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Some properties and continuity of transitive binary relational sets

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Abstract: In the present paper, we introduce and study the continuity and some properties for a set equipped with a transitive binary relation which we call t-set. Also, we give a characterization of a continuous directed complete posets via continuous t-sets. Furthermore, some properties of algebraic t-sets are considered. Our work is inspired by the slogan: "Order theory is the study of transitive relations" due to Abramsky and Jung [1]. The corresponding results of our results due to Nino-Salcedo [8], Heckmann [4] and Zhang [11] are generalized.

Key words: Continuous lattice, Algebraic lattice, Continuous domain, Algebraic domain

1. Introduction and Preliminaries

Abramsky and Jung [1], introduced a method to construct a canonical partially ordered set from a pre-ordered set and said: "many notions from the theory of partially ordered sets make sense even if reflexivity fails". From this point of view, the present paper introduces and studies the continuity and some properties for a set with a transitive binary relation (so called a t-set). It is worth to mention that the continues lattices and algebraic lattices were studied in detail in [3]. Also, some types of continues posets (domains) and algebraic posets (domains) were studied in detail in [1, 4, 5, 8, 11].

This paper is organized as follows: In Section 1, some preliminaries are given. In Section 2, some basic properties of t-sets are discussed. The concept of continuous t-sets is introduced and studied in Section 3. In Section 4, a characterization of continuous directed complete posets [8] were given via continues t-set. Finally, in Section 5, we give a conclusion to show that our results are a generalization of the corresponding results in pre-ordered set (resp. abstract base, continuous information system).

Definition 1.1. Let "$\leq$" be a binary operation on a nonempty set X.

(1) "$\leq$" is reflexive if for all $x, y \in X, x \leq x$ [7];

(2) "$\leq$" is antisymmetric if for all $x, y \in X, x \leq y$ and $y \leq x$, then $x = y$ [7];

(3) "$\leq$" is transitive if for all $x, y, z \in X, x \leq y$ and $y \leq z$ then $x \leq z$ [7];

(4) "$\leq$" is symmetric if for all $x, y \in X, x \leq y$, then $y \leq x$ [7];

(5) "$\leq$" is interpolative if for all $x, z \in X$ with $x \leq z$ there exists $y \in X$ such that $x \leq y \leq z$ [4, 9].

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Definition 1.2. Let X be a nonempty set with a binary relation "$\leq$" on X. The pair $(X, \leq)$ is said to be:

(1) a partially ordered set (poset for short) [7] if "$\leq$"' satisfies the conditions (1), (2) and (3) in Definition 1.1 above;

(2) a pre-ordered (quasi ordered) set [7] if "$\leq$" satisfies the conditions (1) and (3) in Definition 1.1 above;

(3) an equivalence set [7] if "$\leq$" satisfies the conditions (1), (3) and (4) in Definition 1.1 above;

(4) a continuous information system [6, 9] if "$\leq$" satisfies the conditions (3) and (5) in Definition 1.1 above.

(5) an abstract base [10] if "$\leq$" satisfies the conditions (3) in Definition 1.1 above and the following condition holds for every $x \in X$ and every finite subset A of X. If for every $y \in A, y \leq x$, then there exists $z \in X$ such that $y \leq z \leq x$.

Definition 1.3. [2]. Let $(X, \leq)$ be a poset and $A \subseteq X$. Then

(1) the lower (resp. upper) bounded subset in X of A is denoted by $lb(A)$(resp. $ub(A)$) and defined as follows: $lb(A) = \{x \in X : \forall y \in A, x \leq y\}$ (resp.$ub(A) = \{x \in X : \forall y \in A, y \leq x\}$). Each element in $lb(A)$(resp. $ub(A)$) is called a lower (resp. an upper) bounded of A;

(2) the subset of least (resp. largest) elements of a subset A is denoted by $le(A)$(resp. $la(A)$) and defined as follows:
$le(A) = \{x \in A : \forall y \in A, x \leq y\}$ (resp.$la(A) = \{x \in A : \forall y \in A, y \leq x\}$). Each element in $le(A)$(resp. $la(A)$) is called a least (resp. largest) element of A;

(3) the infimum (resp. supremum) of a subset A of X is denoted by $\inf(A)$(resp. $\sup(A)$) and defined as follows: $\inf(A) = la(lb(A))$ (resp.$\sup(A) = le(ub(A))$).
Each element in $\inf(A)$(resp. $\sup(A)$) is called an infimum (resp. a supremum) of A;

(4) the lower (resp. upper) closure of A is denoted by $\downarrow(A)$ (resp. $\uparrow(A)$ and defined as follows:
$\downarrow(A) = \{x \in X : \exists y \in A, x \leq y\}$ (resp.$\uparrow(A) = \{x \in X : \exists y \in A, y \leq x\}$).

Definition 1.4. [8]. Let $(X, \leq)$ be a poset and $A, B \subseteq X$. Then A is said to be:

(1) a directed subset if $A \neq \emptyset$ and for every distinct points $x, y \in A$, there exists $z \in (A \cap ub\{x, y\})$;

(2) a cofinal in B if $A \subseteq B \subseteq \downarrow(A)$.

Definition 1.5. [8]. A poset $(X, \leq)$ is a domain if for every directed subset A of X,$\sup(A)$ exists.

Definition 1.6. [8]. Let $(X, \leq)$ be a domain and $x, y \in X$. We say x is way below y denoted by $x \ll y$ if for every directed subset D of X such that $\sup(D)$ exists and $y \leq \sup(D)$ there exists $d \in D$ such that $x \leq d$. The family of all elements of X that way-below x is denoted and defined as follows: $\Downarrow x = \{y \in X : y \ll x\}$).

Definition 1.7. [8]. Let $(X, \leq)$ be a poset and $x \in X$. If $x \ll x$, then x is said to be compact (isolated) and we write $x \in K(X)$. The family of all isolated points that below $x \in X$ is denoted and defined by:
$\downarrow^{\circ} x = \{y \in X : y \ll y, y \leq x\}$.

Definition 1.8. [8]. A poset $(X, \leq)$ is chain-complete if every directed subset has a supremum, equivalently, if $(X, \leq)$ is a domain in the sense of Heckmann [4].

Definition 1.9. [5]. A subset B of X is base (basis) for a poset $(X, \leq)$ if for every $d \in X$ the set $B \cap \Downarrow d$ is directed subset of X and $\sup(B \cap \Downarrow d) = d$

Definition 1.10. [8]. A poset $(X, \leq)$ is continuous in the sense Nino-Salcedo if:

(1) $(X, \leq)$ is chain-complete;

(2) $\Downarrow x$ is directed for every $x \in X$;

(3) $\sup(\Downarrow x) = x$ for every $x \in X$.

Definition 1.11. [4]. A domain $(X, \leq)$ is continuous in the sense of Heckmann if for every $x \in X$, there exist a directed subset D of $\Downarrow x$ such that $x = \sup(D)$.

Definition 1.12. [11]. A poset $(X, \leq)$ is continuous in the sense Zhang if for every $x \in X$

(1) $\Downarrow x$ is directed;

(2) $\sup(\Downarrow x) = x$.

Definition 1.13. [5]. A poset X is continuous in the sense of Kummetz if it has a base.

Definition 1.14. [8]. A poset $(X, \leq)$ is an algebraic poset in the sense Nino-Salcedo if the following conditions are satisfied:

(1) $(X, \leq)$ is chain-complete;

(2) for every $x \in X, \downarrow^\circ x$ is directed subset; and

(3) for every $x \in X$, $x = \sup(\downarrow^\circ x)$.

Definition 1.15. [8]. A domain $(X, \leq)$ is an algebraic in the sense of Heckmann if for every $x \in X$, there exist a directed subset D of $\downarrow^\circ x$ such that $x = \sup(D)$.

For more well-known basic concepts in posets we refer to [7] and for lattices we refer to [2].

2. Some properties of transitive binary relational sets

Definition 2.1. A transitive binary relational set (t-set for short) is a pair $(X, \leq)$ where X is a non-empty set and "$\leq$" is transitive binary relation on X. For example, partially ordered sets, pre-ordered sets, equivalence sets, and continuous information systems are t-sets.

Notation 2.1. *Any abstract base is a continuous information system (Indeed, suppose $x \in X$ and $\{y\}$ be a finite subset of X, where $y \in X$. If $y \leq x$, then there exists $z \in X$ such that $y \leq z \leq x$. Hence, "$\leq$" is interpolative binary relation on X. The converse need not be true as we illustrate by the following example.*

Example 2.1. *Let $X = \{a, b, x\}$ and $\leq = \{(a, a), (b, b), (a, x)(b, x)\}$. Then "$\leq$" is transitive and interpolative binary relation on X. Now,suppose $A = \{a, b\}$. Hence $(X, \leq)$ is not an abstract base.*

Notation 2.2. *Every pre-ordered set is an abstract base. The converse need not be true as we illustrate by the following example.*

Example 2.2. *Let* $X = \{a, b, c, d, e\}$ *and* $\leq = \{(a, a)\}$. *Then* $(X, \leq)$ *is an abstract base and "*$\leq$*" is not reflexive. Hence* $(X, \leq)$ *is not a pre-ordered set.*

Notation 2.3. *In any t-set* $(X, \leq)$, *any subset* A *of* X, $la(A), le(A), \sup(A)$ *and* $\inf(A)$ *need not be singletons.*

Notation 2.4. *The authors in [8] considered a set* X *equipped with a transitive binary relation and defined the upper (resp. lower) bounded as in Definition 1.2(1) above but they defined the supremum(resp. infimum) of a subset* A *of* X *as the unique least element (resp. unique largest element) of the set of upper bounds (resp. lower bounds) of* A *if it exists. The supremum (resp. infimum) of a subset* A *of* X *in our sense need not be unique.*

Proposition 2.1. *Let* $(X, \leq)$ *be a t-set and* $A, B \subseteq X$. *Then*

(1) $\downarrow(\emptyset) = \emptyset$ *and* $\downarrow(X) \subseteq X$;

(2) $\uparrow(\emptyset) = \emptyset$ *and* $\uparrow(X) \subseteq X$;

(3) $A \subseteq B$, *then* $\downarrow(A) \subseteq \downarrow(B)$;

(4) $A \subseteq B$, *then* $\uparrow(A) \subseteq \uparrow(B)$;

(5) $\downarrow\downarrow(A) \subseteq \downarrow(A)$ *and* $\uparrow\uparrow(A) \subseteq \uparrow(A)$;

(6) If $A \subseteq \downarrow(B)$, *then* $\downarrow(A) \subseteq \downarrow(B)$;

(7) If $A \subseteq \uparrow(B)$, *then* $\uparrow(A) \subseteq \uparrow(B)$.

Proposition 2.2. *Let* $\{A_j : j \in J\}$ *be a family of subsets of a t-set* $(X, \leq)$. *Then:*

(1) $\downarrow(\bigcup_{j\in J} A_j) = \bigcup_{j\in J}\{\downarrow(A_j)\}$;

(2) $\uparrow(\bigcup_{j\in J} A_j) = \bigcup_{j\in J}\{\uparrow(A_j)\}$;

(3) $\downarrow(\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J}\{\downarrow(A_j)\}$;

(4) $\uparrow(\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J}\{\uparrow(A_j)\}$.

Proof. (1) $x \in\downarrow(\bigcup_{j\in J} A_j)$ if and only if there exists $y \in \bigcup_{j\in J} A_j$ such that $x \leq y$ if and only if there exists $A_{j_\circ}$ such that $j_\circ \in J, y \in A_{j_\circ}, x \leq y$ if and only if $x \in (\downarrow A_{j_\circ})$ for some $j_\circ \in J$ if and only if $x \in \bigcup_{j\in J}(\downarrow(A_j))$.

(2) $x \in\uparrow(\bigcup_{j\in J} A_j)$ if and only if there exists $y \in \bigcup_{j\in J}(A_j)$ such that $y \leq x$ if and only if there exists $A_{j_\circ}$ such that $j_\circ \in J, y \in A_{j_\circ}, y \leq x$ if and only if $x \in (\uparrow A_{j_\circ})$ for some $j_\circ \in J$ if and only if $x \in \bigcup_{j\in J}(\uparrow(A_j))$.

(3) Suppose that $x \in\downarrow(\bigcap_{j\in J} A_j)$. Then there exists $y \in \bigcap_{j\in J} A_j$ such that $x \leq y$. So, for all $j \in J$ there exists $y \in A_j$ such that $x \leq y$. Hence $x \in\downarrow(A_j)$ for all $j \in J$. Therefore, we get $x \in \bigcap_{j\in J}(\downarrow(A_j))$.

(4) Suppose that $x \in\uparrow(\bigcap_{j\in J} A_j)$. Then there exists $y \in \bigcap_{j\in J} A_j$ such that $y \leq x$. So, for all $j \in J$ there exists $y \in A_j$ such that $y \leq x$. Hence $x \in\uparrow(A_j)$ for all $j \in J$. Therefore, we have that $x \in \bigcap_{j\in J}(\uparrow(A_j))$.

□

Proposition 2.3. *Let* $(X, \leq)$ *be a t-set and* $A, B \subseteq X$. *If* B *is directed subset and cofinal in* A, *then* A *is directed subset and* $\sup(A) = \sup(B)$.

Proof. First, we prove that A is directed subset. Since $B \subseteq A$, then $A \neq \emptyset$. Let $l, m \in A$ such that $l \neq m$. Then there exist $b_1, b_2 \in B$ such that $l \leq b_1, m \leq b_2$ and $b \in ub(\{b_l, b_2\}) \cap A)$. Hence A is a directed subset. Second, one can deduce that $ub(A) = ub(B)$(Indeed, since $B \subseteq A$, then $ub(A) \subseteq ub(B)$. Now, suppose $y \notin ub(A)$. Then there exists $a \in A$ such that $a \nleq y$. Hence, there exists $a \in \downarrow (B)$ such that $a \nleq y$. Thus, there exists $b \in B$ such that $a \leq b$ and $a \nleq y$ or $b \in B$ such that $b \nleq y$. So, $y \notin ub(B)$. Hence $ub(B) \subseteq ub(A)$). Therefore $\sup(A) = \sup(B)$. □

Proposition 2.4. *Let $(X, \leq)$ be a t-set and $x, y, z \in X$. Then, we have*

(1) if $x \leq y$ and $y \ll z$, then $x \ll z$;

(2) if $x \ll y$ and $y \leq z$, then $x \ll z$;

(3) if $\sup(\{y\}) \neq \emptyset$ and $x \ll y$, then $x \leq y$;

(4) if $\sup(\{y\}) \neq \emptyset$ or $\sup(\{z\}) \neq \emptyset$, $x \ll y$ and $y \ll z$, then $x \ll z$.

Proof. (1) Let D be a directed subset of X such that $z \in\downarrow (\sup(D))$. Then there exists $d \in D$ such that $y \leq d$. Then, $x \leq d$. Therefore, $x \ll z$.

(2) Let D be a directed subset of X such that $z \in\downarrow (\sup(D))$. Then there exists $k \in \sup(D)$ such that $z \leq k$. Thus $y \leq k$ and so $y \in\downarrow \sup(D)$. Hence there exists $l \in D$ such that $x \leq l$. Therefore, $x \ll z$.

(3) Let $D = \{y\}$ be a directed subset of X and $x \ll y$. Then there exists $d \in D$ such that $x \leq d$. But $y = d$. Therefore, $x \leq y$.

(4) From (1) and (2) above we can prove (4).

□

3. Continuous t-sets

Definition 3.1. A t-set $(X, \leq)$ is said to be continuous if for every $x \in X$, the following conditions are satisfied:

(1) $\sup(\{x\}) \neq \emptyset$;

(2) $\Downarrow x$ is a directed subset of X;

(3) $x \in\downarrow (\sup(\bigcup\{\sup(\Downarrow a) : a \in\Downarrow x\}))$.

Lemma 3.1. *In a t-set $(X, \leq)$, suppose $\Downarrow x$ be a directed subset for every $x \in X$. Then for every $z \in X$, $D = \bigcup\{\Downarrow a : a \in\Downarrow z\}$ is a directed subset of X.*

Lemma 3.2. *Let $(X, \leq)$ be a t-set. Then for every $x \in X$, we have that*

$$ub(\cup\{(\Downarrow a) : a \in\Downarrow x\}) = ub(\cup\{\sup(\Downarrow a) : a \in\Downarrow x\}).$$

Theorem 3.1. *If $(X, \leq)$ is a continuous t-set, then the way below relation "$\ll$" is interpolative.*

Proof. From Lemmas 3.1 and 3.2, we have $z \in\downarrow (\sup(\bigcup\{\Downarrow a : a \in\Downarrow z\}))$ and $(\bigcup\{\Downarrow a : a \in\Downarrow z\})$ is directed. Then there exists $x \in\Downarrow a$ for some $a \in\Downarrow z$ such that $x \leq a$. From Proposition 2.4(1), we have that $x \ll a$. Therefore, $x \ll a \ll z$. □

Note that Theorem 3.1 is a generalization of the corresponding result in Proposition 6.7.4 [3], Lemma 2.15 (i) [8] and Proposition 1.17 (3) [11].

From Proposition 2.4(4) and Theorem 3.1, we have the following result.

Theorem 3.2. *If* $(X,\leq)$ *is continuous t-set, then* $(X,\leq)$ *is continuous information system.*

Corollary 3.1. *If* $(X,\leq)$ *is continuous poset in the sense of Zhang (resp., continuous poset in the sense of Nino-Salcedo, continuous domain in the sense of Heckmann), then* $(X,\ll)$ *is continuous information system.*

Lemma 3.3. *Let* $(X,\leq)$ *be a t-set. If for all* $x\in X$ *,*$\sup(\{x\})\neq\emptyset$ *and If "*$\ll$*" is interpolative, then for every* $x\in X$ *we have* $\Downarrow x=\bigcup\{\Downarrow a : a\in\Downarrow x\}$*.*

Proof. First, suppose that $z\in\bigcup\{\Downarrow a : a\in\Downarrow z\}$. Then there exists $a\in\Downarrow x$ such that $z\ll a$. From Proposition 2.4(4), $z\ll x$. Then $z\in\Downarrow x$. Second, let $z\in\Downarrow x$. Then $z\ll x$. Since "$\ll$" is interpolative, then there exists $a\in X$ such that $z\ll a\ll x$. Therefore, $z\in\bigcup\{\Downarrow a : a\in\Downarrow z\}$. □

Applying Lemma 3.1, Theorems 3.1 and 3.2, we have the following theorem.

Theorem 3.3. $(X,\leq)$ *is a continuous t-set if and only if the following conditions are satisfied:*

(1) for all $x\in X$*,* $\sup(\{x\})\neq\emptyset$*;*

(2) for all $x\in X$*,* $\Downarrow x$ *is directed;*

(3) "$\ll$*" is interpolative ;*

(4) for all $x\in X$*,* $x\in\downarrow(\sup(\Downarrow x))$*.*

Proof. First, Conditions (1) and (2) above are common.

⇒: From Theorem 3.1 "$\ll$" is interpolative and from Lemma 3.3, Condition (4) is satisfied.

⇐: From Lemma 3.3, Condition (3) in Definition 3.1 is satisfied. □

Corollary 3.2. $(X,\leq)$ *is a continuous domain in the sense of Heckmann (continuous poset in the sense of Nino-Salcedo) (resp. continuous poset in the sense of Zhang) if and only if* $(X,\leq)$ *is a domain (resp. poset) and*

(1) for all $x\in X$*,* $\Downarrow x$ *is directed;*

(2) "$\ll$*" is interpolative;*

(3) for all $x\in X$*,* $x\in\downarrow(\sup(\Downarrow x))$*.*

In the proof of the following theorem we use the concept of cofinal between two sets to introduce a characterization of continuous t-sets.

Theorem 3.4. $(X,\leq)$ *is a continuous t-set if and only if the following conditions are satisfied:*

(1) for all $x\in X$*,* $\sup(\{x\})\neq\emptyset$*;*

(2) "$\ll$*" is interpolative;*

(3) For every $x\in X$*, there exists a directed subset* D *of* $\Downarrow x$ *such that* $x\in(\sup(D))$*.*

Proof. $\Rightarrow$: From Theorem 3.3, Conditions (1) and (2) above are satisfied. Condition (3) is satisfied if we put $D = \Downarrow x$.

$\Leftarrow$: Now conditions (1) and (3) in Theorem 3.3 are given above. We need to prove that D is cofinal in $\Downarrow x$. First $D \subseteq \Downarrow x$ and D is directed. Second, suppose $y \in \Downarrow x$. Since $x \in \downarrow (\sup(D))$, then there exists $d \in D$ such that $y \in \downarrow D$. Then from Proposition 2.3, $\Downarrow x$ is directed and $\sup(\Downarrow x) = \sup(D)$. Hence conditions (2) and (4) in Theorem 3.3 are satisfied. □

Corollary 3.3. *$(X, \leq)$ is a continuous domain in the sense of Heckmann (continuous poset in the sense of Nino-Salcedo) (resp. continuous poset in the sense of Zhang) if and only if $(X, \leq)$ is a domain (resp. poset) and*

(1) "$\ll$" is interpolative ;

(2) for every $x \in X$, there exists directed subset D *of $\Downarrow x$ such that $\downarrow \sup(D)$ (resp. $\sup(D)$) exists and $x \in \downarrow (\sup(D))$.*

Theorem 3.5. *$(X, \leq)$ is a continuous t-set if and only if the following statements are true:*

(1) for all $x \in X$, $\sup(\{x\}) \neq \emptyset$;

(2) for every $x \in X$, there exists directed subset D *of $\bigcup\{\Downarrow a : a \in \Downarrow x\}$ such that $x \in \downarrow (\sup(D))$.*

Proof. $\Rightarrow$: From Theorem 3.4 and Lemma 3.3 one can have that $\Downarrow x = \bigcup\{\Downarrow a : a \in \Downarrow x\}$ so that from condition (3) in theorem 3.4 one have directly condition (2) above.

$\Leftarrow$: Since D is cofinal in $\bigcup\{\Downarrow a : a \in \Downarrow x\}$ (Indeed, $D \subseteq \bigcup\{\Downarrow a : a \in \Downarrow x\}$. Let $z \in \bigcup\{\Downarrow a : a \in \Downarrow x\}$. Then $z \ll a$ for some $a \in \Downarrow x$ so that from Proposition 2.4(4) $z \ll x$. Since D is directed and $x \in \downarrow (\sup(D))$, then there exists $d \in D$ such that $z \leq d$, i.e. $z \in \downarrow (D)$. , then from Proposition 2.3, $\cup\{\Downarrow a : a \in \Downarrow x\}$ is directed and $\sup(D) = \sup(\bigcup\{\Downarrow a : a \in \Downarrow x\})$.Hence Condition (3) in Theorem 3.4 is satisfied. Also, one can prove that "$\ll$" is interpolative (Indeed, suppose that $x \ll z$. Then from Condition (2) above we have $z \in \downarrow (\sup(\bigcup\{\Downarrow a : a \in \Downarrow x\})))$. Thus, there exists $d \in \bigcup\{\Downarrow a : a \in \Downarrow x\}$ such that $x \leq d \ll a \ll z$. So, from Proposition 2.4 (1), $x \ll a$. Then $x \ll a \ll z$.). Hence Condition (2) in Theorem 3.4 is satisfied. Therefore $(X, \leq)$ is a continuous t-set. □

Note that Theorem 3.5 is a generalization of Proposition 6.7.2 [4] and Lemma 2.14 [8].

Corollary 3.4. *$(X, \leq)$ is a continuous poset in the sense of Zhang if and only if for every $x \in X$, there exists directed subset* D *of $\bigcup\{\Downarrow a : a \in \Downarrow x\}$ such that $x = \sup(D)$.*

From Lemma 3.3, one can write $\Downarrow a$ in Theorem 3.5 and Corollary 3.4 instead of $\bigcup\{\Downarrow a : a \in \Downarrow x\}$.

Definition 3.2. Let $(X, \leq)$ be a t-set. A subset B of X is called a base for X if the following conditions are satisfied:

(1) for all $x \in X$, $\sup(\{x\}) \neq \emptyset$;

(2) for all $x \in X$, there exists a directed subset D_x of B such that $D_x \subseteq \bigcup\{\Downarrow a : a \in \Downarrow x\}$ and $x \in \downarrow (\sup D_x)$.

Theorem 3.6. *$(X, \leq)$ is continuous t-set if and only if it has a base.*

Proof. $\Rightarrow$: From Theorem 3.5, put $B = \bigcup_{x \in X}(\{\bigcup \Downarrow a : a \in \Downarrow x\})$.

$\Leftarrow$: Condition (2) is in Theorem 3.5 is satisfied directly from the definition of the base of a t-set. □

From Lemma 3.3 we have the following theorem.

Theorem 3.7. *If $(X, \leq)$ is continuous poset in the sense of Kummetz [5], then the concept of a base in Definition 3.2 coincides with the concept of a base due to Kummetz given in Definition 1.9.*

Corollary 3.5. *$(X, \leq)$ is a continuous domain in the sense of Zhang if and only if it is a continuous poset in the sense of Kummetz.*

Corollary 3.6. *$(X, \leq)$ is a continuous domain in the sense of Heckmann (continuous poset in the sense of Nino-Salcedo) if and only if it is a continuous poset in the sense of Kummetz i.e. it has a base.*

Corollary 3.7. *$(X, \leq)$ is a continuous domain in the sense of Zhang if and only if it has a base.*

Lemma 3.4. *Let $(X, \leq)$ be a continuous t-set. If "$\leq$" is antisymmetric, then*

(1) the supremum of any set is unique whenever it exists; and

(2) for every $x \in X$, $x = \sup(\Downarrow x)$.

Proof. (1) Obvious.

(2) Now, from Theorem 3.3 (4), $x \in \downarrow (\sup(\Downarrow x))$ and so $x \leq \sup(\Downarrow x)$. Conversely, for every $a \in \Downarrow x$ we have $a \ll x$. From Proposition 2.4(3),we have for every $a \in \Downarrow x$, $a \leq x$. So, $\sup(\Downarrow x) \leq x$ for every $x \in X$.

□

Lemma 3.5. *If $(X, \leq)$ is continuous t-set and if "$\leq$" is reflexive, then a subset D of X is directed if and only if for every $x, y \in D$ there exists $z \in ub(\{x, y\}) \cap D$.*

Proof. $\Rightarrow$: Let $\{x\} \subseteq D$. Since "$\leq$" is reflexive, then $x \in ub(\{x\}) \cap D$;

$\Leftarrow$: Obvious. □

Theorem 3.8. *$(X, \leq)$ is a continuous domain in sense of Heckmann (continuous poset in sense of Nino-Salcedo) if and only if $(X, \leq)$ is a continuous domain t-set with "$\leq$" is reflexive and antisymmetric.*

Proof. From Lemmas 3.4 and 3.5, the proof is obtained. □

Corollary 3.8. *A continuous domain in sense of Heckmann, continuous poset in sense of Nino-Salcedo, continuous poset in sense of Kummetz and continuous poset in sense of Zhang are equivalent notions.*

4. Algebraic t-sets

Definition 4.1. A t-set $(X, \leq)$ is algebraic if the following conditions are satisfied:

(1) for all $x \in X, \downarrow^{\circ} x$ is directed;

(2) for all $x \in X, x \in \downarrow (\sup(\downarrow^{\circ} x))$.

Theorem 4.1. *If $(X, \leq)$ is algebraic t-set, then "$\ll$" is interpolative.*

Proof. Let $x, z \in X$ such that $x \ll z$. From Condition (2) above and from the fact that $z \in \downarrow (\sup(\downarrow^\circ z))$, there exists $l \in \downarrow^\circ z$ such that $x \leq l$. Since $l \ll l$ and $x \leq l$, then from Proposition 2.4(1) and (2) there exists $l \in X$ such that $x \ll l \ll z$. □

Theorem 4.2. *If $(X, \leq)$ is algebraic t-set, then $(X, \ll)$ is continuous information system.*

Proof. Applying Theorem 4.1, it rests to prove that "$\ll$" is a transitive binary relation on X. Suppose $x, y, z \in X$ such that $x \ll y$ and $y \ll z$. Since $z \in \downarrow (\sup(\downarrow^\circ x))$, then there exists $l \in \downarrow^\circ z$ such that $y \leq l$ and $l \leq z$. Hence $y \leq z$. Since $x \ll y$ and $y \leq z$, then from Proposition 2.4(2), we have $x \ll z$. □

In Proposition 2.4(4), we need the property that for every $x \in X$, $\sup\{x\} \neq \emptyset$, to prove that "$\ll$" is transitive for a continuous t-set but as we illustrated in the proof of Theorem 4.1 we do not need this property to prove that "$\ll$" is transitive for an algebraic t-set.

Corollary 4.1. *If $(X, \leq)$ is an algebraic domain in the sense of Heckmann (resp. an algebraic poset in the sense of Nino-Salcedo), then $(X, \leq)$ is a continuous information system.*

Theorem 4.3. *If $(X, \leq)$ is algebraic t-set if and only if for every $x \in X$, there exist a directed subset D of $\downarrow^\circ x$ such that $x \in \downarrow (\sup(D))$.*

Proof. $\Rightarrow$: Suppose that $x \in X$. We need to prove that D is cofinal in $\downarrow^\circ x$. Now, $D \subseteq \downarrow^\circ x$. Assume $z \in \downarrow^\circ x$. So, from Proposition 2.4(2) $z \ll x$. Since $x \in \downarrow (\sup(D))$, there $d \in D$ such that $z \leq d$. Thus $z \in \downarrow D$. Hence, from Proposition 2.3, the result holds.

$\Leftarrow$: For all $x \in X$, take $d = \downarrow^\circ x$ and the result holds. □

Note that Theorem 4.3 is a generalization of Proposition 6.2.2 [3].

Theorem 4.4. *Let $(X, \leq)$ be algebraic t-set and $x \leq y$. Then for every $z \in \sup(\downarrow^\circ y)$, $x \leq z$.*

Proof. First, we prove that $\downarrow^\circ x \subseteq \downarrow^\circ y$. Suppose that $x \leq y$ and $z \in \downarrow^\circ x$. Then $z \ll z$ and $z \leq x$. Hence, $z \ll z$ and $z \leq y$. So, $z \in \downarrow^\circ y$. Therefore, $\downarrow^\circ x \subseteq \downarrow^\circ y$. Now, for every $l \in \sup(\downarrow^\circ x)$ and for every $z \in \sup(\downarrow^\circ y)$, we have $l \leq z$. So, $x \in l_\circ$ for some $l_\circ \in \sup(\downarrow^\circ x)$ because $x \in \downarrow (\sup(\downarrow^\circ x))$. Therefore, $x \leq z$ for every $z \in \sup(\downarrow^\circ y)$. □

Corollary 4.2. *If $(X, \leq)$ is an algebraic domain and "$\leq$" is antisymmetric, then $x \leq y$ if and only if $\downarrow^\circ x \subseteq \downarrow^\circ y$.*

Proof. $\Rightarrow$: Follows from Theorem 4.4 (1).
$\Leftarrow$: Since "$\leq$" is antisymmetric and $(X, \leq)$ is domain, then $\sup(\downarrow^\circ y)$ exists and unique, say equal z. Since $(X, \leq)$ is algebraic t-set, then $z = y$. From Theorem 4.4 (2), the result holds. □

Corollary 4.2 is a generalization of the corresponding result in Proposition 6.2.3 [3] since the reflexivity of "$\leq$" is not assumed.

Theorem 4.5. *If $(X, \leq)$ is algebraic t-set, then for every $x \in X$ there exists a directed subset D of $\Downarrow x$ such that $x \in \downarrow (\sup D)$.*

Proof. The result holds if we prove that $\downarrow^\circ x \subseteq \Downarrow x$ for every $x \in X$. Suppose that $z \in \downarrow^\circ x$. Then we have $z \ll z$ and $z \ll x$. Therefore, from Proposition 2.4 (2), $z \in \Downarrow x$. □

Corollary 4.3. *If $(X, \leq)$ is algebraic t-set and for every $x \in X$, $\sup\{x\} \neq \emptyset$, then $(X, \leq)$ is continuous t-set.*

Note that Theorem 4.5 is a generalization of Proposition 6.7.3 [4].

Theorem 4.6. *Let $(X, \leq)$ be a t-set. Then $(X, \leq)$ is algebraic if the following conditions are satisfied :*

(1) For every $x \in X$, $\Downarrow x$ is a directed subset;

(2) For every $x \in X$, $x \in \downarrow(\sup(\Downarrow x))$;

(3) For all $x, y \in X$ such that $x \ll y$, there exists $k \in K(X)$ such that $x \leq k \leq y$.

Proof. We need to prove that for all $x \in X$, $\downarrow^\circ x$ is cofinal in $\Downarrow x$. Suppose that $z \in \downarrow^\circ x$. Then $z \ll z$ and $z \leq x$. From Proposition 2.4 (2), we have $z \ll x$, i.e. $z \in \Downarrow x$. Hence $\downarrow^\circ x \subseteq \Downarrow x$. Let $l \in \Downarrow x$. So, $l \ll x$. Thus, there exists $k \in K(X)$ with $l \leq k \leq x$. From Proposition 2.4(2), one can deduce that $k \in \downarrow^\circ x$. Hence $l \in \downarrow(\downarrow^\circ x)$. Then from Proposition 2.3, $\downarrow^\circ x$ is directed and $\sup(\Downarrow x) = \sup(\downarrow^\circ x)$. □

Corollary 4.4. *If $(X, \leq)$ is continuous t-set and Theorem 4.6 (3) is satisfied, then X is algebraic.*

Proof. From Theorem 3.3 and Theorem 4.6, the results follow. □

Corollary 4.5. *Let $(X, \leq)$ be a domain. Then $(X, \leq)$ is algebraic if and only if $(X, \leq)$ is continuous and for every $x, y \in X$, with $x \ll y$, there exists $k \in K(X)$ such that $x \leq k \leq y$.*

Note that Corollary 4.4, 4.5 are generalizations of the corresponding result in Proposition 4.5.1 [3].

5. Conclusion

(1) Lemma 3.3 and Theorems 3.1-3.6 and 4.1-4.6 can be obtained if we replace the condition t-set by pre-ordered set (resp. abstract base, continuous in formation system).

(2) In a similar way, one can verify that the concept of algebraic t-set is identical with the concept of algebraic domain if the reflexivity, antisymmetric and chain-complete conditions are satisfied for t-set.

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