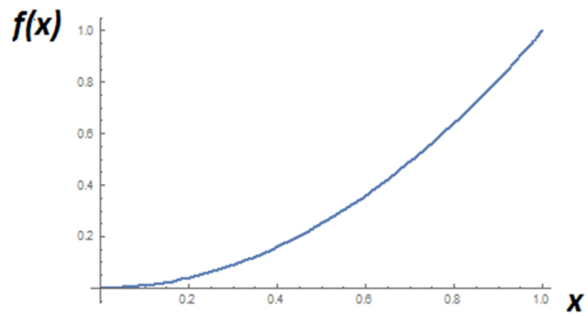


**Figure 1**. Original plane $(x, f(x) = x^2)$.

(space) shown in Figure 2, for $\gamma = 0.6$. Since the tangent space $Y(X)$ of the curve at a point $X_0$ is defined by the line

$$Y(X) = F(X_0) + \frac{dF(X_0)}{dX}(X - X_0), \tag{18}$$

in the fractional plane for $\gamma = 0.6$, the derivative

$$\frac{dF(X)}{dX} = \frac{24(\Gamma(3-\gamma)\ X)^{3/(2-\gamma)}}{(2-\gamma)\Gamma(6-\gamma)}, \tag{19}$$

at $X_0 = 0.6$ is equal to $D(F(X_0)) = 1.34$ (Figure 3).

Let us point out that the geometry may not be transferred in the original space, since formulation of the fractional differential is not possible in $(x, f(x))$. Functions may only be transferred from the fractional $\Lambda$-space to the original space.

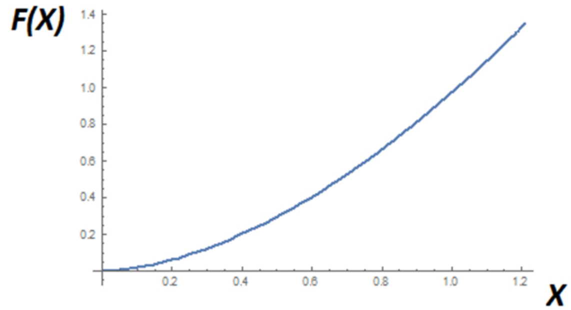


**Figure 2**. Curve in fractional $\Lambda$-space $(X, F(X))$ for $\gamma = 0.6$.

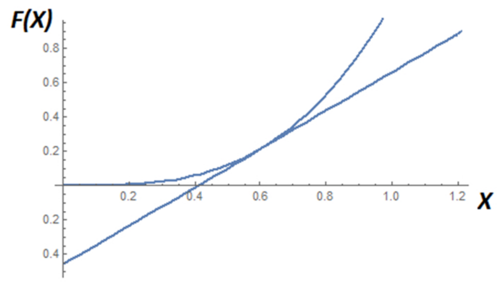


**Figure 3**. Curve with its tangent space in $\Lambda$-fractional plane.

In addition it is pointed out that for the derivation it is not necessary to express the various functions with respect to $X$. Indeed, since

$$\frac{dF(X)}{dX} = \frac{dF(X(x))/dx}{dX(x)/dx}, \tag{20}$$

there is no need to perform the substitution of $x$ for the $X$ variable. Calculus through the $x$ variable is simpler but also effective.

## 3. The $\Lambda$-Fractional Van der Pol Oscillator

For the VdP oscillator differential equation in $\Lambda$-space,

$$\frac{d^2Y}{dT^2} - \mu(1 - Y^2)\frac{dY}{dT} + Y = 0, \tag{21}$$

$T(t)$ is time and $Y(T(t))$ the corresponding displacement in $\Lambda$-space, and $\mu$ is a real-valued parameter. It is pointed out that the solution in the $\Lambda$-space is the conventional one according to the theory presented in the preceding section. The function T(t) is defined by the relation, see Eq. (14),

$$T(t) = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}. \tag{22}$$

Solving Eq. (21) numerically for various parameters and initial conditions, first in the fractional $\Lambda$-space, the graphs of the solution $Y(T)$ versus $T$, and the phase portrait $dY/dT$ versus $Y(T)$, for various values of the friction parameter $\mu$ in $\Lambda$-space are presented in the following.

For the perfect VdP oscillator, without damping $(\mu = 0)$, and initial conditions $Y(0) = 1,\ (dY/dT)(0) = 0$, the numerical solution is shown in Figure 4, where the trace along with the phase space are also presented.

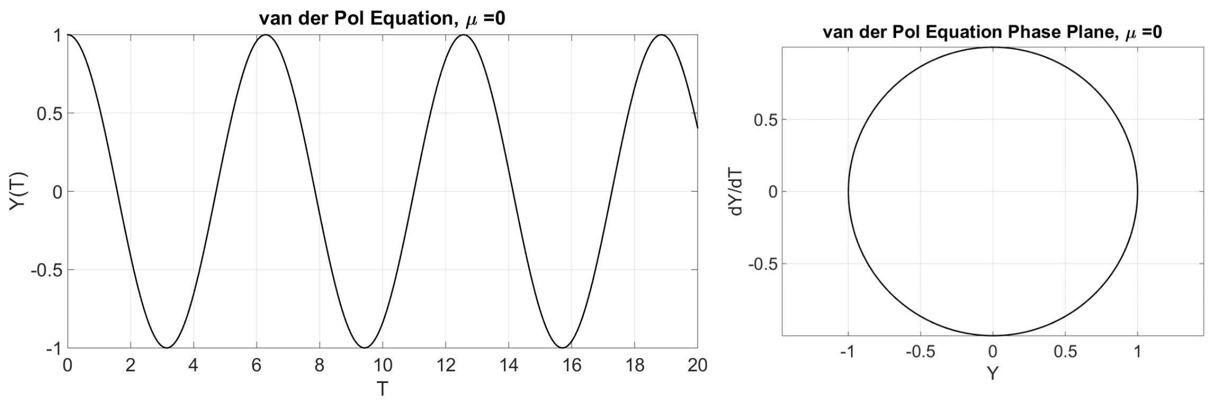


**Figure 4**. Solution $Y(T)$ versus time $T(t)$ and phase portrait in $\Lambda$-space for the conservative system $(\mu = 0)$.

In order to transfer the solution from $\Lambda$-space to the initial space $(t,\ y(t))$, Eq. (9) and the procedure to obtain Eq. (16) are employed, such that

$$y(t) = {}_0^{RL}D_t^{1-\gamma}Y(t) = \frac{1}{\Gamma(\gamma)}\frac{d}{dt}\int_0^t \frac{Y(s)}{(t-s)^{1-\gamma}}\,ds. \tag{23}$$

Figure 5 shows the solution of the trace $y(t)$, with respect to time $t$, and corresponding phase-space diagram in the initial space $(t,\ y(t))$, for various values of the fractional dimension $\gamma$.

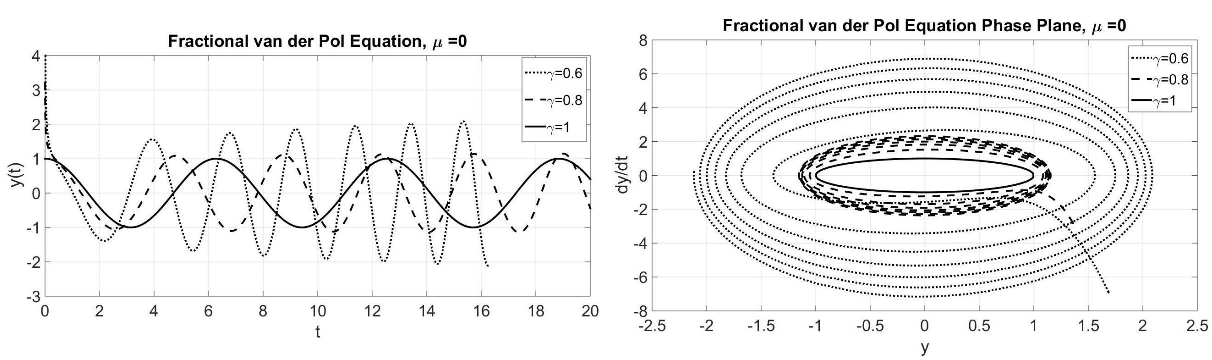


**Figure 5**. Solution and phase space of the conservative VdP equation for various fractional orders $\gamma$.

The influence of fractional dimension in the chaotic behaviour of the system becomes more pronounced in the case of $\gamma = 0.4$, for which the displacement diagram and phase portrait are shown in Figure 6. It is evident that displacements along with velocities are far greater for smaller fractional dimension $\gamma$. It is evident

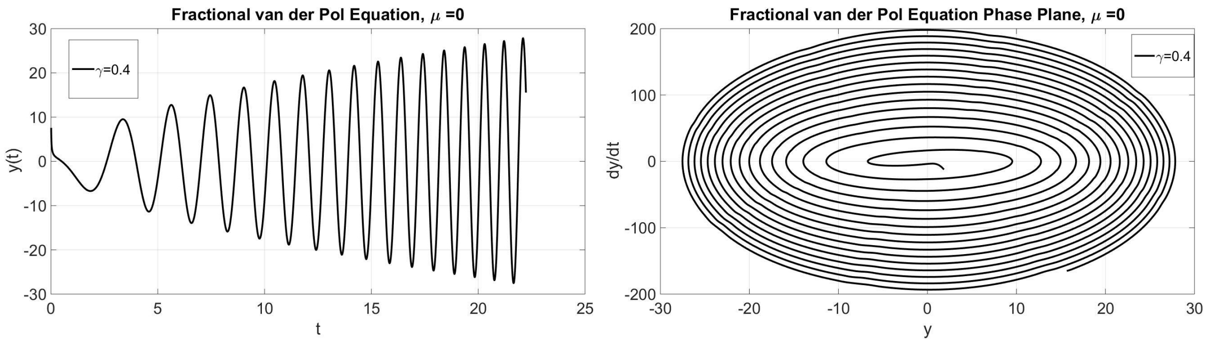


**Figure 6**. Solution of VdP equation and phase space of the conservative oscillator in the initial space for $\gamma = 0.4$.

that the influence of the fractional order $\gamma$ is quite strong. Indeed, the increase of $\gamma$ contributes in smaller amplitudes and longer periods.

Proceeding now to the study of the VdP oscillator under the influence of friction in the fractional $\Lambda$-space, the case $\mu = 1$ is examined first, with initial conditions $Y(0) = 1$, $(dY/dT)(0) = 0$. Since in $\Lambda$-space the solution is the same as in the conventional case, Eq. (21) yields the numerical solution presented in Figure 7. After transferring the solution from $\Lambda$-space to the initial space $(t,\ y(t))$, as previously described, the trace

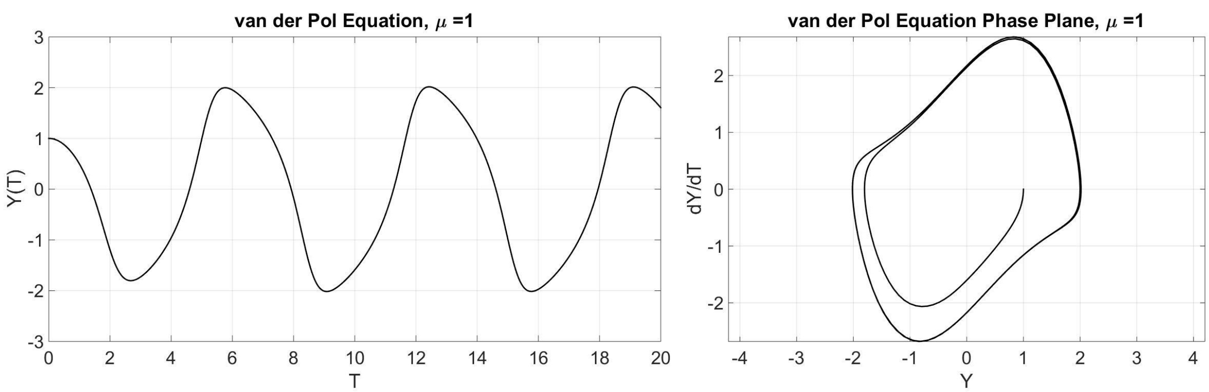


**Figure 7**. Solution of the VdP equation and its phase portrait in $\Lambda$-space for $\mu = 1$.

of y(t) for various values of $\gamma$ in real time space is shown in Figure 8. With increasing fractional order, it is evident that the width of oscillation is decreasing while the period is increasing. Furthermore, Figure 9 shows the corresponding phase portraits in the intitial space.

The effect of fractional dimension is more evident for the case $\gamma = 0.4$ where the various responses are extremely stronger. Figure 10 shows the trace of the function with respect to time in the initial space $(y,\ t)$, along with the corresponding phase portrait for an oscillator with friction coefficient $\mu = 1$. These results accentuate the major influence of the fractional order $\gamma$ on the motion of the VdP oscillator.

Increasing the friction coefficient to $\mu = 10$, and considering initial conditions $Y(0) = 1$ and $(dY/dT)(0) = 0$, the response of the VdP oscillator in $\Lambda$-space is shown in Figure 11. To emphasise the effect of the fractional order, solutions are then transferred to the initial (real time) space, recalling Eqs. (9, 16), numerically computed, and plotted along with the corresponding phase spaces for various values of $\gamma$ (Figure 12).

Increasing the influence of friction further in $\Lambda$-space, such that $\mu = 100$, and setting the initial conditions

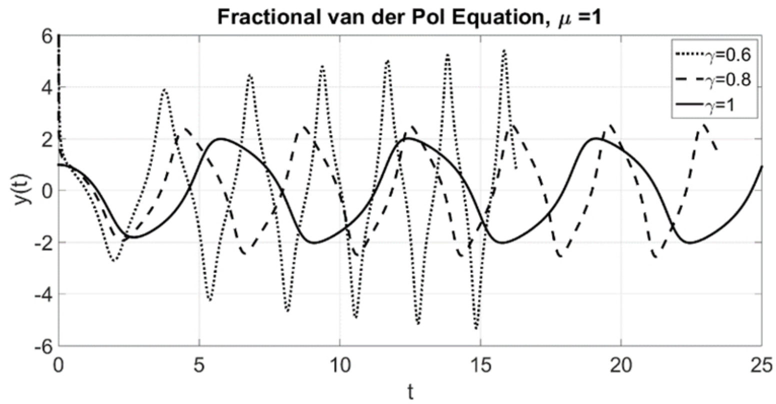


**Figure 8**. Trace $y(t)$ in the initial space for $\mu = 1$ and for various fractional dimensions.

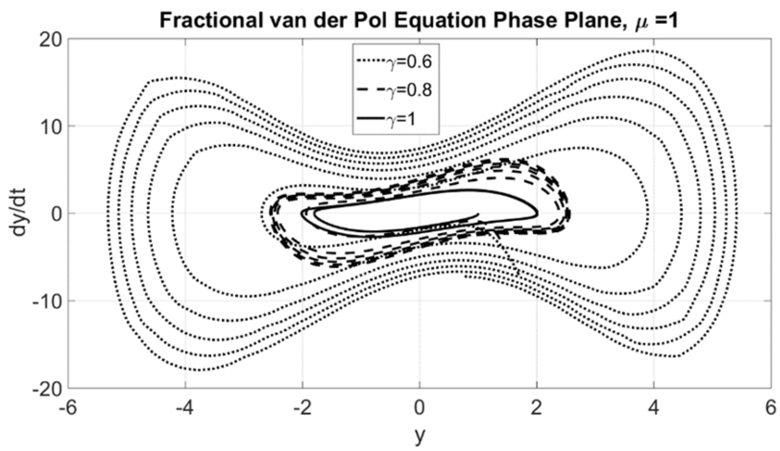


**Figure 9**. Phase portraits of the VdP oscillator in the initial space for $\mu = 1$ and for various fractional dimensions.

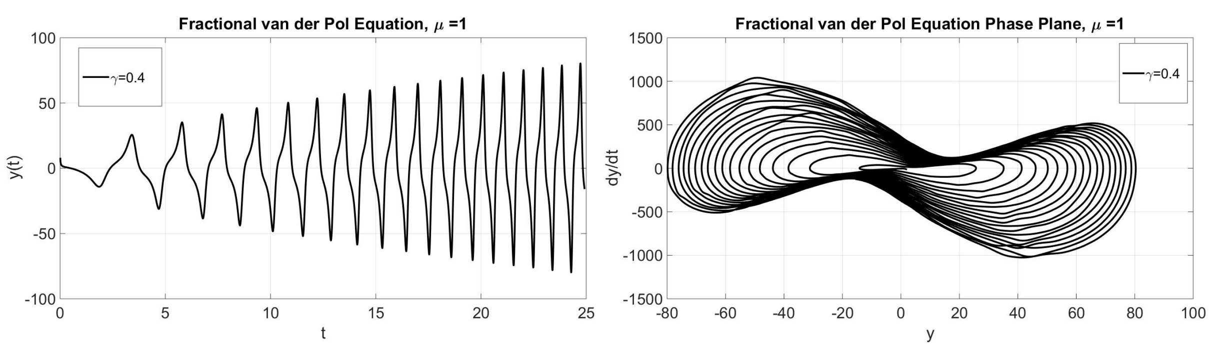


**Figure 10**. Trace $y(t)$ and phase portrait in the initial space for $\gamma = 0.4$ and $\mu = 1$.

as $Y(0) = 1$ and $(dY/dT)(0) = 0$, the numerical solution presented in Figure 13, with the trace and associated phase space, is obtained in the fractional $\Lambda$-space. Recalling Eqs. (9, 16), solutions are subsequently transferred to the initial (real time) space, numerically computed, and plotted along with the corresponding phase spaces for various values of $\gamma$ (Figure 14) in order to demonstrate the effect of the fractional order.

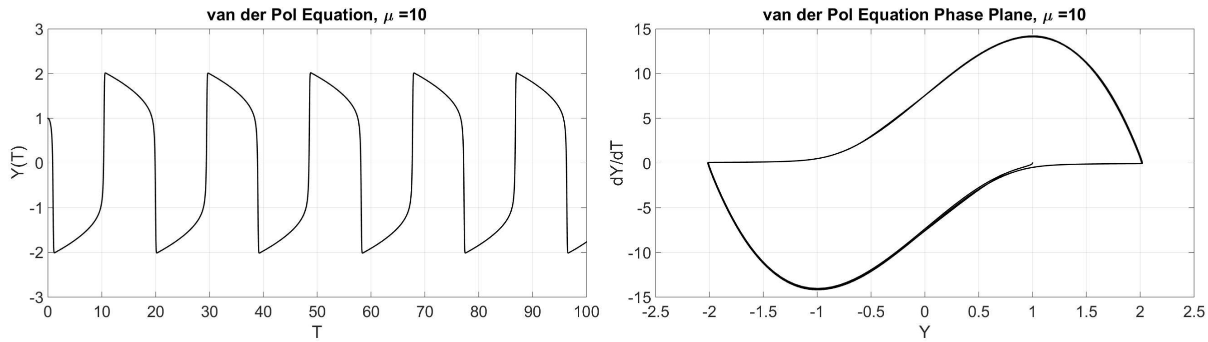


**Figure 11**. Solution and phase space of the VdP oscillator in $\Lambda$-space for $\gamma = 1$ and $\mu = 10$.

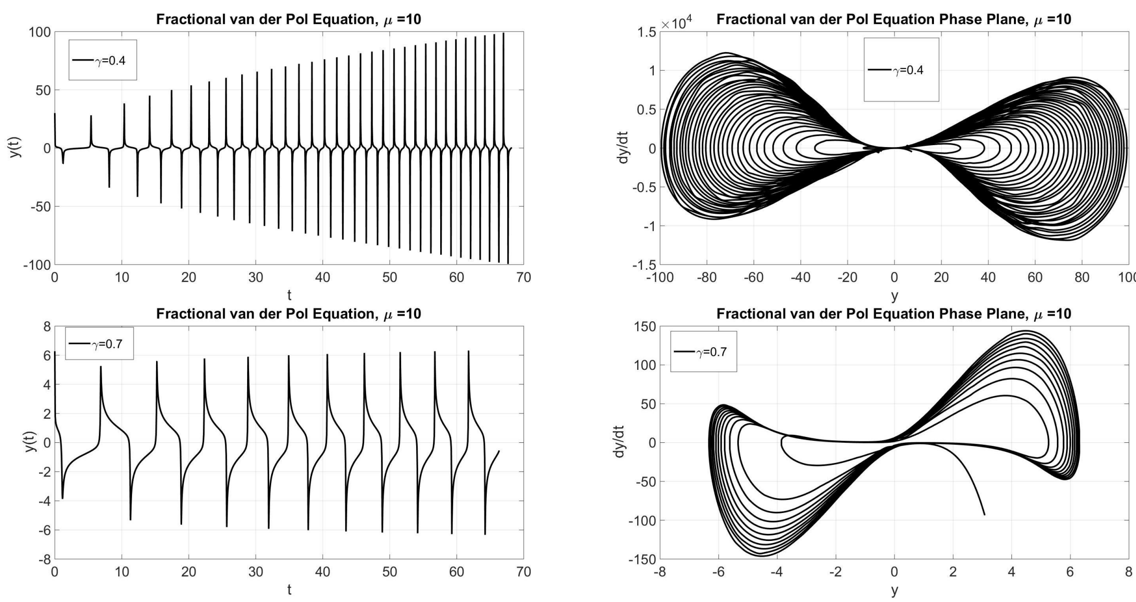


**Figure 12**. Solutions and phase spaces of the VdP oscillator in the initial space for $\mu = 10$ and for various fractional orders.

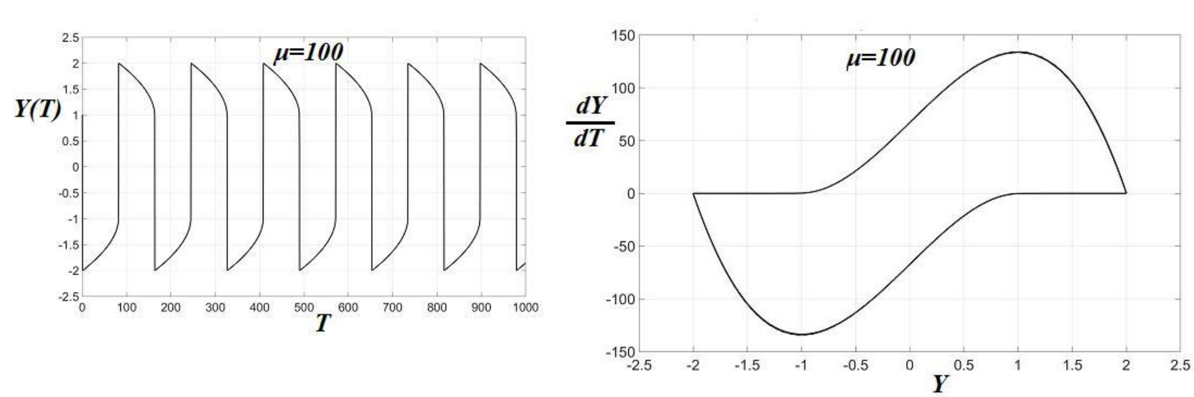


**Figure 13**. Solution and phase space of the VdP oscillator in $\Lambda$-space for $\gamma = 1$ and $\mu = 100$.

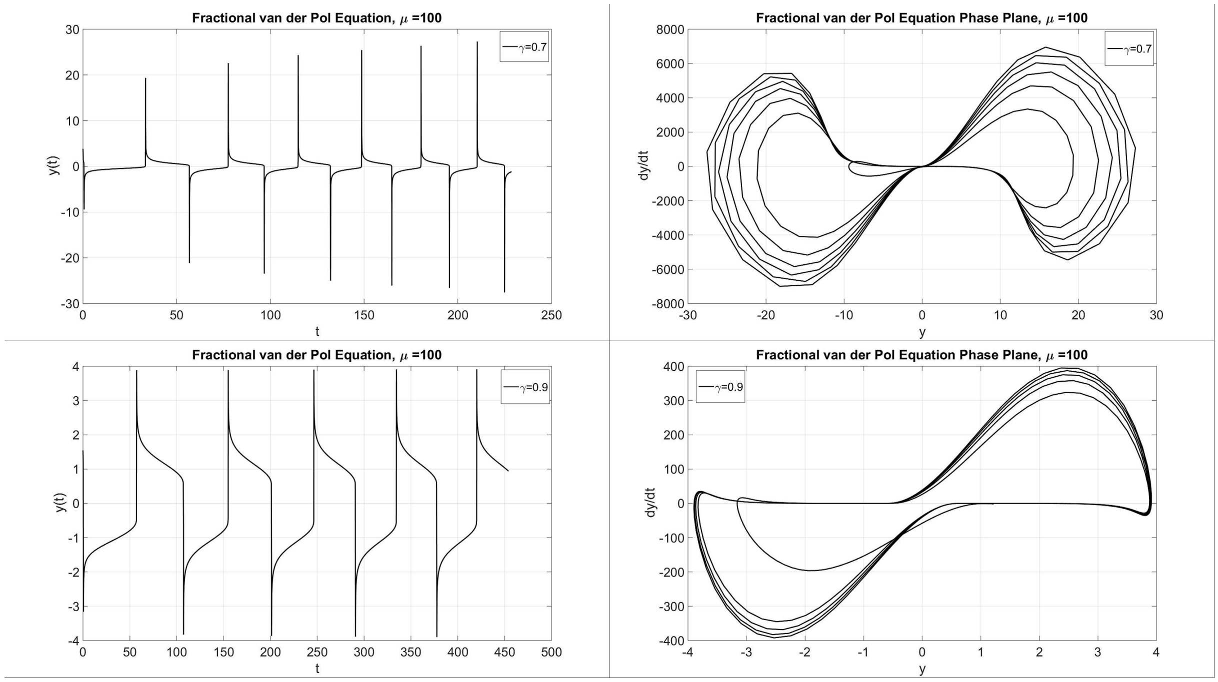


**Figure 14**. Solutions and phase spaces of the VdP oscillator in the initial space for $\mu = 100$ and for various fractional orders.

As it has already been pointed out, our purpose was to demonstrate how the fractional VdP equation may be solved and discussed considering the fractional order. Therefore, in the present work, we do not expand to other topics concerning the fractional VdP equation, like, for example, bifurcations etc.

## 4. The Forced $\Lambda$-Fractional Van der Pol Oscillator

Let us consider the differential equation of the forced VdPol oscillator in the fractional $\Lambda$-space, that is,

$$\frac{d^2Y}{dT^2} - \mu(1 - Y^2)\frac{dY}{dT} + Y = F\ cos(\omega T), \tag{24}$$

which is the VdP Eq. (21) with the presence of a forcing term on the right hand side of Eq. (24). Specifically, $F$ is the amplitude of the force applied with frequency $\omega$. In the following, Eq. (24) is solved numerically and solutions along with corresponding phase spaces are presented first in the fractional $\Lambda$-space and then in the initial space, for various parameter values.

For the forced VdP oscillator without damping $(\mu = 0)$, force amplitude $F = 10$, frequency $\omega = 3$, and initial conditions $Y(0) = 1, (dY/dT)(0) = 0$, the numerical solution in the fractional $\Lambda$-space yields the trace and phase space presented in Figure 15. Transferring the results to the initial space, see Eq.(23), the solutions and corresponding phase spaces for various $\gamma$ values are illustrated in Figure 16, which exemplifies and renders easily understood the influence of the fractional order $\gamma$ on the forced VdP oscillator without friction.

Proceeding to the study of the VdP equation with small friction and smaller force amplitude, with all other parameters remaining the same, the numerical solution in the $\Lambda$-space and associated phase space for $\mu = 1, F = 5, \omega = 3$, and initial conditions $Y(0) = 1, (dY/dT)(0) = 0$ is presented in Figure 17. Recalling

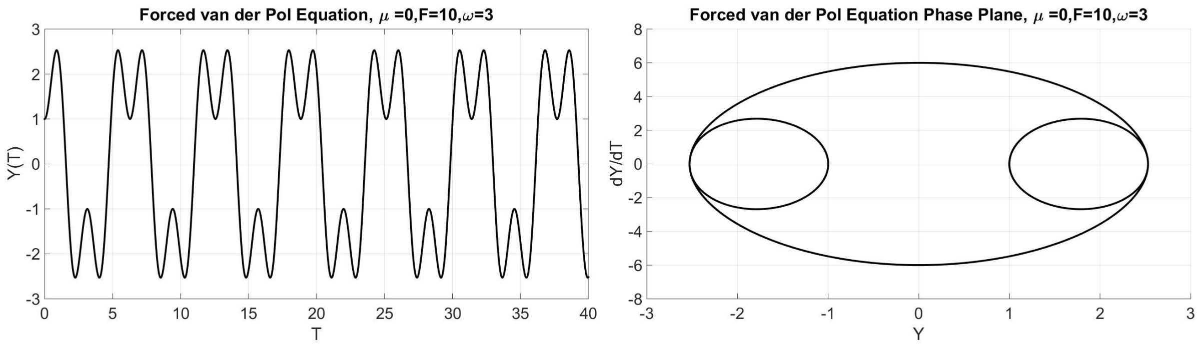


**Figure 15**. Solution and phase of the forced VdP oscillator in $\Lambda$-space for $\gamma = 1, \mu = 0, F = 10$ and $\omega = 3$.

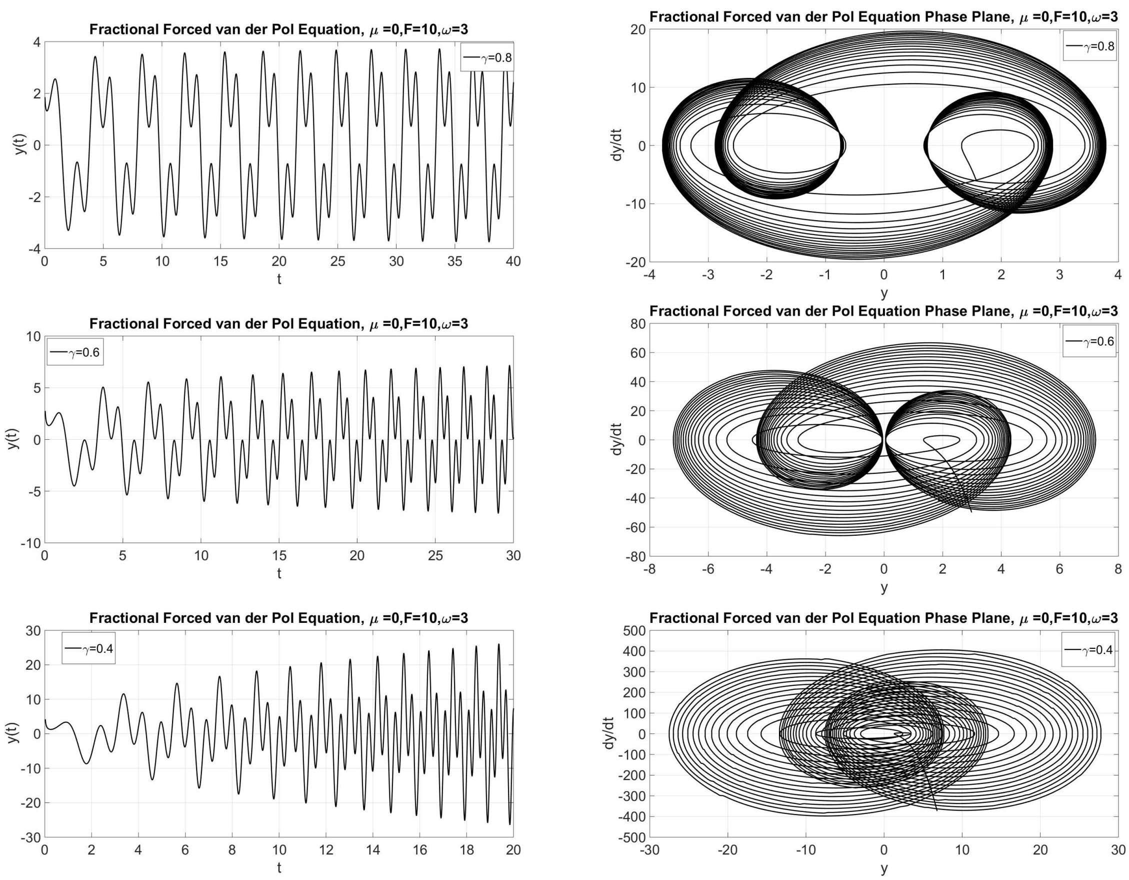


**Figure 16**. Solutions and phase spaces of the forced VdP oscillator in the initial space for fractional orders $\gamma = 0.8, 0.6, 0.4$; $\mu = 0$, $F = 10$ and $\omega = 3$.

Eq.(23), the results are transferred to the initial space and the solution $y(t)$ and corresponding phase space for various $\gamma$ values are presented in Figure 18.

Increasing the friction coefficient and force amplitude such that $\mu = 10$ and $F = 10$, with $\omega = 3$, $Y(0) = 1$, $(dY/dT)(0) = 0$, Eq. (24) of the forced VdP oscillator is solved numerically in the fractional $\Lambda$-space and the solution trace along with the corresponding phase space are shown in Figure 19. In turn, the corresponding

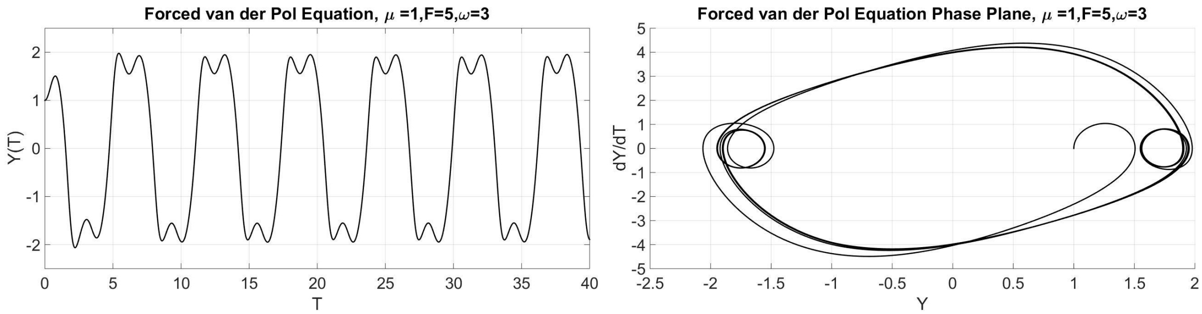


**Figure 17**. Solution and phase space of the forced VdP oscillator in $\Lambda$-space for $\mu = 1,\ F = 5$ and $\omega = 3$.

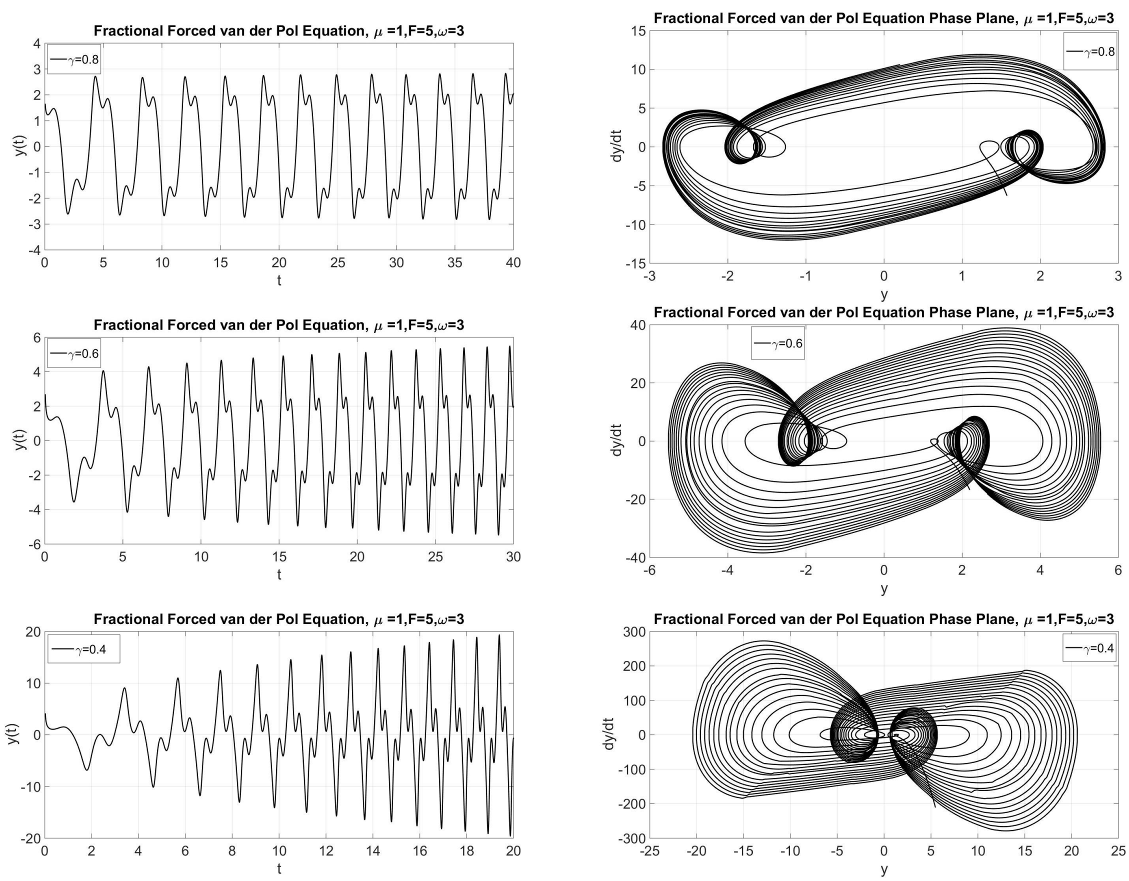


**Figure 18**. Solutions and phase spaces of the forced VdP oscillator in the initial space for fractional orders $\gamma = 0.8, 0.6, 0.4;\ \mu = 1,\ F = 5$ and $\omega = 3$.

solution $y(t)$ and phase space $(dy/dt,\ y)$ in the initial (real time) space, for various values of $\gamma$, are presented in Figure 20.

Finally, the effect of friction in the forced VdP oscillator is increased further, such that $\mu = 100$. Equation (24) is then solved numerically for $F = 5,\ \omega = 0.2, Y(0) = 1, (dY/dT)(0) = 0$, and the resulting trace and phase space are shown in Figure 21. The corresponding solution and phase space in the initial (real time) space, for various values of $\gamma$, are illustrated in Figure 22.

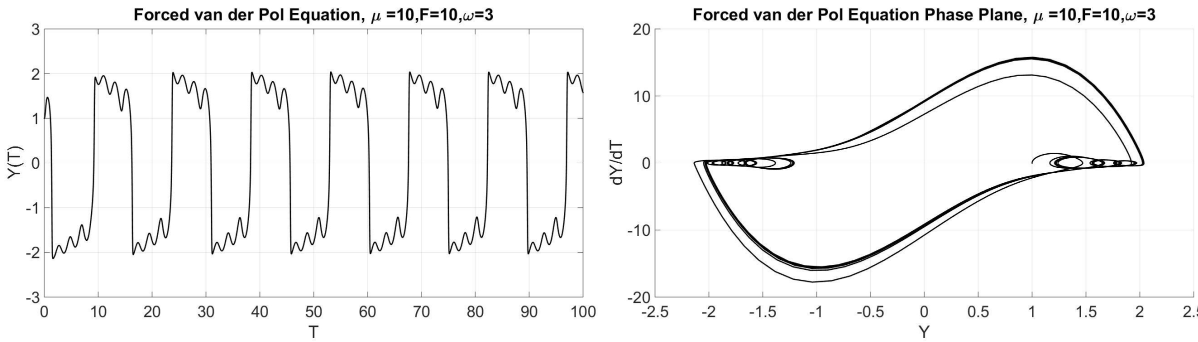


**Figure 19**. Solutions and phase spaces of the forced VdP oscillator in $\Lambda$-space for $\mu = 10,\ F = 10$ and $\omega = 3$.

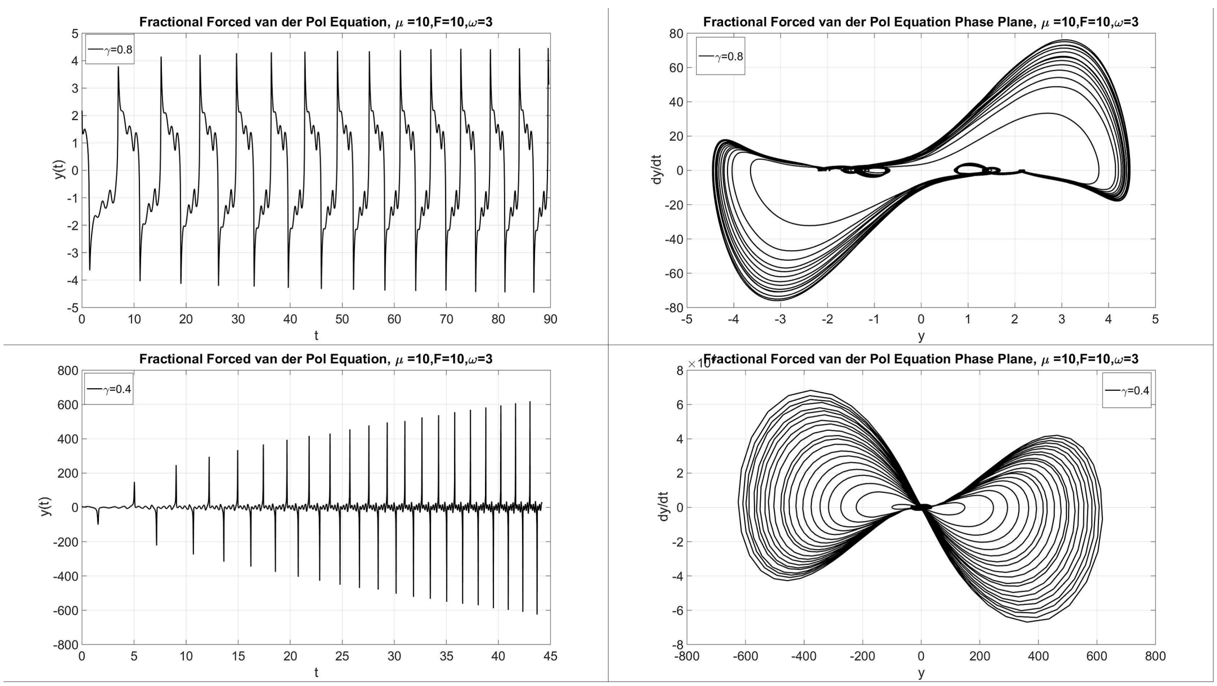


**Figure 20**. Solutions and phase spaces of the forced VdP oscillator in the initial space for fractional orders $\gamma = 0.8, 0.6, 0.4$; $\mu = 10,\ F = 10$ and $\omega = 3$.

## 5. Conclusion

The analysis of the fractional VdP oscillator, without and with a forcing term, introduced in the present work, explores the influence of the fractional order $\gamma$ on the oscillator's behaviour. The latter is thoroughly investigated for various $\gamma$ and for different values of the system's parameters, and then graphically exemplified through a series of figures of the solutions and associated phase portraits. It is evident from the figures that the influence of the fractional order on the fractional VdP equation is quite important since the oscillator's response becomes more pronounced with decreasing $\gamma$ values.

The $\Lambda$-fractional analysis satisfies the prerequisites of Differential Topology and offers mathematical accuracy. Furthermore, the $\Lambda$-fractional derivative is the unique fractional derivative corresponding to a differential and, therefore, able to generate Fractional Differential Geometry. It is quite important to point out that there is no Fractional Differential Geometry with the well known fractional derivatives, although

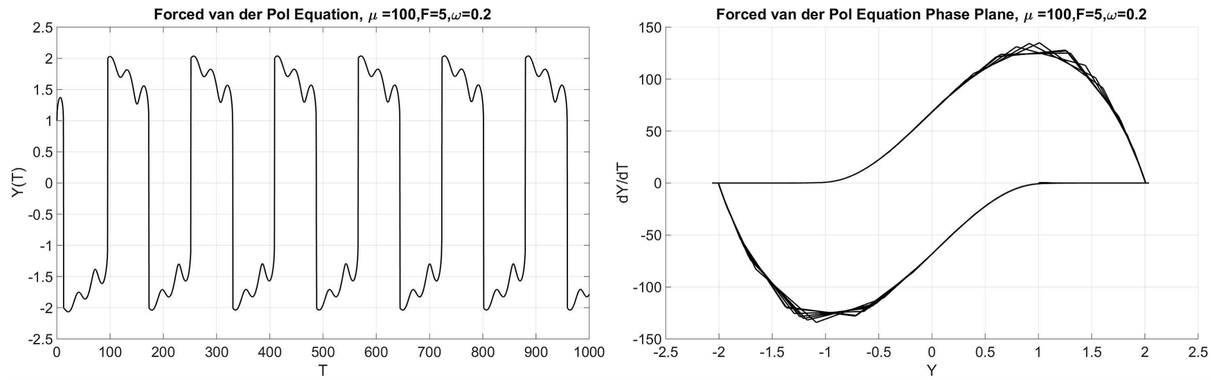


**Figure 21**. Solution and phase space of the forced VdP oscillator under strong friction in $\Lambda$-space $\mu = 100,\ F = 5$ and $\omega = 0.2$.

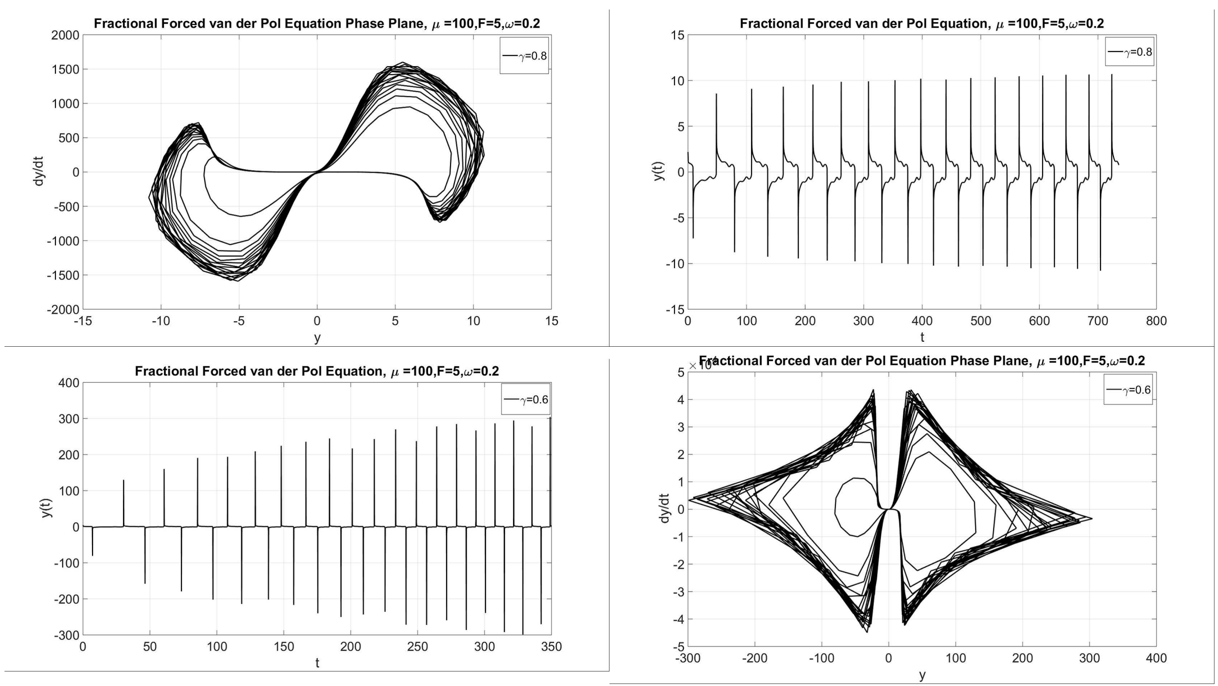


**Figure 22**. Solution and phase space of the forced VdP oscillator under strong friction in the initial space for fractional orders $\gamma = 0.8, 0.6;\ \mu = 100,\ F = 5$ and $\omega = 0.2$.

Fractional Calculus is used in solving real problems in physics, mechanics, etc.

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