An existence and uniqueness of solution for \( p \)-Laplacian Kirchhoff type equation with singular term

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Abstract: This work is devoted to study the existence of positive solution for a class of \( p \)-Laplacian Kirchhoff type equation with singular nonlinearity:

\[
\begin{align*}
L_p(u) &= f(x)|u|^{-\gamma} - \lambda|u|^{p^* - 2}u & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega,
\end{align*}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n (n \geq 3) \), \( \lambda > 0 \) is a real parameter. Here \( \gamma \in (0, 1) \) is a constant, \( a, b \geq 0 \) such that \( a + b > 0 \) are parameters, the weight function \( f : \Omega \to \mathbb{R} \) is positive and belonging to the Lebesgue space \( L^\alpha(\Omega) \) with \( \alpha := \frac{p^*}{p^* + \gamma - 1} \), and \( p^* := \frac{np}{n - p} \) is the Sobolev critical exponent in the Euclidian embedding \( W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \).

The operator is defined as

\[
L_p(u) := -\left( a \int_\Omega |\nabla u|^p dx + b \right)^{p-1} \Delta_p u + \ell(x)|u|^{p^* - 2}u,
\]

and the operator \( \Delta_p \) is the \( p \)-Laplacian for \( 1 < p < n \). Our approach relies on the variational methods and some analysis’ techniques.

Key words: Kirchhoff type equation, \( p \)-Laplacian Kirchhoff type equation, Critical exponent of Sobolev

1. Introduction and Motivation

In this paper, we study the existence and uniqueness of positive solution to the following nonlinear elliptic \( p \)-Laplacian Kirchhoff equation:

\[
\begin{align*}
L_p(u) &= f(x)|u|^{-\gamma} - \lambda|u|^{p^* - 2}u & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega,
\end{align*}
\]

where, throughout this work, \( \Omega \subset \mathbb{R}^n (n \geq 3) \) is a smooth bounded domain, \( \lambda > 0 \) is a real parameter. Here \( \gamma \in (0, 1) \) is a constant, \( a, b \geq 0 \) such that \( a + b > 0 \) are parameters, the weight function \( f : \Omega \to \mathbb{R} \) is positive and belonging to the Lebesgue space \( L^\alpha(\Omega) \) with \( \alpha := \frac{p^*}{p^* + \gamma - 1} \), and \( p^* := \frac{np}{n - p} \) is the Sobolev critical exponent.
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in the Euclidian embedding \( W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \). The operator is defined as

\[
L_p(u) := - \left( a \int_{\Omega} |\nabla u|^p \, dx + b \right)^{p-1} \Delta_p u + \ell(x)|u|^{p-2} u.
\]

Such that \( \ell \in L^2(\Omega) \cap L^{\infty}(\Omega) \) and the operator \( \Delta_p \) is the \( p \)-Laplacian which be defined as for all \( u \in W^{1,p}(\Omega) \):

\[
\Delta_p u := - \text{div} (|\nabla u|^{p-2} \nabla u) = \sum_{1 \leq i \leq n} \frac{\partial}{\partial x_i} \left( |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right).
\]

The equation \((P_1)\) is related to the stationary analogue of the Kirchhoff equation

\[
u_{tt} - \left( a \int_{\Omega} |\nabla u|^p \, dx + b \right) \Delta_p u = g(x,u) \quad \text{in } \Omega,
\]

with \( \Omega \subset \mathbb{R}^n(n \geq 3) \) is a smooth bounded domain, which was proposed by Kirchhoff in 1883 in [11] as an extension of the classical D’Alembert’s wave equation

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{k} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = g(x,u),
\]

for free vibrations of elastic strings. The parameters in above equation have physical significant meanings as follows: \( L \) is the length of the string, \( E \) is the area of the cross section, \( \rho \) is the Young modulus of the material, \( \rho_0 \) is the mass density and is the initial tension.

In 2006, the authors in [8] considered two problems of the Kirchhoff type:

\[
\left\{ \begin{array}{ll}
- [M \left( \int_{\Omega} |\nabla u|^p \, dx \right)]^{p-1} \Delta_p u = f(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right. \quad \text{(P4)}
\]

and

\[
\left\{ \begin{array}{ll}
- [M \left( \int_{\Omega} |\nabla u|^p \, dx \right)]^{p-1} \Delta_p u = f(x,u) + \lambda |u|^{s-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{array} \right. \quad \text{(P5)}
\]

Where \( \Omega \subset \mathbb{R}^n(n \geq 3) \) is a smooth bounded domain for \( 1 < p < n, s \geq p^* := \frac{np}{n-p} \) and \( M \) and \( f \) are two continuous functions. Using Pass-Montain Theorem, they obtained the existence of positive solutions of problems (P4) and (P5).

In 2009, the authors in [7] have considered the existence and multiplicity of solutions to a class of \( p \)-Kirchhoff type equation:

\[
\left\{ \begin{array}{ll}
- [M \left( \int_{\Omega} |\nabla u|^p \, dx \right)]^{p-1} \Delta_p u = f(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{array} \right. \quad \text{(P6)}
\]

They obtained the existence and multiplicity result of non trivial solution of the problem (P6).

Recently, the first author, S. Benmansour and Kh. Tahri in [21] showed the existence, nonexistence and multiplicity results for the following \( p \)-Laplacian Kirchhoff equation:

\[
\left\{ \begin{array}{ll}
- (a \int_{\Omega} |\nabla u|^p \, dx + b) \Delta_p u = \mu |u|^{p-2} u + \lambda |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right. \quad \text{(P7)}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^n(n > 3), \lambda, \mu > 0$ are real parameters and $a, b \geq 0 : a + b > 0$ are positive constants, $\Delta_p u$ is the $p$–Laplacian operator for $1 < p < n$.

They established the following results:

**Theorem 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n(n > 3)$, assume $a, b \geq 0 : a + b > 0$ and $p^* = 4$ then the following assertions are true:

(i). Assume that $a > 0, b > 0, 0 < \mu < aK(n,p)^2$ and $0 < \lambda < b\lambda_1$, then the equation $(P_7)$ has no positive nontrivial solution.

(ii). Assume that $a \geq 0, b > 0, 0 < \mu < aK(n,p)^2$ and $\lambda > b\lambda_1$, then the equation $(P_7)$ has a positive nontrivial solution.

(iii). Assume that $a \geq 0, b > 0, 0 < \mu < aK(n,p)^2$, then for any $k \in \mathbb{Z}^+$, there exists $\Lambda_k > 0$ such that the equation $(P_7)$ has at least $k$ pairs of nontrivial solutions for $\lambda > \Lambda_k > 0$.

Some interesting studies for Kirchhoff type problems in a bounded domain of $\mathbb{R}^n(n \geq 3)$ by critical points theory and variational methods can be found in [1], [2], [3], [4], [5], [6], [10], [12], [13], [14], [15], [16], [17], [18], [20], [22], [23].

The following theorem is our main result.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n(n \geq 3)$ be a smooth bounded domain and assuming that $(H_i)_{1 \leq i \leq 2}$ are hold. The problem $(P_1)$ possesses a positive solution. Moreover, this solution is a global minimizer solution.

This paper proceeds as follows. In the next section, we prove the energy functional $J_\lambda$ satisfies some geometric conditions. In section 3, by using critical point theory, we get the main result of this paper.

### 2. Variational Setting

Let $X = W^{1,p}_0(\Omega)$ be the usual Sobolev space, equipped with the norm

$$
\|u\| = (\int_\Omega |\nabla u|^p \, dx)^{\frac{1}{p}}
$$

and $\|u\|_p = (\int_\Omega |u|^p \, dx)^{\frac{1}{p}}$ denotes the norm in $L^p(\Omega)$.

A function $u \in X$ is said to be a weak solution of problem $(P_1)$ if $u > 0$ in $\Omega$ and there holds

$$
0 = (a + b \int_\Omega |\nabla u|^p \, dx)^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + \int_\Omega \ell(x)|u|^{p-2} u \varphi \, dx
$$

$$
+ \lambda \int_\Omega u \varphi \, dx - \int_\Omega f(x)u^{-\gamma} \varphi \, dx.
$$

(VF)

for all $\varphi \in X$.

We shall look for (weak) solutions of $(P_1)$ by finding critical points of the energy functional $J_\lambda : X \to \mathbb{R}$ given by

$$
J_\lambda(u) = \frac{1}{bp^2} \left[ (a + b \int_\Omega |\nabla u|^p \, dx)^{p} - a^p \right] + \frac{1}{p} \int_\Omega \ell(x)|u|^p \, dx
$$

$$
+ \frac{\lambda}{1 + q} \int_\Omega |u|^{1+q} \, dx - \frac{1}{1 - \gamma} \int_\Omega f(x)|u|^{1-\gamma} \, dx,
$$

3
for all $u \in X$. By analyzing the associated minimization problems for the functional $J_\lambda$, one can study weak solutions for $(P_1)$. As we know, the functional $J_\lambda$ fails to be Fréchet differentiable because of the singular term, then we cannot apply the critical point theory to obtain the existence of solutions directly.

Consider $\lambda_1$ the first eigenvalue of the problem:

$$
\begin{cases}
-\Delta_p u = \lambda_1 |u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

According to the work developed by Peral in [19] it has been shown that:

$$
\lambda_1 := \inf_{u \in W^{1,p}_0(\Omega) - \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx}.
$$

the first eigenvalue is isolated and simple and its corresponding first eigenfunction named $\phi_1$ is positive.

Let $\nu_1$ be the first eigenvalue of the following eigenvalue problem:

$$
\begin{cases}
L_p(u) = \nu |u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Where

$$
L_p(u) := - \left( a \int_{\Omega} |\nabla u|^p dx + b \right)^{p^{-1}} \Delta_p u + \ell(x)|u|^{p-2}u.
$$

According to the work developed by the authors of [9] it has been shown that:

$$
\nu_1 := \inf_{u \in W^{1,p}_0(\Omega) - \{0\}} \frac{\int_{\Omega} uL_p(u) dx}{\int_{\Omega} |u|^p dx}.
$$

The first eigenvalue is simple and strictly positive and its corresponding first eigenfunction named $\Psi_1$ is simple and strictly positive also, and the operator $L_p$ possesses unbounded eigenvalues sequence:

$$
\nu_1 < \nu_2 \leq \ldots \leq \nu_n \to +\infty \text{ as } n \to +\infty.
$$

Then we have the following proposition as a caracterisation of the values of the sequence $(\nu_k)_{k \geq 1}$.

**Proposition 2.1.** If $\nu$ is an eigenvalue of the operator $L_p$, then there exist $\nu_k$ and $u_k$ such that

$$
\nu := \nu_k \left( a \int_{\Omega} |\nabla u_k|^p dx + b \right)^{p^{-1}}.
$$

Throughout this paper, we make the following assumptions:

$(H_1)$ $0 < \gamma < 1, 0 < q \leq p^* - 1$.

$(H_2)$ $f \in L^{p^*+\gamma-1} (\Omega)$ avec $f(x) \geq 0$ pour tout $x \in \Omega$. 


3. Some useful Lemmas

In this section, we will recall and prove some lemmas which are crucial in the proof of the main theorem.

**Lemma 3.1.** The energy functional $J_\lambda$ has a minimum $\alpha$ in $X$ with $\alpha < 0$.

*Proof.* Since $0 < \gamma < 1$, $\lambda \geq 0$, by the Hölder inequality, we have

$$
\int_\Omega f(x)|u|^{1-\gamma} \, dx \leq (\int_\Omega |f(x)|^{\frac{p^*}{p-1+\gamma}} \, dx)^{\frac{p}{p^*}} (\int_\Omega |u|^{(1-\gamma)(\frac{p^*}{p-1+\gamma})} \, dx)^{\frac{1-\gamma}{p^*}}.
$$

Furthermore, by the Sobolev embedding theorem, we obtain that

$$
\frac{1}{1-\gamma} \|u\|_{p^*}^{1-\gamma} \leq C \|u\|_p^{1-\gamma}.
$$

Hence

$$
J_\lambda(u) = -\frac{a^p}{bp^2} + \frac{1}{bp^2} (a + b \|u\|_p^p) + \frac{1}{bp^2} \int_\Omega \ell(x)|u|^{p} \, dx + \frac{\lambda}{1+q} \int_\Omega |u|^{1+q} \, dx - \frac{1}{1-\gamma} \int_\Omega f(x)|u|^{1-\gamma} \, dx.
$$

Hence

$$
J_\lambda(u) \geq \frac{1}{bp^2} (a + b \|u\|_p^p) - C \|f\|_{\frac{p^*}{p-1+\gamma}} \|u\|_p^{1-\gamma}.
$$

where $C > 0$ is a constant.

This implies that $J_\lambda$ is coercive and bounded from below on $X$.

Then

$$
\alpha = \inf_{u \in X} J_\lambda
$$

is well defined.

Moreover, since $0 < \gamma < 1$ and $f(x) > 0$ for almost every $x \in \Omega$, we have $J_\lambda(t\delta) < 0$ for all $\delta \neq 0$ and small $t > 0$.

Thus, we obtain

$$
\alpha = \inf_{u \in X} J_\lambda < 0.
$$

The proof is complete.

The validity of the next lemma will be crucial in the sequel.

**Lemma 3.2.** Assume that conditions (H1) and (H2) hold. Then $J_\lambda$ attains the global minimizer in $X$, that is, there exists $u_* \in X$ such that

$$
J_\lambda(u_*) = \alpha < 0.
$$

*Proof.* From Lemma 1, there exists a minimizing sequence $\{u_n\} \subset X$ such that

$$
\lim_{n \to \infty} J_\lambda(u_n) = \alpha < 0.
$$
Since \( J_\lambda(u_n) = J_\lambda(|u_n|) \), we may assume that \( u_n \geq 0 \) for almost every \( x \in \Omega \).

By (1), the sequence \( \{u_n\} \) is bounded in \( X \).

Since \( X \) is reflexive, we may extract a subsequence that for simplicity we call again \( \{u_n\} \), there exists \( u_\ast \geq 0 \) such that

\[
\begin{align*}
  u_n &\rightharpoonup u_\ast \quad \text{weakly in } X, \\
  u_n &\rightarrow u_\ast \quad \text{strongly } L^s(\Omega), \ 1 \leq s \leq p^*, \\
  u_n(x) &\rightarrow u_\ast(x) \quad \text{p.p. in } \Omega,
\end{align*}
\]

as \( n \to \infty \). As usual, letting \( w_n = u_n - u_\ast \), we need to prove that \( \|w_n\| \to 0 \) as \( n \to \infty \). By Vitali’s theorem, we find

\[
\lim_{n \to \infty} \int f(x)|u_n|^{1-\gamma} \, dx = \int f(x)|u_\ast|^{1-\gamma} \, dx
\]

and

\[
\lim_{n \to \infty} \int \ell(x)|u_n|^p \, dx = \int \ell(x)|u_\ast|^p \, dx.
\]

Moreover, by the weak convergence of \( \{u_n\} \) in \( X \) and Brézis-Lieb’s Lemma, one obtains

\[
\|u_n\|^p = \|w_n\|^p + \|u_\ast\|^p + o(1)
\]

where \( o(1) \) is an infinitesimal as \( n \to \infty \).

Hence, in the case that \( 0 < q < p^* - 1 \), from (4)-(7), we deduce that

\[
\alpha = \lim_{n \to \infty} J_\lambda(u_n)
\]

\[
= \lim_{n \to \infty} \left( -\frac{a^p}{bp^2} + \frac{1}{bp^2} (a + b\|u_n\|^p) + \frac{1}{p} \int \ell(x)|u_n|^p \, dx \right)
\]

\[
+ \frac{\lambda}{1+q} \int |u_n|^{1+q} \, dx - \frac{1}{1-\gamma} \int f(x)|u_n|^{1-\gamma} \, dx
\]

\[
= \lim_{n \to \infty} \left( -\frac{a^p}{bp^2} + \frac{1}{bp^2} (a + b(\|w_n\|^p + \|u_\ast\|^p)) + \frac{1}{p} \int \ell(x)|u_\ast|^p \, dx \right)
\]

\[
+ \frac{\lambda}{1+q} \int |u_\ast|^{1+q} \, dx - \frac{1}{1-\gamma} \int f(x)|u_\ast|^{1-\gamma} \, dx
\]

\[
= J_\lambda(u_\ast) + \lim_{n \to \infty} \left( -\frac{a^p}{bp^2} + \frac{1}{bp^2} (a + b\|w_n\|^p) \right)
\]

\[
\geq J_\lambda(u_\ast) \geq \inf_{u_n \in X} J_\lambda(u_n) = \alpha
\]

which implies

\[
J_\lambda(u_\ast) = \alpha.
\]
In the case that \( q = p^* - 1 \), it follows from (5)–(8) that
\[
\alpha = J_\lambda(u_\ast) + \lim_{n \to \infty} \left( -\frac{a^p}{bp^2} + \frac{1}{bp^2} (a + b\|w_n\|^p)p + \frac{\lambda}{p^*}\|w_n\|_{p^*}^p \right)
\geq J_\lambda(u_\ast) \geq \alpha
\]
which yields
\[
J_\lambda(u_\ast) = \alpha.
\]
Thus
\[
\inf_{u_n \in X} J_\lambda(u_n) = J_\lambda(u_\ast)
\]
and this completes the proof of Lemma 2. The proof is complete.

We are now in a position to prove Theorem 2.

4. Proof of Theorem 2

We only need to prove that \( u_\ast \) is a weak solution of (\( P_1 \)) and \( u_\ast > 0 \) in \( \Omega \). Firstly, we show that \( u_\ast \) is a weak solution of (\( P_1 \)). From Lemma 1, we see that
\[
\min_{u_n \in X} J_\lambda(u_n) = J_\lambda(u_\ast), \quad \forall \varphi \in X
\]
thus
\[
J'_\lambda(u_\ast + t\varphi)|_{t=0} = 0.
\]
This implies that
\[
0 = (a + b) \int_\Omega |\nabla u|^p dx - \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx + \int_\Omega \ell(x)|u|^{p-2}u\varphi dx
+ \lambda \int_\Omega u^q dx - \int_\Omega f(x)u^{-\gamma}\varphi dx.
\]
for all \( \varphi \in X \), Thus, \( u_\ast \) is a weak solution of (\( P_1 \)).

Secondly, we prove that \( u_\ast > 0 \) for almost every \( x \in \Omega \).
Since \( J_\lambda(u_\ast) = \alpha < 0 \), we obtain \( u_\ast \geq 0 \) and \( u_\ast \neq 0 \).
Then, \( \forall \phi \in X \) and \( \phi \geq 0 \) and \( t > 0 \), we have
\[
0 \leq \frac{J_\lambda(u_\ast + t\phi) - J_\lambda(u_\ast)}{t}
= -\frac{a^p}{bp^2} + \frac{1}{bp^2} (a + b\|u_\ast\|^p)p - \frac{\lambda}{p^*}\|u_\ast\|_{p^*}^p
+ \frac{1}{p} \int_\Omega \ell(x)|u_\ast + t\phi|^p - |u_\ast|^p dx
+ \lambda \int_\Omega \frac{(u_\ast + t\phi)^{1+q} - u_\ast^{1+q}}{t} dx
+ \frac{\lambda}{1+q} \int_\Omega \frac{1}{t} \left( (u_\ast + t\phi)^1 - u_\ast^1 \right) dx
- \frac{1}{1-\gamma} \int_\Omega \frac{f(x)(u_\ast + t\phi)^{1-\gamma} - u_\ast^{1-\gamma}}{t} dx.
\]
Using the Lebesgue Dominated Convergence Theorem, we have
\[
\lim_{t \to 0^+} \frac{1}{p} \int_\Omega \ell(x) \frac{(u_* + t\phi)^p - u_*^p}{t} dx = \frac{1}{p} \int_\Omega \ell(x) u_*^{p-1} \phi dx.
\] (8)

and
\[
\lim_{t \to 0^+} \frac{1}{1 + q} \int_\Omega \frac{(u_* + t\phi)^{1+q} - u_*^{1+q}}{t} dx = \int_\Omega u_*^q \phi dx.
\] (9)

For any \( x \in \Omega \), we denote \( g(t) = f(x) \frac{[u_* + t\phi(x)][u_*^1 - u_*^{1-\gamma}]}{(1 - \gamma)t} \)

Then
\[
g'(t) = f(x) \frac{u_*^{1-\gamma}(x) - [\gamma t\phi(x) + u_*^1][u_* + t\phi]^\gamma}{t^2(1 - \gamma)} \leq 0
\]

which implies that \( g(t) \) is non increasing for \( t > 0 \).

Moreover, we have
\[
\lim_{t \to 0^+} g(t) = (u_* + t\phi(x))_{t=0} = f(x) u_*^{-\gamma}(x) \phi(x).
\]

for every \( x \in \Omega \), which may be \( +\infty \) when \( u_* = 0 \) and \( \phi(x) > 0 \).

Consequently, by the Monotone Convergence Theorem, we obtain
\[
\lim_{t \to 0^+} \frac{1}{(1 - \gamma)} \int_\Omega f(x) \frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t} dx = \int_\Omega f(x) u_*^{-\gamma} dx.
\]

which may equal to \( +\infty \).

Combining this with (8 and 9), let \( t \to 0 \), it follows from (7) that
\[
0 \leq (a + b) \int_\Omega |u_*\phi|^p dx \int_\Omega |\nabla u_*|^p - 2\nabla u_* \nabla \phi dx
+ \int_\Omega \ell(x) |u_*|^{p-2} u_* \phi dx + \lambda \int_\Omega u_*^q \phi dx - \int_\Omega f(x) u_*^{-\gamma} \phi dx.
\] (10)

Then, we have
\[
\int_\Omega f(x) u_*^{-\gamma} \phi dx
\leq (a + b \|u_*\|^p) \int_\Omega |\nabla u_*|^p - 2\nabla u_* \nabla \phi dx + \frac{1}{p} \int_\Omega \ell(x) u_*^{p-1} \phi dx + \lambda \int_\Omega u_*^q \phi dx
\]

for all \( \phi \in X \) with \( \phi > 0 \).
Let $\phi_1 \in X$ be the first eigenfunction of the operator $-\Delta_p$ with $\phi_1 > 0$ and $\|\phi_1\| = 1$. Particularly, taking $\phi = \phi_1$ in (6), one gets

$$\int_{\Omega} f(x) u_*^{-\gamma} \phi_1 \, dx \leq (a + b \|u_*\|^p) \int_{\Omega} |\nabla u_*|^{p-2} \nabla u_* \nabla \phi_1 \, dx + \frac{1}{p} \int_{\Omega} \ell(x) u_*^{p-2} \phi_1 \, dx$$

$$+ \lambda \int_{\Omega} |u_*|^q \, dx$$

$$\leq (a + b \|u_*\|^p)^{p-1} \left[ \int_{\Omega} |\nabla u_*|^{(p-1)(\frac{p}{p-1})} \, dx \right] \int_{\Omega} |\nabla \phi_1|^p \, dx \right]$$

$$+ \frac{1}{p} \left[ \int_{\Omega} \ell(x) u_*^{p} \, dx \right]^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla \phi_1|^p \, dx \right)^{\frac{1}{p}}$$

$$+ \lambda \int_{\Omega} |u_*|^q \, dx$$

$$\leq (a + b \|u_*\|^p)^{p-1} \left[ \int_{\Omega} |\nabla u_*|^{(p-1)(\frac{p}{p-1})} \, dx \right] \int_{\Omega} |\nabla \phi_1|^p \, dx \right]$$

$$+ \frac{1}{p} \left( \int_{\Omega} \ell(x) \phi_1 \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} \phi_1 \, dx \right)^{\frac{1}{p}}$$

$$+ \lambda \int_{\Omega} |u_*|^q \, dx$$

$$\leq (a + b \|u_*\|^p)^{p-1} \left[ \int_{\Omega} |\nabla u_*|^{(p-1)(\frac{p}{p-1})} \, dx \right] \int_{\Omega} |\nabla \phi_1|^p \, dx \right]$$

$$+ \frac{1}{p} \left( \int_{\Omega} \ell(x) \phi_1 \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} \phi_1 \, dx \right)^{\frac{1}{p}}$$

$$+ \lambda \int_{\Omega} |u_*|^q \, dx$$

$$\leq (a + b \|u_*\|^p)^{p-1} \left[ \int_{\Omega} |\nabla u_*|^{(p-1)(\frac{p}{p-1})} \, dx \right] \int_{\Omega} |\nabla \phi_1|^p \, dx \right]$$

$$+ \frac{1}{p} \left( \int_{\Omega} \ell(x) \phi_1 \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} \phi_1 \, dx \right)^{\frac{1}{p}}$$

$$+ \lambda \int_{\Omega} |u_*|^q \, dx$$

$$< \infty$$

which implies that $u_* > 0$ for almost every $x \in \Omega$.

Moreover, according to Lemma 2, we have

$$J_\lambda(u_*) = \inf_{u \in X} J_\lambda(u).$$

Thus $u_*$ is a global minimizer solution.
Assuming that $\varphi$ and $\psi$ are two distinct weak solutions of problem (1), then we test equation $(VF)$ by $(\varphi - \psi)$:

$$\int_{\Omega} L_p(\varphi)(\varphi - \psi)dx = \int_{\Omega} f(x)|\varphi|^{-\gamma}(\varphi - \psi)dx - \lambda \int_{\Omega} |\varphi|^{p-2}\varphi(\varphi - \psi)dx$$

(12)

and

$$\int_{\Omega} L_p(\psi)(\varphi - \psi)dx = \int_{\Omega} f(x)|\psi|^{-\gamma}(\varphi - \psi)dx - \lambda \int_{\Omega} |\psi|^{p-2}\psi(\varphi - \psi)dx$$

(13)

Then we have

$$\int_{\Omega} L_p(\varphi)(\varphi - \psi)dx = (a \|\varphi\|^p + b)^{p-1} \int_{\Omega} |\nabla \varphi|^{p-2}\nabla \varphi \nabla (\varphi - \psi)dx$$

$$+ \int_{\Omega} \ell(x)|\varphi|^{p-2}\varphi(\varphi - \psi)dx$$

and

$$\int_{\Omega} L_p(\psi)(\varphi - \psi)dx = (a \|\psi\|^p + b)^{p-1} \int_{\Omega} |\nabla \psi|^{p-2}\nabla \psi \nabla (\varphi - \psi)dx$$

$$+ \int_{\Omega} \ell(x)|\psi|^{p-2}\psi(\varphi - \psi)dx.$$

From (12), one obtains

$$\int_{\Omega} L_p(\varphi)(\varphi - \psi)dx =$$

$$(a \|\varphi\|^p + b)^{p-1} \left[ \|\varphi\|^p - \int_{\Omega} |\nabla \varphi|^{p-2}\nabla \varphi \nabla \psi dx \right]$$

(14)

$$+ \int_{\Omega} \ell(x)|\varphi|^{p} dx - \int_{\Omega} \ell(x)|\varphi|^{p-2}\varphi dx$$

and also, from (13), one obtains

$$\int_{\Omega} L_p(\psi)(\varphi - \psi)dx =$$

$$(a \|\psi\|^p + b)^{p-1} \left[ -\|\psi\|^p + \int_{\Omega} |\nabla \psi|^{p-2}\nabla \varphi \nabla \psi dx \right]$$

(15)

$$- \int_{\Omega} \ell(x)|\psi|^{p} dx + \int_{\Omega} \ell(x)|\psi|^{p-2}\varphi dx.$$
Combining (14) and (15), we have

\[
\int_{\Omega} L_p(\varphi)(\varphi - \psi) dx - \int_{\Omega} L_p(\psi)(\varphi - \psi) dx = \\
\|\varphi\|^p (a\|\varphi\|^p + b)^{p-1} + \|\psi\|^p (a\|\psi\|^p + b)^{p-1} \\
- (a\|\varphi\|^p + b)^{p-1} \int_{\Omega} |\nabla \varphi|^2 \nabla \varphi \nabla \psi dx \\
- (a\|\psi\|^p + b)^{p-1} \int_{\Omega} |\nabla \psi|^2 \nabla \varphi \nabla \psi dx \\
+ \int_{\Omega} \ell(x) (|\varphi|^p + |\psi|^p) dx - \int_{\Omega} \ell(x) (|\varphi|^{p-2} + |\psi|^{p-2}) \varphi \psi dx.
\]

With the same computation for the right term, we have

\[
\int_{\Omega} L_p(\varphi)(\varphi - \psi) dx - \int_{\Omega} L_p(\psi)(\varphi - \psi) dx = \\
\int_{\Omega} f(x) (|\varphi| - |\psi|) (\varphi - \psi) dx
\]

(16)

Put

\[
Q_1(\varphi, \psi) := \|\varphi\|^p (a\|\varphi\|^p + b)^{p-1} - (a\|\varphi\|^p + b)^{p-1} \int_{\Omega} |\nabla \varphi|^2 \nabla \varphi \nabla \psi dx, \\
Q_2(\varphi, \psi) := \|\psi\|^p (a\|\psi\|^p + b)^{p-1} - (a\|\psi\|^p + b)^{p-1} \int_{\Omega} |\nabla \psi|^2 \nabla \varphi \nabla \psi dx,
\]

and

\[
Q_3(\varphi, \psi) := \int_{\Omega} \ell(x) (|\varphi|^p + |\psi|^p) dx - \int_{\Omega} \ell(x) (|\varphi|^{p-2} + |\psi|^{p-2}) \varphi \psi dx.
\]

Using Holder inequality, we have

\[
Q_1(\varphi, \psi) \geq \|\varphi\|^{p-1} (a\|\varphi\|^p + b)^{p-1} (\|\varphi\| - \|\psi\|), \\
Q_2(\varphi, \psi) \geq \|\psi\|^{p-1} (a\|\psi\|^p + b)^{p-1} (\|\psi\| - \|\varphi\|), \\
\]

and

\[
Q_3(\varphi, \psi) \geq (\|\varphi\|_p - \|\psi\|_p) ||\ell||_{\infty} \left[ ||\varphi||^{p-1}_p - ||\psi||^{p-1}_p \right].
\]

We divided into three cases:

1. Case if \(\|\varphi\| - \|\psi\| > 0\), then

\[
Q_1(\varphi, \psi) - Q_2(\varphi, \psi) + Q_3(\varphi, \psi) > 0.
\]

2. Case if \(\|\varphi\| - \|\psi\| < 0\), then

\[
Q_1(\varphi, \psi) - Q_2(\varphi, \psi) + Q_3(\varphi, \psi) < 0.
\]

3. Case if \(\|\varphi\| - \|\psi\| = 0\), then

\[
Q_1(\varphi, \psi) - Q_2(\varphi, \psi) + Q_3(\varphi, \psi) \geq 0.
\]
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Since $\gamma \in (0, 1)$ and $p > 0$, it is well known the following inequalities:

$$\forall x, y > 0 : \begin{cases} (x^p - y^p)(x - y) \geq 0, \\
(x^{-\gamma} - y^{-\gamma})(y - x) \geq 0. \end{cases}$$

Thus

$$\int_{\Omega} L_p(\varphi)(\varphi - \psi)dx - \int_{\Omega} L_p(\psi)(\varphi - \psi)dx \geq 0.$$

Consequently, we obtain a contradiction with the equation (16).

Then

$$\varphi = \psi.$$

This completes the proof of the theorem 1.2.

References


