Asia Mathematika
Volume: 5 Issue: 3 , (2021) Pages: 22 – 29
Available online at **www.asiamath.org**

On Analytic Functions defined by Combination of Operators

A. A. YUSUF[1]* and M. DARUS[2]
[1]Department of Mathematics, federal University of Africulture, Abeokuta, Ogun State, Nigeria.
ORCID: 0000-0002-6322-0970
[2]Department of Mathematical Sciences Faculty of Science and Technology
Universiti Kebangsaan Malaysia. Bangi 43600 Selangor.
ORCID: 0000-0001-9138-916X

Received: 24 Jun 2021 • **Accepted:** 17 Sep 2021 • **Published Online:** 30 Dec 2021

Abstract: In this work, we introduce a new class of analytic functions defined by a combination of two operator. We obtain univalency condition of the new class, its integral representations, sufficient inclusion conditions and coefficient inequalities.

Key words: Analytic functions, starlike, bounded turning and univalent functions.

1. Introduction

Let A denote the class of analytic functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{1}$$

in the unit disk $\{\mathbb{U} = |z| < 1\}$. Let P denote the class of the functions

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \tag{2}$$

analytic in U, satisfying $Rep(z) > 0$ and by $P(\beta)$ if $p(z) > \beta$ for some real number $0 \leq \beta < 1$..
It is well-known that $f(z) \in A$ is a starlike function of order β, if

$$Re\frac{zf^{'}(z)}{f(z)} \in p(\beta).$$

denoted as $S^*(\beta)$ (see [18]) and $Ref^{'}(z) \in p(\beta)$ denoted as $R(\beta)$ referred to as the class of bounded turning of order β, (see [17]).
In [1], Abdulhalim generalized the class of bazilevic function consisting of functions satisfying the geometric condition

$$Re\frac{D^nf(z)^\alpha}{z^\alpha} > 0, z \in \mathbb{U}. \tag{3}$$

denoted as $B_n(\alpha)$, where $D^nf(z) = D(D^{n-1}f(z))$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and proved that the class contains only univalent functions in the unit disk.

©Asia Mathematika, DOI: 10.5281/zenodo.5809168
*Correspondence: yusufaa@funaab.edu.ng

A further generalization, $T_n^\alpha(\beta)$, was introduced by Opoola [11] which consists of functions satisfying the geometric condition

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta.$$

He proved the inclusion property and the univalency of functions in the class.
Using the salagean differential operator $D^n f(z)$ and and inverse of integral operator

$$\mathcal{L}_{\sigma,\gamma} f(z) = \frac{(\alpha+\gamma)^{-\sigma} t^{\gamma-1}}{z^\gamma \Gamma - \sigma} \int_0^z (log\frac{z}{t})^{-\sigma-1} f(t)^\lambda dt.$$

(see [8], [16]), on $f(z)^\alpha$, we have

$$D^n(\mathcal{L}_{\sigma,\gamma} f(z)^\alpha) = z^\alpha \alpha^n + \sum_{k=2}^{\infty} \left(\frac{\alpha+\gamma+k-1}{\alpha+\gamma}\right)^\sigma (\alpha+k-1)^n A_k(\alpha) z^{\alpha+k-1}. \tag{4}$$

where A_k for $k = 2, 3, \cdots$ depends on the coefficients a_k of $f(z)$ and the index α.
We denote

$$\mathcal{L}_{\sigma,\gamma}(D^n f(z)^\alpha) = D^n(\mathcal{L}_{\sigma,\gamma} f(z)^\alpha) = L^n_{\sigma,\gamma} f(z)^\alpha. \tag{5}$$

$n \in N \cup \{0\}, \sigma > 0, \gamma > -1, \ \alpha > 0$.

Remark 1.1. $L^n_{1,0} = D^{n+1} f(z)^\alpha$, $L^0_{1,0} = Df(z)^\alpha = zf'(z)^\alpha$. *If* $\alpha = 1$, *then* $L^0_{1,0} = zf'(z)$.

From the series expansions of the operator $\mathcal{L}_{\sigma,\gamma}$ on $f(z)^\alpha$, we have the recursive relation

$$z(\mathcal{L}_{\sigma,\gamma} f(z)^\alpha)' = (\alpha+\gamma)\mathcal{L}_{\sigma+1,\gamma} f(z)^\alpha - \gamma \mathcal{L}_{\sigma,\gamma} f(z)^\alpha. \tag{6}$$

Applying D^n on (6), we have

$$L^{n+1}_{\sigma,\gamma} f(z)^\alpha = (\alpha+\gamma) L^n_{\sigma+1,\gamma} f(z)^\alpha - \gamma L^n_{\sigma,\gamma} f(z)^\alpha. \tag{7}$$

Using the salagean anti-derivative define as $I_n = I(I_{n-1} f(z)) = \int_0^z \frac{I_{n-1} f(t)}{t} dt$ and

$\mathcal{J}_{\sigma,\gamma} f(z) = \frac{(\alpha+\gamma)^\sigma t^{\gamma-1}}{z^\gamma \Gamma \sigma} \int_0^z (log\frac{z}{t})^{\sigma-1} f(t) dt$, (see [8], [16]) on $f(z)^\alpha$.
Therefore

$$I_n(\mathcal{J}_{\sigma,\gamma} f(z)^\alpha) = \frac{z^\alpha}{\alpha^n} + \sum_{k=2}^{\infty} \left(\frac{\alpha+\gamma}{\alpha+\gamma+k-1}\right)^\sigma \frac{A_k(\alpha)}{(\alpha+k-1)^n} z^{\alpha+k-1}. \tag{8}$$

We denote

$$I_n(\mathcal{J}_{\sigma,\gamma} f(z)^\alpha) = \mathcal{J}_{\sigma,\gamma}(I_n f(z)^\alpha) = J^n_{\sigma,\gamma} f(z)^\alpha. \tag{9}$$

It can be seen that

$$L^n_{\sigma,\gamma}(J^n_{\sigma,\gamma} f(z)^\alpha) = J^n_{\sigma,\gamma}(L^n_{\sigma,\gamma} f(z)^\alpha) = f(z)^\alpha. \tag{10}$$

The concept of combining operators in theory of geometric function has been a very useful tool and this has been considered by many researchers to introduce subclasses of analytic and meromorphic functions, (see [2–7]).
Using the operator $L^n_{\sigma,\gamma}$, we introduce a new class defined as follows:

Definition 1. An analytic function $f \in A$ is said to belong to the class $B^{n,\alpha}_{\sigma,\gamma}(\beta)$ if and only if

$$\frac{L^n_{\sigma,\gamma} f(z)^\alpha}{\alpha^n z^\alpha} > \beta. \tag{11}$$

$n \in N \cup \{0\}, \sigma > 0, \gamma > -1,\ \alpha > 0.$

Remark 1.2. *If $\sigma = 1,\ \gamma = 0, n = 0$ and $\alpha = 1$ we have the class of analytic function satisfying*

$$Ref^{'}(z) > \beta \tag{12}$$

which is the class of functions of bounded turning of order β denoted as $R(\beta)$.

2. Prelimary Lemmas

Lemma 2.1. *[9] Let $p(z)$ be holomorphic in E with $p(0) = 1$. Suppose that*

$$Re\left(1 + \frac{zp^{'}(z)}{p(z)}\right) > \frac{3\beta - 1}{2\beta}.$$

Then

$$Rep(z) > 2^{1-\frac{1}{\beta}}, \frac{1}{2} \le \beta < 1, z \in U. \tag{13}$$

and the constant $2^{1-\frac{1}{\beta}}$ is the best possible.

Lemma 2.2. *[10] Let $u = u_1 + u_2 i,\ v = v_1 + v_2 i$ and $\Phi(u, v)$ a complex valued function satisfying*
(i) $\Phi(u, v)$ is continuous in a domain Ω of C^2.
(ii) $(1, 0) \in \Omega$ and $Re\Phi(1, 0) > 0$.
(iii) $Re\Phi(\beta + (1-\beta)u_2 i, v_1) \le \beta$ when $(\beta + (1-\beta)u_2 i, v_1) \in \Omega$ If $p \in P$ such that $(p(z), zp^{'}(z)) \in \Omega$ and $Re(p(z), zp^{'}(z)) > \beta$ for $z \in \mathbb{U}$. Then $Rep(z) > \beta$ in $\mathbb{U}$.

Lemma 2.3. *[12] Let $p \in P$. where $p(z) = 1 + c_1 z + p_2 z^2 + \cdots$, then*

$$|p_k| \le 2, k = 1, 2, 3, \cdots. \tag{14}$$

Lemma 2.4. *[11] Let $p \in P$. then for any real or complex number μ, we have sharp inequalities*

$$\left| p_2 - \mu \frac{p_1^2}{2} \right| \le 2 \max\{1, |1 - \mu|\}. \tag{15}$$

3. Main Results

Theorem 3.1. *Let* $f \in B^{n;\alpha}_{\sigma,\gamma}(\beta)$, *then* $f(z)$ *has the integral representation*

$$f(z) = J^n_{\sigma,\gamma}[\alpha^n z^n (p(z))]^{\frac{1}{\alpha}}$$

Proof. Since $f \in B^{n,\alpha}_{\sigma,\gamma}(\beta)$, then there exists $p \in P(\beta)$ such that

$$\frac{L^n_{\sigma,\gamma} f(z)^\alpha}{\alpha^n z^\alpha} = p(z)$$

and

$$L^n_{\sigma,\gamma} f(z)^\alpha = \alpha^n z^\alpha p(z)$$

applying the antiderivatie operator $J^n_{\sigma,\gamma}$, we obtain

$$f(z) = J^n_{\sigma,\gamma}[\alpha^n z^n (p(z))]^{\frac{1}{\alpha}}$$

□

Theorem 3.2. $B^{n;\alpha}_{\sigma,\gamma}(\beta) \subset T^\alpha_n(\beta)$, *for* $\alpha > 0$.

Proof. Let

$$p(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}$$

Then , from (11), we obtain

$$Re\left(p(z)^2 + \frac{zp(z)p^{'}(z)}{\alpha}\right) > \beta.$$

We define

$$\Phi(u, v) = u^2 + \frac{uv}{\alpha}, \alpha > 0.$$

Clearly, $\Phi(u, v)$ satisfies the condition of Lemma 2.2. whenever $2v_1 < -(1-\beta)(1+u_2^2)$, we have

$$Re\Phi(\beta + (1-\beta)u_2 i, v_1) = \beta^2 - (1-\beta)^2 u_2^2 - \frac{\beta(1-\beta)(1+u_2^2)}{2\alpha} < \beta < \beta.$$

Hence by lemma 2.2, we have $Rep(z)$, implies that $Re\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta$ and the proof completes. □

Corollary 3.1. *For* $n \geq 1$, *the class* $B^{n,\alpha}_{\sigma,\gamma}(\beta)$ *consists of univalent functions.*

Theorem 3.3. *If* $f \in A$ *satisfies*

$$Re\left(\frac{L^{n+1}_{\sigma,\gamma} f(z)^\alpha}{L^n_{\sigma,\gamma} f(z)^\alpha}\right) > \frac{2\alpha\beta + \beta - 1}{2\beta}. \tag{16}$$

Then

$$Re\frac{L^n_{\sigma,\gamma} f(z)^\alpha}{\alpha^n z^n} > 2^{1-\frac{1}{\beta}}, \frac{1}{2} \leq \beta < 1, z \in \mathbb{U}.$$

Proof. Let

$$\frac{L_{\sigma,\gamma}^{n+1} f(z)^{\alpha}}{\alpha^n z^n} = p(z),$$

then we have that

$$\frac{zp^{'}(z)}{p(z)} = \frac{L_{\sigma,\gamma}^{n+1} f(z)^{\alpha}}{L_{\sigma,\gamma}^{n} f(z)^{\alpha}} - \alpha.$$

By the condition of the theorem,

$$Re\left(1 + \frac{zp^{'}(z)}{p(z)}\right) = Re\left(\frac{L_{\sigma,\gamma}^{n+1} f(z)^{\alpha}}{L_{\sigma,\gamma}^{n} f(z)^{\alpha}} - \alpha + 1\right) > \frac{3\beta - 1}{2\beta}$$

and this is equivalent to

$$Re\left(\frac{L_{\sigma,\gamma}^{n+1} f(z)^{\alpha}}{L_{\sigma,\gamma}^{n} f(z)^{\alpha}}\right) > \frac{2\alpha\beta + \beta - 1}{2\beta}.$$

Thus by lemma 2.1, $Rep(z) > 2^{1-\frac{1}{\beta}}$, $\frac{1}{\beta} \leq \beta < 1, z \in \mathbb{U}$. □

Corollary 3.2. *If $f \in A$ satisfies the condition, then $f \in B_{\sigma,\gamma}^{n,\alpha}(2^{1-\frac{1}{\beta}})$.*

If $n = 0$, $\alpha = 0$, we have

Corollary 3.3. *Suppose*

$$Re\left(\frac{zf^{''}(z)}{f^{'}(z)} + 1\right) > \frac{\beta - 1}{2\beta}.$$

Then

$$Ref^{'}(z) > 2^{1-\frac{1}{\beta}}.$$

If $n = 0$, $\alpha = 1/2$, we have

Corollary 3.4. *Suppose*

$$Re\left(\frac{zf^{''}(z)}{f^{'}(z)} + 1\right) > \frac{3\beta - 1}{2\beta}.$$

Then

$$Ref^{'}(z) > 2^{1-\frac{1}{\beta}}.$$

If $n = 0$, $\alpha = 1$ and $\beta = 1/2$, we have

Corollary 3.5. *Suppose*

$$Re\left(\frac{zf^{''}(z)}{f^{'}(z)} + 1\right) > \frac{1}{2}$$

. *Then*

$$Ref^{'}(z) > \frac{1}{2}.$$

Theorem 3.4. *Let* $f \in B^{n;\alpha}_{\sigma,\gamma}(\beta)$, *then*

$$|a_2| \leq \tfrac{2\alpha^{n-1}(1-\beta)}{(\alpha+1)^n}\left(\tfrac{\alpha+\gamma}{\alpha+\gamma+1}\right)^{\sigma}.$$

$$|a_3| \leq \frac{\alpha^{n-1}(\alpha+\gamma)^{\sigma}(1-\beta)}{(\alpha+2)^n(\alpha+\gamma+2)^{\sigma}} max\{1, |\mathbf{M_1}|\} \tag{17}$$

where $\mathbf{M_1} = \dfrac{2(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma} + (1-\alpha)\alpha^n(\alpha+\gamma)^{\sigma}(\alpha+2)^n(\alpha+\gamma+2)^{\sigma}}{(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma}}$

The bounds are best possible. Equalities are obtained also by

$$f(z)^{\alpha} = \left\{J^n_{\sigma,\gamma}\left[\alpha^n z^{\alpha}\left(\beta + (1-\beta)\frac{1+z}{1-z}\right)\right]\right\}^{\dfrac{1}{\alpha}}$$

$$= z + \frac{\alpha^n}{(\alpha+1)^n}\left(\frac{\alpha+\gamma}{\alpha+\gamma+1}\right)^{\sigma} z^2 +$$

$$\frac{\alpha^n(\alpha+\gamma)^{\sigma}}{(\alpha+2)^n(\alpha+\gamma+2)^{\sigma}}\left\{\frac{(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma} + (1-\alpha)\alpha^n(\alpha+\gamma)^{\sigma}(\alpha+2)^n(\alpha+\gamma+2)^{\sigma}}{(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma}}\right\} z^3 + \cdots$$

Proof. Let $f \in B^{n;\alpha}_{\sigma,\lambda}(\lambda))$, then there exists $p \in P_{\beta}$ such that

$$\frac{L^n_{\sigma,\gamma} f(z)^{\alpha}}{\alpha^n z^{\alpha}} = p(z) = 1 + (1-\beta)c_1 z + (1-\beta)c_2 z^2 + (1-\beta)c_3 c^3 + \cdots \tag{18}$$

$$L^n_{\sigma,\gamma} f(z)^{\alpha} = \alpha^n z^{\alpha} + \alpha^n(1-\beta)c_1 z^{\alpha+1} + \alpha^n(1-\beta)c_2 z^{\alpha+2} + \alpha^n(1-\beta)c_3 z^{\alpha+3} + \alpha^{n-1}(1-\beta)c_4 z^{\alpha+4} + \cdots$$

Using the anti-derivative of the operator $L^n_{\sigma,\gamma}$ denoted as $J^n_{\sigma,\gamma}$, we have that

$$\begin{aligned} f(z)^{\alpha} \quad &= z^{\alpha} + \tfrac{\alpha^n(1-\beta)}{(\alpha+1)^n}\left(\tfrac{\alpha+\gamma}{\alpha+\gamma+1}\right)^{\sigma} c_1 z^{\alpha+1} + \tfrac{\alpha^n(1-\beta)}{(\alpha+2)^n}\left(\tfrac{\alpha+\gamma}{\alpha+\gamma+2}\right)^{\sigma} c_2 z^{\alpha+2} \\ &+ \tfrac{\alpha^n(1-\beta)}{(\alpha+3)^n}\left(\tfrac{\alpha+\gamma}{\alpha+\gamma+3}\right)^{\sigma} c_3 z^{\alpha+3} + \tfrac{\alpha^n(1-\beta)}{(\alpha+4)^n}\left(\tfrac{\alpha+\gamma}{\alpha+\gamma+4}\right)^{\sigma} c_4 z^{\alpha+4} \cdots \end{aligned}$$

Given that

$$\begin{aligned} f(z)^{\alpha} \quad &= z^{\alpha} + \alpha a_2 z^{\alpha+1} + \left(\alpha a_3 + \frac{\alpha(\alpha-1)}{2}a_2^2\right) z^{\alpha+2} + \left(\alpha a_4 + \alpha(\alpha-1)a_2 a_3 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{6}a_2^3\right) z^{\alpha+3} \\ &+ \left(\alpha a_5 + \alpha(\alpha-1)a_2 a_4 + \tfrac{\alpha(\alpha-1)}{2}a_3^2 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{2}a_2^2 a_3 + \tfrac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{12}a_2^4\right) z^{\alpha+4} + \cdots \end{aligned}$$

By comparing the coefficient , we have

$$a_2 = \frac{\alpha^{n-1}}{(\alpha+1)^n}\left(\frac{\alpha+\gamma}{\alpha+\gamma+1}\right)^{\sigma} c_1$$

By Lemma 2.3, we obtained the bound of a_2, also

$$a_3 = \frac{\alpha^n(\alpha+\gamma)^\sigma(1-\beta)}{(\alpha+2)^n(\alpha+\gamma+2)^\sigma}\left[c_2 - \frac{\alpha^n(\alpha-1)(\alpha+\gamma)^\sigma(\alpha+2)^n(\alpha+\gamma+2)^\sigma}{(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma}}\frac{c_1^2}{2}\right]$$

By Lemma 2.4 and with $\rho = \dfrac{\alpha^n(\alpha-1)(\alpha+\gamma)^\sigma(\alpha+2)^n(\alpha+\gamma+2)^\sigma}{(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma}}$, we obtained the bound on the third coefficient of these function. By letting

$$p(z) = \beta + (1-\beta)\frac{1+z}{1-z}$$

from the integral representation we have the equality attained by the extremal function given. □

Theorem 3.5. *Let* $f \in B^{n;\alpha}_{\sigma,\gamma}(\beta)$. *Then*

$$|a_3 - \rho a_2^2| \leq \frac{\alpha^{n-1}(1-\beta)(\alpha+\gamma)^\sigma}{(\alpha+2)^n(\alpha+\gamma+2)^\sigma} max\{1, |\mathbf{M_2}|\} \tag{19}$$

where $\mathbf{M_2} = \dfrac{2(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma} + (1+2\rho-\alpha)\alpha^{n-1}(\alpha+\gamma)^\sigma(\alpha+2)^n(\alpha+\gamma+2)^\sigma}{(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma}}$

Proof. From the computation and by comparing coefficient with respect to z, then

$$a_2 = \frac{\alpha^{n-1}(1-\beta)}{(\alpha+1)^n}\left(\frac{\alpha+\gamma}{\alpha+\gamma+1}\right)^\sigma c_1 \tag{20}$$

and

$$a_3 = \frac{\alpha^n(\alpha+\gamma)^\sigma c_2}{(\alpha+2)^n(\alpha+\gamma+2)^\sigma} + \frac{(1-\alpha)\alpha^{2(n-2)}(\alpha+\gamma)^{2\sigma}}{(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma}}\frac{c_1^2}{2} \tag{21}$$

Hence

$$|a_3 - \rho a_2^2| = \frac{\alpha^{n-1}(\alpha+\gamma)^\sigma}{(\alpha+2)^n(\alpha+\gamma+2)^\sigma}c_2 - \frac{(\alpha-1+2\rho)(\alpha+2)^n\alpha^{n-1}(\alpha+\gamma)^\sigma(\alpha+\gamma+2)^\sigma}{(\alpha+1)^{2n}(\alpha+\gamma+1)^{2\sigma}}\frac{c_1^2}{2} \tag{22}$$

by lemma 2.4 we have the required inequality. □

4. Conclusion

In this work we have been able to determine the univalency condition of the new class, its integral representations, sufficient inclusion conditions and coefficient inequalities of a subclass of analytics functions defined by combination of two operators.

Acknowledgment

The author would like to thank the anonymous referee whose comments improved the original version of this manuscript.

References

[1] Abdulhalim S. On a class of analytic functions involving the Salagean differential operator. Tankang Journal of mathematics 1992; 23(1): 51-58.

[2] Al-Kaseasbeh, M. and M. Darus, Concave meromorphic functions involving constructed operators. Acta Universitatis Apulensis, 2017; (52),11-19.

[3] Al-Kaseasbeh, M. and Darus, M. Inclusion and convolution properties of a certain class of analytic functions, Eurasian Mathematical Journal. 8(4), 11-17.

[4] Al-Kaseasbeh, M and Darus, M. Meromorphic functions involved constructed differential operator, Janabha-Vijana Parishad of India. 2017; 47(1),63-76.

[5] Al-Kaseasbeh M., Darus M. and Al-Kaseasbeh S., Certain differential sandwich theorem involved constructed differential operator. International Information Institute 2016; 19(10), 4663-4670.

[6] Al-Kaseasbeh M. and M. Darus, On an operator defined by the combination of both generalized operators of Salagean and Ruscheweyh. Far East Journal of Mathematical Sciences 2015; 97(4)443-455.

[7] M. Al-Kaseasbeh and M. Darus, On subclass of harmonic univalent functions defined by generalized Salagean operator and Ruscheweyh operator. Transylvanian Journal of Mathematics and Mechanics 2015; 7(2), 95-100.

[8] Babalola, K.O. Subclasses of Analytic Functions defined by the Inverse of Certain Integral Operator, Analele Universitatis Oradea Fasc. Matematica, Tom XIX, Issue No.2012; 1: 255-264.

[9] Babalola, K. O. and T. O. Opoolo, Iterated integral transforms of Caratheodory functions and their applications to analytic and Univalent functions, Tamkang Journal of Mathematics, 2006; 37(4), 355-366.

[10] Babalola, K. O. On λ pseudo-starlike functions, Journal of Classical Analysis, 2013; 3(2), 137-147.

[11] Babalola, K. O. and T. O. Opoola, On the coefficients of a certain class of analytic functions, Advances in Inequalities for Series, (2008), 1-13.

[12] Caratheodory, C. Theory of functions of a complex variable, II. (1960). Chelsea Publishing Co. New York.

[13] Lin, L. J. and S. Owa. Properties of the Salagean Operator. Georgian Mathematics Journal 1998; 5(4), 361-366.

[14] Opoola, T. O. On a new subclass of Univalent functions. Mathematica (cluj) 1994; 36: 59 (2), 195-200.

[15] Ravichandran, V; Selvaraj, C and Rajalaksmi, R. Sufficient conditions for Starlike functions of order α. Journal of Inequalities, Pure and Applied Mathematics. 2012; 3(5): 1 - 6.

[16] Salagean, G. S. Subclasses of Univalent functions. Lecture notes in maths, 1013, 362-372. Springer-verlag, Berlin, New York.

[17] Singh, R. On Bazilevic functions. Proceedings of the American Mathematical Society. 1973; 38(2): 261-271.

[18] Yamaguchi, K. On functions satisfying Re $\frac{f(z)}{z} > 0$. Proceedings of American Mathematical Society 1966; 17: 588-591.