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On somewhat near continuity and some applications

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Abstract: We start with studying the concept of somewhat nearly open sets and use it to give more properties and applications to somewhat near continuity of functions proposed by Z. Piotrowski. Among the applications, we show that sets with Baire property are preserved under the image of somewhat nearly continuous quasiopen injections, and Baire spaces are preserved under the preimage of somewhat nearly continuous quasiopen countable fiber complete functions.

Key words: Quasicontinuous, almost quasicontinuous, semicontinuous, precontinuous, β-continuous, somewhat continuous, somewhat nearly continuous, Baire space

1. Introduction

The concept of continuity is one of the most significant tools in all of mathematics. Many classes of near-continuity have been introduced since the dawn of modern mathematics. Quasicontinuity, near continuity, somewhat continuity, α-continuity, and almost quasicontinuity are the most well-known classes. In 1932, Kempisty [1] introduced the notion of quasicontinuity for extending some classical results of Hahn and Baire concerning separately continuous real-valued functions of many variables. In the same year, Banach considered near continuity while proving Closed Graph Theorem [2, Theorem 4, p40] under the name of almost continuity. somewhat continuous functions [3] are given by Frolik while investigating the Baire spaces preservation under mappings, see also [4]. In 1965, a stronger notion to both quasicontinuity and near continuity was introduced by Njastad called α-continuity [5]. It is known that a function is α-continuous iff it is both nearly continuous and quasicontinuous. Later, Borsik [6] defined the class of almost quasicontinuous functions, this class is implied by near continuity and quasicontinuity. As a direct generalization of somewhat continuity and almost quasicontinuity, Piotrowski [7] introduced a new class, named somewhat nearly continuous functions, while working on separate versus joint continuity problems as well as on the Closed Graph Theorem. Due to the importance of this class, we continue the work of Piotrowski and give further properties and characterizations.

2. Preliminaries

The letters $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ represent the set of natural, rational, and real numbers, respectively, throughout this work. An arbitrary topological space is referred to as "space." The closure and interior of A with regard to X are indicated by $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$ (or simply $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, for a subset A of a space (X,τ). a subset A of a space X is said to be co-dense if $\mathrm{Int}(A)=\emptyset$, preopen [8] if $A\subseteq \mathrm{Int}(\mathrm{Cl}(A))$, semiopen [9] if $A\subseteq \mathrm{Cl}(\mathrm{Int}(A))$, α-open [5] if $A\subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, β-open [10] or semipreopen [11] if $A\subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,

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somewhat open (briefly sw-open) [7] if $\operatorname{Int}(A) \neq \emptyset$ or $A = \emptyset$. The complement of each preopen (resp. semiopen, α-open, β-open, sw-open) set is preclosed (resp. semi-closed, α-closed, β-closed, sw-closed).

The intersection of all β-closed sets in X containing A is β-closure of A which is denoted by $\operatorname{Cl}_\beta(A)$. The union of all semiopen sets in X contained in A is called semi-interior of A and is denoted by $\operatorname{Int}_s(A)$. The collection of all preopen (resp. semiopen, α-open, β-open) subsets of X is denoted by $PO(X)$ (resp. $SO(X)$, $\alpha O(X)$, $\beta O(X)$).

Remark 2.1. *It is well-known that for a space X, $\tau \subseteq \alpha O(X) \subseteq PO(X) \cup SO(X) \subseteq \beta O(X)$.*

Definition 2.1. A point $x \in X$ is said to be in the β-closure of a subset A of a space X if $U \cap A \neq \phi$ for each β-open set U containing x.

Lemma 2.1. *[12, Proposition 1.1] For a subset A of a space X, $\operatorname{Cl}_\beta(A) = A \cup \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(A)))$.*

Lemma 2.2. *[11, Theorem 3.22] For a subset A of a space X, $\operatorname{Cl}_\beta(\operatorname{Int}(A)) = \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(A)))$.*

Lemma 2.3. *Let A be a subset of a space X.*

(i) A is semiopen iff $\operatorname{Cl}(A) = \operatorname{Cl}(\operatorname{Int}(A))$.

(ii) A is β-open iff $\operatorname{Cl}(A) = \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(A)))$.

Proof. (i) If A is semiopen, then $A \subseteq \operatorname{Cl}(\operatorname{Int}(A))$ and so $\operatorname{Cl}(A) \subseteq \operatorname{Cl}(\operatorname{Int}(A))$. For other side of inclusion, we always have $\operatorname{Int}(A) \subseteq A$. Therefore $\operatorname{Cl}(\operatorname{Int}(A)) \subseteq \operatorname{Cl}(A)$. Thus $\operatorname{Cl}(A) = \operatorname{Cl}(\operatorname{Int}(A))$.

Conversely, assume that $\operatorname{Cl}(A) = \operatorname{Cl}(\operatorname{Int}(A))$, but $A \subseteq \operatorname{Cl}(A)$ always, so $A \subseteq \operatorname{Cl}(\operatorname{Int}(A))$. Hence A is semiopen.

(ii) Theorem 2.4 in [11]. □

Lemma 2.4. *Let A be a nonempty subset of a space X.*

(i) If A is semiopen, then $\operatorname{Int}(A) \neq \emptyset$.

(ii) If A is β-open, then $\operatorname{Int}(\operatorname{Cl}(A)) \neq \emptyset$.

Proof. (i) Suppose otherwise that if A is a semiopen set such that $\operatorname{Int}(A) = \emptyset$, by Lemma 2.3 (i), $\operatorname{Cl}(A) = \emptyset$ which implies that $A = \emptyset$. Contradiction!

(ii) Similar to (i). □

Lemma 2.5. *Let A be a subset of a space X. Then $\operatorname{Int}(A) \neq \emptyset$ iff $\operatorname{Int}_s(A) \neq \emptyset$.*

Proof. Given a subset A of X. If $\operatorname{Int}(A) \neq \emptyset$, then $\emptyset \neq \operatorname{Int}(A) \subseteq \operatorname{Int}_s(A)$ and so $\operatorname{Int}_s(A) \neq \emptyset$.

Conversely, if $\operatorname{Int}_s(A) \neq \emptyset$, then for some $x \in A$, there exists a semiopen set G containing x such that $G \subseteq A$. By Lemma 2.4 (i), $\emptyset \neq \operatorname{Int}(G) \subseteq \operatorname{Int}(A)$. Therefore $\operatorname{Int}(A) \neq \emptyset$. □

Lemma 2.6. *[13, Theorem 2.4] Let Y be a subspace of a space X and let $A \subseteq Y$. If Y is semiopen in X, then A is semiopen in Y iff A is semiopen in X.*

Lemma 2.7. *Let A, B be subsets of X. If A is α-open and B is preopen (resp. β-open), then $A \cap B$ is preopen (resp. β-open) in X.*

Proof. Lemma 2.1 in [14] (resp. Theorem 2.7 in [10]). □

Lemma 2.8. *Let A, B be subsets of X. If A is α-open and B is preopen (resp. β-open), then $A \cap B$ is preopen (resp. β-open) in A.*

Proof. Lemma 2.1 in [15] (resp. Lemma 2.5 in [10]). □

Lemma 2.9. *Let A, D be subsets of space X.*

(i) If A is open and D is dense, then $\mathrm{Cl}(A \cap D) = \mathrm{Cl}(A)$

(ii) If D is open dense, then $\mathrm{Cl}(A) \cap D = Cl_D(A \cap D)$.

Proof. Standard.

□

Lemma 2.10. *Let A be a subset of a space X. If A is semiopen, then it has the Baire property.*

Proof. Given a semiopen set A. By [9, Theorem 7], $A = O \cup N$, where O is open and N is nowhere dense such that $O \cap N = \emptyset$. Therefore $A = O \Delta N$. Hence A has the Baire property.

□

3. Somewhat Nearly Open Sets

Definition 3.1. [7] a subset A of a space X is said to be somewhat nearly open (briefly swn-open) if $\mathrm{Int}(\mathrm{Cl}(A)) \neq \emptyset$ or $A = \emptyset$.

The complement of each swn-open set is called swn-closed. That is, a set F is swn-closed if $\mathrm{Cl}(\mathrm{Int}(F)) \neq X$ or $F = X$.

Remark 3.1. *Let $A, B \subset X$ such that $A \neq \emptyset, B \neq X$. Then*

(a) A is swn-open iff $\mathrm{Int}_s(\mathrm{Cl}(A)) \neq \emptyset$, see Lemma 2.5.

(b) A is swn-open iff there is an open (or a semiopen) set U such that $\emptyset \neq U \subseteq \mathrm{Cl}(A)$.

(c) B is swn-closed iff there is a closed (or a semiclosed) set F such that $\mathrm{Int}(B) \subseteq F \subsetneq X$.

Proposition 3.1. *Any union of swn-open sets is swn-open.*

Proof. Let $\{A_\alpha : \alpha \in \Delta\}$ be any collection of swn-open subsets of a space X. Now

$$\begin{aligned} \mathrm{Int}(\mathrm{Cl}(\bigcup_{\alpha\in\Delta} A_\alpha)) &\supseteq \mathrm{Int}(\bigcup_{\alpha\in\Delta} \mathrm{Cl}(A_\alpha)) \\ &\supseteq \bigcup_{\alpha\in\Delta} \mathrm{Int}(\mathrm{Cl}(A_\alpha)) \neq \emptyset. \end{aligned}$$

Thus $\bigcup_{\alpha\in\Delta} A_\alpha$ is swn-open. □

Remark 3.2. *The intersection of two swn-open sets need not be swn-open. For instance, take swn-open sets $A = [0, 1]$ and $B = [1, 2]$ in $\mathbb{R}$. Then $\mathrm{Int}(\mathrm{Cl}[A \cap B]) = \emptyset$.*

We remark that, in general, the family of swn-open subsets of a space X does not from a topology.

Next we put Remark 2.1 and Lemma 2.4 into the following diagram which shows the relation between swn-open and the most well-known types of open sets.

$$\begin{array}{ccc}
\alpha\text{-open} & \longrightarrow & \text{preopen} \\
\downarrow & & \downarrow \\
\text{semiopen} & \longrightarrow & \beta\text{-open} \\
\downarrow & & \downarrow \\
sw\text{-open} & \longrightarrow & swn\text{-open}
\end{array}$$

Diagram I:Relationship between generalized types of open sets

In general, none of these implications can be replaced by equivalence as shown below:

Example 3.1. *Consider $\mathbb{R}$ with the natural topology. Let $A = [0,1] \cap \mathbb{Q}$. Then A is swn-open but not sw-open. If $B = \mathcal{C} \cup [2,3]$, where $\mathcal{C}$ is the ternary Cantor set, then B is swn-open but not β-open. Examples for other none implications can be found in [16] or in the literature.*

Remark 3.3. *From the above example, one can conclude that the intersection of an swn-open set with an open, a closed or a dense set may not be swn-open.*

Proposition 3.2. *Let A, D be subsets of a space X. If A is open and D is dense, then $A \cap D$ is swn-open in X.*

Proof. If $A = \emptyset$, then $A \cap D = \emptyset$ which is swn-open. Let A be nonempty and let D be dense. By Lemma 2.9 (i), $\text{Int}(\text{Cl}(A \cap D)) = \text{Int}(\text{Cl}(A)) \neq \emptyset$. That is $A \cap D$ is swn-open. □

Proposition 3.3. *Let A, D be subsets of a space X. If A is swn-open and D is open dense, then $A \cap D$ is swn-open in D.*

Proof. Let A be swn-open and let D be open dense. By using Lemma 2.9 (ii), one can get $\text{Int}_D(\text{Cl}_D(A \cap D)) = \text{Int}(\text{Cl}_D(A \cap D)) \cap D = \text{Int}[\text{Cl}(A) \cap D] \cap D = \text{Int}(\text{Cl}(A)) \cap D$. Since $\text{Int}(\text{Cl}(A))$ is nonempty open and D is dense, they must have nonempty intersection. Therefore $\text{Int}_D(\text{Cl}_D(A \cap D)) \neq \emptyset$ and so $A \cap D$ is swn-open in D. □

Proposition 3.4. *Let A be a subset of a space X. Then either A is sw-open or swn-closed.*

Proof. Given $A \subseteq X$. If A is not sw-open, then $\text{Int}(A) = \emptyset$. Therefore $\text{Cl}(\text{Int}(A)) = \emptyset \neq X$ and so A is swn-closed.

On the other hand, if A is not swn-closed, then $\text{Cl}(\text{Int}(A)) = X$. Surely $\text{Int}(A) \neq \emptyset$. Thus A is sw-open. □

Corollary 3.1. *Each subset of a space X is either swn-open or swn-closed.*

Here, we remark the following:

(i) Each nowhere dense is swn-closed. The converse is false.

(ii) A set is swn-open if and only of its closure is swn-open.

(iii) The interior of an swn-open set need not be swn-open.

Proposition 3.5. *Let A be a semiclosed subset of a space X. Then A is swn-open iff A is sw-open.*

Proof. By Lemma 2.3 (i), A is semiclosed iff $\mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(A)$. The rest is clear. □

Proposition 3.6. *Let Y be a semiopen subspace of a space X and let $A \subseteq Y$. Then A is swn-open in Y iff A is swn-open in X.*

Proof. Let A be swn-open in Y. There exist a semiopen subset H in Y such that $H \subseteq \mathrm{Cl}_Y(A)$. Now, $H = H \cap Y \subseteq \mathrm{Cl}(A) \cap Y \subseteq \mathrm{Cl}(A)$. Since Y is semiopen, by Lemma 2.6, H is semiopen in X and $H \subseteq \mathrm{Cl}(A)$. Hence A is swn-open in X.

Conversely, assume that A is swn-open in X. Suppose otherwise that A is not swn-open in Y. Let G be a semiopen set in X with $G \cap Y \neq \emptyset$. Then there is a nonempty semiopen set $H \subseteq G \cap Y$ such that $A \cap H = \emptyset$. Since H is semiopen in Y, there is a semiopen set U in X such that $H = U \cap Y$. Now, we have $U \subseteq G$ and $A \cap U = \emptyset$. This means that A is not swn-open in X, which is impossible. Hence the result. □

Proposition 3.7. *Let Y be a dense subspace of a space X and let $A \subseteq Y$. Then A is swn-open in Y iff A is swn-open in X.*

Proof. Similar to Proposition 3.6 □

Proposition 3.8. *Let X be a space. a subset A of X is β-open iff $A \cap U$ is swn-open for each open set U in X.*

Proof. Given a β-open set A and an arbitrary open set U. By Lemma 2.7, $A \cap U$ is β-open and consequently it is swn-open by Lemma 2.4 (ii).

Conversely, let $x \in A$ and assume that $A \cap U$ is swn-open for each open set U in X. That is $\mathrm{Int}(\mathrm{Cl}(A \cap U)) \neq \emptyset$. Now we have $\emptyset \neq \mathrm{Int}(\mathrm{Cl}(A \cap U)) \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \cap Int(\mathrm{Cl}(U)) = \mathrm{Int}(\mathrm{Cl}(A)) \cap U$, which implies that $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and so $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. This proves that A is β-open. □

Proposition 3.9. *Let X be a space. a subset A of X is preopen iff $A \cap U$ is swn-open for each α-open set U in X.*

Proof. Let A be preopen and let U be any α-open. By Lemma 2.8, $A \cap U$ is preopen and so it is swn-open, (see Diagram I).

Conversely, let $x \in A$. Suppose that $A \cap U$ is swn-open for each α-open U in X. Then $\emptyset \neq \mathrm{Int}(\mathrm{Cl}(A \cap U) \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(U)) = \mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$. Since U is α-open, by Lemma 2.1, $\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Cl}_\beta(U) \neq \emptyset$ for each α-open U. This means that $\mathrm{Int}(\mathrm{Cl}(A)) \cap U \cap V \neq \emptyset$ for each β-open set V in X containing x. Therefore $x \in \mathrm{Cl}_\beta(\mathrm{Int}(\mathrm{Cl}(A)) \cap U) \subseteq \mathrm{Cl}_\beta(\mathrm{Int}(\mathrm{Cl}(A)))$ and so, by Lemma 2.2, $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) = \mathrm{Int}(\mathrm{Cl}(A))$. Hence A is preopen. □

Definition 3.2. [17] a space X is said to be

(1) irresolvable if any two dense subsets intersect.

(2) strongly irresolvable if each open subspace is irresolvable.

Theorem 3.1. *The following are equivalent for a space X:*

(1) *strongly irresolvable,*

(2) *each open subspace is irresolvable,*

(3) *each preopen subset of X is α-open,*

(4) *each β-open subset of X is semiopen,*

(5) *each preopen subset of X is semiopen,*

(6) *each dense subset of X is semiopen,*

(7) *each dense subset of X has an interior dense,*

(8) *each co-dense subset of X is nowhere dense,*

(9) *each swn-open subset of X is sw-open,*

(10) *each subset of X has a nowhere dense boundary.*

(11) *each subset is the union of an open set and a nowhere dense set.*

Proof. [18, Theorem 17]. □

Lemma 3.1. *The following are equivalent for a space X:*

(1) *X is strongly irresolvable,*

(2) $\text{Int}(\text{Cl}(A \cap B)) = \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(B))$ *for each subsets A, B of X.*

Proof. (1) $\implies$ (2) The first direction is clear. That is $\text{Int}(\text{Cl}(A \cap B)) \subseteq \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(B))$.

On the other hand, given any two sets A, B in X. By Theorem 3.1 (1), $\text{Int}(\text{Cl}(A)) = \text{Int}(\text{Cl}(\text{Int}(A)))$. Now, we have

$$\begin{aligned} \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(B)) &= \text{Int}(\text{Cl}(\text{Int}(A))) \cap \text{Int}(\text{Cl}(B)) \\ &\subseteq \text{Cl}(\text{Int}(A)) \cap \text{Int}(\text{Cl}(B)) \\ &\subseteq \text{Cl}[\text{Int}(A) \cap \text{Int}(\text{Cl}(B))] \\ &\subseteq \text{Cl}[A \cap B]. \end{aligned}$$

Then taking the interior of both sides, we get $\text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(B)) \subseteq \text{Int}(\text{Cl}(A \cap B))$. Thus $\text{Int}(\text{Cl}(A \cap B)) = \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(B))$.

(2) $\implies$ (1) Assume that (2) is true. Let A be a subset of X. Now

$$\begin{aligned} \text{Int}(\partial(A)) &= \text{Int}[\text{Cl}(A) \cap \text{Cl}(X \setminus A)] \\ &= \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(X \setminus A)) \\ &= \text{Int}[\text{Cl}[A \cap (X \setminus A)]] \\ &= \emptyset. \end{aligned}$$

By Theorem 3.1 (10), X is strongly irresolvable. □

Theorem 3.2. *Let X be a space. If X is strongly irresolvable, the family $SN(X)$ of all swn-open subsets of X forms a topology.*

Proof. By Proposition 3.1, $SN(X)$ is closed under arbitrary unions and it contains the empty set by definition. Therefore, it is enough to prove that $SN(X)$ is closed under finite intersections. Let $A, B \in SN(X)$. By Lemma 3.1, $\text{Int}(\text{Cl}(A \cap B)) = \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(B))$. By the choice of A, B, $\text{Int}(\text{Cl}(A))$ and $\text{Int}(\text{Cl}(B))$ are nonempty open sets in X. Since X is strongly irresolvable, $\text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(B)) \neq \emptyset$. Thus $\text{Int}(\text{Cl}(A \cap B)) \neq \emptyset$. Hence the proof.

□

4. Somewhat Near Continuity

Definition 4.1. [7] Let X, Y be spaces. A function $f : X \to Y$ is said to be somewhat nearly continuous (briefly swn-continuous) if the inverse image of each open set in Y is swn-open in X.

The above definition can be stated as:

Remark 4.1. *A function $f : X \to Y$ is swn-continuous if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an swn-open set U in X containing x such that $f(U) \subset V$.*

Definition 4.2. For a subset A of a space X, we introduce the following:

(i) $\text{Cl}_{swn}(A) = \bigcap\{F : F \text{ is } swn\text{-closed in } X \text{ and } A \subseteq F\}$.

(ii) $\text{Int}_{swn}(A) = \bigcup\{O : O \text{ is } swn\text{-open in } X \text{ and } O \subseteq A\}$.

Proposition 4.1. *Let X, Y be spaces. For a function $f : X \to Y$, the following are equivalent:*

(1) f is swn-continuous,

(2) $f^{-1}(F)$ is swn-closed set in X, for each closed set F in Y,

(3) $f(\text{Cl}_{swn}(A)) \subset \text{Cl}(f(A))$, for each subset A of X,

(4) $\text{Cl}_{swn}(f^{-1}(B)) \subset f^{-1}(Cl(B))$, for each subset B of Y,

(5) $f^{-1}(\text{Int}(B)) \subset \text{Int}_{swn}(f^{-1}(B))$, for each subset B of Y,

Proof. Follows from the definition of swn-continuity. □

Theorem 4.1. *Let X, Y be spaces. For a function $f : X \to Y$, the following are equivalent:*

(1) f is swn-continuous,

(2) if for each open subset V of Y with $f^{-1}(V) \neq \emptyset$, there exists a nonempty open set U in X such that $U \subseteq \text{Cl}(f^{-1}(V))$,

(3) if for each closed subset F of Y with $f^{-1}(F) \neq X$, there exists a proper closed E in X such that $\text{Int}(f^{-1}(F)) \subseteq E$,

(4) if for each open dense subset D of X, then $f(D)$ is dense in $f(X)$.

Proof. (1) $\Longrightarrow$ (2) Remark 3.1 and the definition of swn-continuity.

(2) $\Longrightarrow$ (3) Let F be a closed set in Y such that $f^{-1}(F) \neq X$. Then $Y \setminus F$ is open in Y with $f^{-1}(Y \setminus F) \neq \emptyset$. By (2), there exists an open set U in X such that $\emptyset \neq U \subseteq \mathrm{Cl}(f^{-1}(Y \setminus F)) = X \setminus \mathrm{Int}(f^{-1}(F))$. This implies that $\mathrm{Int}(f^{-1}(F)) \subseteq X \setminus U \neq X$. If $E = X \setminus U$, then E is a proper closed set that satisfies the required property.

(3) $\Longrightarrow$ (4) Let D be open dense in X. We need to prove that $f(D)$ is dense in $f(X)$. Suppose otherwise that $f(D)$ is not dense in $f(X)$. There exists a proper closed set F such that $f(D) \subseteq F \subset f(X)$. Therefore $D \subseteq f^{-1}(F)$. By (3), there exist a proper closed set E in X such that $D \subseteq \mathrm{Int}(f^{-1}(F)) \subseteq E \subset X$. This contradicts that D is dense in X. Thus (4) holds.

(4) $\Longrightarrow$ (1) W.l.o.g, let H be an open set in Y with $f^{-1}(H) \neq \emptyset$, because if $f^{-1}(H) = \emptyset$, then it is trivially swn-open. Suppose that $f^{-1}(H)$ is not swn-open. That is $\mathrm{Int}(\mathrm{Cl}(f^{-1}(H)) = \emptyset$. Therefore $\mathrm{Cl}(\mathrm{Int}(X \setminus f^{-1}(H)) = X$. This implies that $\mathrm{Int}(X \setminus f^{-1}(H))$ is dense in X. By (4), $f(X \setminus f^{-1}(H))$ is dense in $f(X)$, i.e., $\mathrm{Cl}(f(X \setminus f^{-1}(H))) = f(X)$. This yields that $\mathrm{Cl}(f(X) \setminus H) = f(X) \setminus H = f(X)$ and so $H = \emptyset$. Contradiction to the choice of H. It follows that $\mathrm{Int}(\mathrm{Cl}(f^{-1}(H))$ must not be empty. Thus $f^{-1}(H)$ is swn-open in X. □

Theorem 4.2. *For a one to one function f from a space X onto a space Y, the following are equivalent:*

(1) f is swn-continuous,

(2) if for each (closed) nowhere dense subset N of X, then $f(N)$ is co-dense in Y.

Proof. (1) $\Longrightarrow$ (2) Let N be a (closed) nowhere dense set in X. We need to show that $f(N)$ is co-dense in Y. Suppose otherwise, then there is a nonempty open set H in Y such that $H \subseteq f(N)$ and so $f^{-1}(H) \subseteq f^{-1}(f(N)) = N$. By (1), $\emptyset \neq \mathrm{Int}(\mathrm{Cl}(f^{-1}(H)) \subseteq \mathrm{Int}(\mathrm{Cl}(N) = \mathrm{Int}(N)$. This proves that is not (closed) nowhere dense in X, which is contradiction. Hence (2) is established.

(2) $\Longrightarrow$ (1) Let H be an open set in Y. If $f^{-1}(H) = \emptyset$, then $f^{-1}(H)$ is swn-open by the definition. Let $f^{-1}(H) \neq \emptyset$. If $f^{-1}(H)$ is not swn-open, then it is nowhere dense in X. By (2), $f(f^{-1}(H))$ is co-dense in Y. That is, $\emptyset = \mathrm{Int}(f(f^{-1}(H))) = H$. This is impossible. Therefore f is swn-continuous. □

Theorem 4.3. *A function f from a space X onto a space Y is swn-continuous if and only $f^{-1}(A)$ is swn-open for each sw-open set A in Y.*

Proof. Assume that f is swn-continuous. Let A be sw-open in Y. If $A = \emptyset$, then $\emptyset = f^{-1}(H)$ is clearly swn-open. Let $A \neq \emptyset$. Then there is a nonempty open set H in Y such that $H \subseteq A$. Therefore $f^{-1}(H) \subseteq f^{-1}(A)$. By assumption, $\emptyset \neq \mathrm{Int}(\mathrm{Cl}(f^{-1}(H))) \subseteq \mathrm{Int}(\mathrm{Cl}(f^{-1}(A)))$. This proves that $f^{-1}(A)$ is swn-open.

Conversely, if G is an open set in Y, then it is sw-open. By assumption, $f^{-1}(G)$ is swn-open. Hence f is swn-continuous. □

5. Comparisons and relationships

After recalling the following definitions, we examine the link between swn-continuous function and other well-known classes of continuity in this part, and then other properties of swn-continuity are presented.

Definition 5.1. A function $f : X \to Y$ is called

(1) quasicontinuous [1] or semicontinuous [9] if the inverse image of each open set in Y is semiopen in X,

(2) nearly continuous [19] or precontinuous [8] if the inverse image of each open set in Y is preopen in X,

(3) α-continuous [5] if the inverse image of each open set in Y is α-open in X,

(4) almost quasicontinuous [6] or β-continuous [10] if the inverse image of each open set in Y is β-open in X,

(5) somewhat continuous [4] (briefly sw-continuous) if the inverse image of each open set in Y is sw-open in X,

(6) contra-semicontinuous [20] if the inverse image of each open set in Y is semiclosed in X,

(7) quasiopen or semiopen [21] if the image of each open set in X is semiopen in Y,

(8) quasiclosed or semiclosed [21] if the image of each closed set in X is semiclosed in Y.

(9) somewhat open [4] (briefly sw-open) if the image of each open set in X is sw-open in Y,

The following is the consequence of Diagram I (see also [7, Diagram I]):

$$\begin{array}{ccc}
\alpha\text{-continuity} & \longrightarrow & \text{near continuity} \\
\downarrow & & \downarrow \\
\text{quasicontinuity} & \longrightarrow & \text{almost quasicontinuity} \\
\downarrow & & \downarrow \\
sw\text{-continuity} & \longrightarrow & swn\text{-continuity}
\end{array}$$

Diagram II:Connection between generalized continuity

In general, equivalency cannot replace any of these implications. Only functions explicitly connected to swn-continuity are given as counterexamples. Additional examples can be found in the literature.

Example 5.1. *Consider $X = Y = \mathbb{R}$ with the natural topologies. If $f : X \to Y$ is defined by*

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q}; \\ 1, & x \in \mathbb{Q}, \end{cases}$$

then f is swn-continuous but not sw-continuous. The inverse image of any open subset of Y containing only 1 is $\mathbb{Q}$ which is not sw-open in X.

Example 5.2. *Let the function $f : X \to Y$ be defined by*

$$f(x) = \begin{cases} x, & \text{if } x \notin \{0,1\}; \\ 0, & \text{if } x = 1; \\ 1, & \text{if } x = 0, \end{cases}$$

where $X = Y = \mathbb{R}$ with the natural topologies. Then f is swn-continuous because the inverse image of any interval always contains some interval, so the interior of its closure cannot be empty, while f cannot be almost quasicontinuous. Take the open set $G = (-\varepsilon, \varepsilon)$, where $\varepsilon < 1$. Therefore $f^{-1}(G) = (-\varepsilon, 0) \cup (0, \varepsilon) \cup \{1\}$. But $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(G)))) = [-\varepsilon, \varepsilon]$ and so $f^{-1}(G) \nsubseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(G))))$. In conclusion, f cannot be almost quasicontinuous.

Proposition 5.1. *Let X, Y be spaces and let D be an open dense subspace of X. If $f : X \to Y$ is swn-continuous on X, then $f|_D$ is swn-continuous on D.*

Proof. Follows from Proposition 3.3. ☐

From Remark 3.3 and Propositions 2.21-2.28 in [7], we have

Remark 5.1. *The restriction of an swn-continuous function to an open (resp. a dense, a closed) subspace need not be swn-continuous.*

Theorem 5.1. *Let X, Y be spaces. A function $f : X \to Y$ is almost quasicontinuous iff $f|_U$ is swn-continuous for each open subset $U \subseteq X$.*

Proof. Assume that f is almost quasicontinuous. Let H be an open subset of Y and let U be an open subset of X. By assumption $f^{-1}(H)$ is β-open in X. By Lemma 2.8, $f^{-1}(H) \cap U$ is β-open in U and thus, by Lemma 2.4 (ii), $f^{-1}(H) \cap U$ is an swn-open subset of U. Hence, $f|_U$ is swn-continuous.

Conversely, suppose that $f|_U$ is swn-continuous for each open subset U of X. Let H be an open set in Y. Then $f^{-1}|_U(H) = f^{-1}(H) \cap U$ is swn-open in U. Since U is an open subset of X and each open is semiopen, by Proposition 3.6, $f^{-1}(H) \cap U$ is swn-open in X for each open U and consequently, by Proposition 3.8, $f^{-1}(H)$ is β-open in X. Thus f is almost quasicontinuous. ☐

Theorem 5.2. *Let X, Y be spaces. A function $f : X \to Y$ is nearly continuous if $f|_U$ is swn-continuous for each α-open subset $U \subseteq X$.*

Proof. By the same steps given in the proof of Theorem 5.1 and using Proposition 3.9, one can obtain the proof. ☐

Theorem 5.3. *Let X, Y be spaces and let $f : X \to Y$ be a function. If X is strongly irresolvable, then f swn-continuous iff f is sw-continuous*

Proof. From Theorem 3.1 (9). ☐

Theorem 5.4. *Let X, Y be spaces. For a function $f : X \to Y$, the following are equivalent:*

(1) f is swn-continuous and contra-semicontinuous,

(2) f is sw-continuous and contra-semicontinuous,

Proof. From Lemma 3.5. ☐

Proposition 5.2. *Let f be a one to one function from a space X onto a space Y. Then f quasiopen iff f is quasiclosed.*

Proof. Obvious. ☐

Theorem 5.5. *For a one to one quasiopen function f from a space X onto a space Y, the following are equivalent:*

(1) f is swn-continuous,

(2) if for each (closed) nowhere dense subset N of X, then $f(N)$ is nowhere dense in Y,

(3) if for each swn-open subset A of Y, then $f^{-1}(A)$ is swn-open in X,

(4) f is almost quasicontinuous.

Proof. (1) $\Longrightarrow$ (2) Let N be a closed nowhere dense set in X. By quasiopenness of f, $\operatorname{Int}(\operatorname{Cl}(f(N))) \subseteq f(N)$ and so $\operatorname{Int}(\operatorname{Cl}(f(N))) = \operatorname{Int}(f(N))$. By Theorem 4.2, $\operatorname{Int}(f(N)) = \emptyset$. Thus $\operatorname{Int}(\operatorname{Cl}(f(N))) = \emptyset$. Hence $f(N)$ is nowhere dense in Y.

(2)$\Longleftrightarrow$(3) Suppose (3) is not true. There exists an swn-open subset A of Y such that $f^{-1}(A)$ is not swn-open, which means that $f^{-1}(A)$ is a nowhere dense in X. By (2), $f(f^{-1}(A)) = A$ is nowhere dense, i.e., A is not swn-open. This is contradiction. Hence (3) must be true. The converse can be proved similarly.

(2) $\Longrightarrow$ (4) Let H be an open set in Y. We want to show that $f^{-1}(H)$ is β-open in X. Let $x \notin \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(f^{-1}(H))))$. Then there is an open set G in X containing x such that $\operatorname{Int}(\operatorname{Cl}(f^{-1}(H))) \cap G = \emptyset$ and so $\emptyset = \operatorname{Int}(\operatorname{Cl}(f^{-1}(H))) \cap \operatorname{Int}(\operatorname{Cl}(G)) \supseteq \operatorname{Int}(\operatorname{Cl}(f^{-1}(H) \cap G))$. Therefore $f^{-1}(H) \cap G$ is nowhere dense in X. By (2), $f(f^{-1}(H) \cap G) = H \cap f(G)$ is nowhere dense in Y. This implies that $\operatorname{Int}(H \cap f(G)) = H \cap \operatorname{Int}(f(G)) = \emptyset$ and so $H \cap \operatorname{Cl}(\operatorname{Int}(f(G))) = \emptyset$. Since f is quasiopen, then $f(G) \subseteq \operatorname{Cl}(\operatorname{Int}(f(G)))$. Therefore $H \cap f(G) = \emptyset$ and then $f^{-1}(H) \cap G = \emptyset$. Thus $x \notin f^{-1}(H)$. This yields that $f^{-1}(H) \subseteq \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(f^{-1}(H))))$, which establishes that, f is almost quasicontinuous.

(4) $\Longrightarrow$ (1) Let V be an open set in Y. If $V = \emptyset$, clearly its inverse is swn-open. Suppose that $V \neq \emptyset$. By (4), $f^{-1}(V) \subseteq \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(f^{-1}(V))))$. By Lemma 2.4 (ii), $\operatorname{Int}(\operatorname{Cl}(f^{-1}(V))) \neq \emptyset$. Thus f is swn-continuous. □

6. Applications

Proposition 6.1. *Let f be a one to one quasiopen swn-continuous function f from a space X onto a space Y. If M is a meager set in X, then $f(M)$ is meager in Y.*

Proof. Let M be a meager set in X. Then $M = \bigcup_{i=1}^{\infty} N_i$ such tat N_i are nowhere set in X for $i = 1, 2, \cdots$. Therefore

$$f(M) = f\left(\bigcup_{i=1}^{\infty} N_i\right) = \bigcup_{i=1}^{\infty} f(N_i).$$

By Theorem 5.5 (2), $f(N_i)$ is nowhere dense for each i. Hence $f(M)$ is meager in Y. □

Theorem 6.1. *Let X, Y be space and let $f : X \to Y$ be a function. If f is quasiopen, the following are equivalent:*

(1) f is swn-continuous,

(2) if for each open dense set D in X, then $\operatorname{Int}(f(N))$ is dense in $f(X)$.

Proof. (1) $\Longrightarrow$ (2) Let D be open dense in X. By Theorem 4.1 (4), $f(D)$ is dense in $f(X)$. Since f is quasiopen, then $f(D) \subseteq \operatorname{Cl}(\operatorname{Int}(f(D)))$ and so $f(X) = \operatorname{Cl}(f(D)) = \operatorname{Cl}(\operatorname{Int}(f(D)))$ by Lemma 2.3 (i). Thus $\operatorname{Int}(f(N))$ is dense in $f(X)$.

(2) $\Longrightarrow$ (1) Straightforward (from Theorem 4.1 (4) $\Longrightarrow$ (1)). □

Theorem 6.2. *Let f be a one to one quasiopen swn-continuous function f from a space X onto a space Y. If $A \subseteq X$ has the Baire property, then $f(A)$ has the Baire property in Y.*

Proof. Let $A \subseteq X$ be a set of Baire property. Then $A = G\Delta N$ for some open G and meager N subsets of X. Now, $f(A) = f(G)\Delta f(N)$. By Proposition 6.1, $f(N)$ is meager. It is enough to show that $f(G)$ has the Baire property. Since G in open and f is quasiopen, so $f(G)$ a semiopen set in Y, by Lemma 2.10, $f(G)$ has the Baire property. Thus $f(A)$ has the Baire property. □

Theorem 6.3. *Let f be a one to one quasiopen swn-continuous function f from a space X onto a Baire space Y. Then X is a Baire space.*

Proof. Assume that G is an open meager subset of X. By Proposition 6.1, $f(G)$ is meager in Y. But f is quasiopen, so $f(G)$ is semiopen in Y. By Lemma 2.4 (i), $\text{Int}(f(G)) \neq \emptyset$. Contradiction to the assumption that Y is Baire. Hence X is a Baire space. □

The above Theorem is a slight generalization of the following result given by Noll:

Corollary 6.1. *Let f be a one to one sw-open open sw-continuous function f from a space X onto a Baire space Y. Then X is a Baire space.*

Proof. Corollary of Theorem 1 in [22], (*c.f.* Theorem 18 in [4]). □

We remark that the "one to one" condition in Theorem 6.3 can be weakened to "countably fiber-complete".

Definition 6.1. [23] Let X, Y be spaces. A function $f : X \to Y$ is called countably fiber-complete if for each centered sequence $\{G_n\}_{n\in\mathbb{N}}$ of open subsets of X, $\bigcap_{n\in\mathbb{N}} G_n \neq \emptyset$, if there is $y \in Y$ such that $f^{-1}(y) \cap G_n \neq \emptyset$ for each n.

Theorem 6.4. *Let f be a quasiopen swn-continuous countably fiber-complete function f from a space X onto a Baire space Y. Then X is a Baire space.*

Proof. Let $\{D_n\}_{n\in\mathbb{N}}$ be a countable collection of dense open subsets of X. We need to show that $\bigcap_{n\in\mathbb{N}} D_n$ is dense in X. Let G be any nonempty open subset of X. Since f is quasiopen, then $f(G)$ is a semiopen subset of Y. By Lemma 2.4 (i), $\text{Int}(f(G)) \neq \emptyset$. Set $H = \text{Int}(f(G))$. By Theorem 6.1, $\{\text{Int}(f(D_n))\}_{n\in\mathbb{N}}$ is a countable collection of dense subsets of Y. Since Y is Baire, $\bigcap_{n\in\mathbb{N}} \text{Int}(f(D_n))$ is dense in Y. It follows that $\bigcap_{n\in\mathbb{N}} \text{Int}(f(D_n)) \cap H \neq \emptyset$. Let $y \in \bigcap_{n\in\mathbb{N}} \text{Int}(f(D_n)) \cap H$. This implies that $\{y\} \cap f(D_n) \cap H \neq \emptyset$ and therefore $f^{-1}\{y\} \cap D_n \cap G \neq \emptyset$ for each n. By countable fiber-completeness of f, $\bigcap_{n\in\mathbb{N}} D_n \cap G \neq \emptyset$, which means that $\bigcap_{n\in\mathbb{N}} D_n$ is dense in X. This proves that X is a Baire space. □

The next example shows that the condition quasiopenness of a function f in Theorems 6.3 and 6.4 cannot be dropped:

Example 6.1. *Let $X = \mathbb{R}$ with the right order topology, i.e., the topology generated by the basis $B_a = \{x : x > a\}$, let $Y = \mathbb{R}$ with the finite complement topology and define $f : X \to Y$ to be the identity function. We claim that f is swn-continuous but not quasiopen. On the other hand Y is a Baire space but X is not. Clearly f is one to one and onto (consequently f satisfies countable fiber-completeness). First we want to show that f is swn-continuous. Given any open set G in Y, then it has the form $G = (-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_n, \infty)$*

for $x_1, x_2, \ldots, x_n \in \mathbb{R}$. *It follows that* $f^{-1}(G)$ *always contains an open set, and each open set in* X *is dense, so* $\mathrm{Int}(\mathrm{Cl}(f^{-1}(G))) \neq \emptyset$. *Thus* f *is swn-continuous. Now, take the open set* $(0, \infty)$ *in* X, *then* $(0, \infty) = f((0, \infty)) \nsubseteq \mathrm{Cl}(\mathrm{Int}(f(0, \infty))) = \emptyset$. *Therefore* f *is not quasiopen. The nowhere sets in* Y *are only finite. Then* Y *cannot be written a countable union of finite sets, so it is a Baire space. While* $X = \bigcup_{r \in \mathbb{Q}} N_r$, *where* $N_r = \{x : x < r\}$ *is nowhere dense for each* r, *[24, p74]. Hence* X *is not a Baire space.*

We shall compare Theorems 6.3 and 6.4 with the following results by Noll and Mirmostafaee and Piotrowski respectively. We claim that our results are superior and the next example proves our claim.

Theorem 6.5. *Let* f *be a one-to-one sw-open sw-continuous function* f *from a space* X *onto a Baire space* Y. *Then* X *is a Baire space.*

Proof. Corollary of Theorem 1 in [22], (*c.f.* Theorem 18 in [4]). □

Theorem 6.6. *[23, Theorem 1.7] Let* f *be an sw open and sw-continuous countably fiber complete function from a space* X *onto a Baire space* Y. *Then* X *is a Baire space.*

Example 6.2. *Let* r *be the natural topology on* $\mathbb{R}$ *and let* θ *be a topology on* $\mathbb{R}$ *generated by* $r \cup \{\mathbb{P}\}$. *Suppose that* $g : (\mathbb{R}, r) \to (\mathbb{R}, \theta)$ *is the identity function. One can easily see that* g *sends open sets to open, so* g *is semiopen and consequently somewhat open. We now check that* g *is somewhat nearly continuous. Take any open set* H *in* θ, *its inverse either contains an open interval or a subset of* $\mathbb{P}$ *and in both cases* $\mathrm{Int}(\mathrm{Cl}(g^{-1}(H))) \neq \emptyset$. *If we take* $\mathbb{P}$ *as an open set in* θ, *then* $\mathrm{Int}(g^{-1}(\mathbb{P})) = \emptyset$ *and so* g *cannot be somewhat continuous. On the other hand, by Baire category theorem both* $(\mathbb{R}, r)$ *and* $(\mathbb{R}, \theta)$ *are Baire spaces.*

In conclusion, we have given two examples which verified that Theorems 6.3 and 6.4 are natural generalizations of Theorems 6.5 and 6.6.

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