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Nonexistence results for semi linear structural damped wave model with nonlinear memory on the Heisenberg group

Ali Hakem[1]*and Svetlin Georgiev[2]
[1]Laboratory ACEDP, Djillali Liabes university,
22000 Sidi Bel Abbes, Algeria. ORCID: 0000-0001-6145-4514
[2]Sofia University, Faculty of Mathematics and Informatics,
Sofia, Bulgaria. ORCID: 0000-0001-8015-4226

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Abstract: In this paper we investigate the following semi-linear structurally damped wave equation with nonlinear memory

$$\begin{cases} u_{tt}-\Delta_{\mathbb{H}}u+(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}u_t=\displaystyle\int_0^t (t-s)^{-\gamma}|u(s)|^p ds, \quad \eta\in\mathbb{R}^{2n+1}, \quad t>0, \\ u(\eta,0)=u_0(\eta) \quad , \quad u_t(\eta,0)=u_1(\eta), \end{cases}$$

where $\sigma\in(0,2)$, $\gamma\in(0,1)$, $p\in\mathbb{R}$, $p>1$ and $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}$. We intend to apply the method of a modified test function to establish the nonexistence of global weak solutions and to overcome some difficulties as well caused by the well-known fractional Laplacian $(-\Delta)^{\frac{\sigma}{2}}$.

Key words: Nonlocal operators, weak solution, Heisenberg group.

1. Introduction and preliminaries

This paper is concerned with the nonexistence of global weak solutions for the following semi-linear wave equation with nonlinear mixed damping term:

$$u_{tt}-\Delta_{\mathbb{H}}u+(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}u_t=\int_0^t (t-s)^{-\gamma}|u(s)|^p ds, \tag{1}$$

subject to the following initial conditions

$$u(\eta,0)=u_0(\eta), \quad u_t(\eta,0)=u_1(\eta), \quad \eta\in\mathbb{R}^{2n+1}, \tag{2}$$

where $0<\gamma<1, \sigma\in(0,2)$, $p>1$, $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2n+1)$-dimensional Heisenberg group. The operator $(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}$ accounts for anomalous diffusion. Our article is motivated by the paper of A. Hakem et al. [3] which deals with the blow-up of solutions for the following Cauchy problem

$$u_{tt}-\Delta u+|u|^{m-1}u_t=\int_0^t (t-s)^{-\gamma}|u(.,s)|^p ds; \quad t>0, \quad x\in\mathbb{R}^n,$$

subject to the initial data

$$u_0(x,0)=u_0(x), \; u_t(x,0)=u_1(x), x\in\mathbb{R}^n,$$

*Correspondence: hakemali@yahoo.com

where the unknown function u is real-valued, $n \geq 1$, $0 < \gamma < 1$, $m > 1$ and $p > 1$. More precisely, in [3] it is proved that if

$$n \leq min\left\{\frac{2(m+(1-\gamma)p)}{(p-1+(1-\gamma)(m-1))}, \quad \frac{2(1+(2-\gamma)p)}{(\frac{(p-1)(2-\gamma)}{p-m}+\gamma-1)(p-1)}\right\},$$

or $p \leq \frac{1}{\gamma}$, then the solution of the above problem does not exist globally in time. Recently, T. Dao and A.Z. Fino in [4] have succeeded to prove blow-up results to determine the critical exponents for the following Cauchy problem for semi-linear structurally damped wave model with nonlinear memory:

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s)|^p ds, \quad x \in \mathbb{R}^n, \quad t > 0 \;, \\ \\ u(x,0) = u_0(x) \quad , \quad u_t(x,0) = u_1(x), \end{cases}$$

where $\mu > 0$, $\sigma \in (0,2)$ for some $\gamma \in (0,1)$ and $p > 1$, by using a modified test function method. Namely, it was shown that, if

$$p \leq p_c = 1 + \frac{2+(1-\gamma)(2-\tilde{\sigma})}{\max\left[n-2+\gamma(2-\tilde{\sigma}),0\right]}, \quad \int_{\mathbb{R}^n} (u_1(x) + (-\Delta)^{\frac{\sigma}{2}} u_0(x) dx > 0,$$

then, there is no global (in time) weak solution. The problem of nonexistence of global solutions in the Heisenberg groups was previously analyzed by several authors from a different point of view in recent years (see [2], [5],[7], [8] and references therein). For instance, see the papers [5], [6], [7], [8],[12] and references therein for a variety of problems related to Heisenberg groups and partial differential equations in Heisenberg groups.

The paper is organized as follows. In the next section we give some auxiliary results for the Heisenberg group $\mathbb{H}$, the operator $\Delta_{\mathbb{H}}$ and Riemann-Liouville fractional differentiation and integration. In Section 3, we formulate and prove our main result.

2. Auxiliary Results

In this section, for the sake of the reader, we give some known facts about the Heisenberg group $\mathbb{H}$ and the operator $\Delta_{\mathbb{H}}$.

The Heisenberg group $\mathbb{H}$ whose points will be denoted by $\eta = (x,y,\tau)$, is the Lie group $(\mathbb{R}^{2n+1}; \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \eta' = \left(x + x', y + y', \tau + \tau' + 2\sum_{i=1}^{n}(x_i \cdot y'_i - x'_i \cdot y_i)\right),$$

for all $\eta = (x,y,\tau), \eta' = (x',y',\tau') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, where $\cdot$ denotes the standard inner product in $\mathbb{R}^n$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields $X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau}$ and $Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau}$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n} \left(X_i^2 + Y_i^2\right).$$

Observe that the vector field $T = \frac{\partial}{\partial \tau}$ does not appear in the equality above. This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T\delta_{ij}, \quad i, j = 1, 2, 3, \ldots, n.$$

The relation above proves that $\mathbb{H}$ is a nilpotent Lie group of order 2. The sub-Laplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H}$ is

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).$$

A natural group of dilatations on $\mathbb{H}$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \qquad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}$.
The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilations δ_λ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta \circ \eta')) = (\Delta_{\mathbb{H}} u)(\eta \circ \eta'), \Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda, \eta, \eta' \in \mathbb{H}.$$

The natural distance from η to the origin is introduced by Folland and Stein, see [6],

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{n} (x_i^2 + y_i^2) \right)^2 \right)^{\frac{1}{4}}.$$

For the reader convenience, we present some definitions and results concerning the fractional integrals and fractional derivatives that will be used hereafter.

We denote by $AC[0, T]$ the space of all functions which are absolutely continuous on $[0, T]$ with $0 < T < \infty$.

Definition 2.1. ([13]) Let $f \in \mathbb{L}^1(0, T)$ with $T > 0$. The Riemann-Liouville left- and right-sided fractional integrals of order $\alpha \in (0, 1)$ are defined by

$$I^\alpha_{0|t} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{-(1-\alpha)} f(s) ds, \quad t > 0, \tag{3}$$

and

$$I^\alpha_{t|T} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (t - s)^{-(1-\alpha)} f(s) ds, \quad t < T, \tag{4}$$

respectively, where Γ is the Euler gamma function.

Definition 2.2. ([13]) Let $f \in AC[0, T]$ with $T > 0$. The Riemann-Liouville left- and right-sided fractional derivatives of order $\in \alpha \in (0, 1)$ are defined by

$$D^\alpha_{0|t} f(t) = \frac{d}{dt} I^{1-\alpha}_{0|t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t - s)^{-\alpha} f(s) ds, \quad t > 0, \tag{5}$$

$$D^{\alpha}_{t|T}f(t) = -\frac{d}{dt}I^{1-\alpha}_{t|T}f(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T (t-s)^{-\alpha}f(s)ds, \quad t<T, \tag{6}$$

respectively.

Proposition 2.1. *([13]) Let $T>0$ and $\alpha\in(0,1)$. The fractional integration by parts formula*

$$\int_0^T f(t)D^{\alpha}_{0|t}g(t)dt = \int_0^T g(t)D^{\alpha}_{t|T}f(t)dt, \tag{7}$$

is valid for every $f\in I^{\alpha}_{t|T}(\mathbb{L}^p(0,T))$ and $g\in I^{\alpha}_{0|t}(\mathbb{L}^q(0,T))$ such that $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$ with $p,q>1$, where

$$I^{\alpha}_{t|T}(\mathbb{L}^p(0,T)) = \left\{f = I^{\alpha}_{t|T}h, \quad h\in\mathbb{L}^p(0,T)\right\},$$

and

$$I^{\alpha}_{0|t}(\mathbb{L}^q(0,T)) = \left\{f = I^{\alpha}_{0|t}h, \quad h\in\mathbb{L}^q(0,T)\right\}.$$

Proposition 2.2. *([9]) Let $T>0$ and $\alpha\in(0,1)$. Then, we have the following identities:*

$$D^{\alpha}_{0|t}I^{\alpha}_{0|t}f(t) = f(t), \quad ae \quad t\in(0,T) \quad \text{for all} \quad f\in\mathbb{L}^r(0,T) \quad \text{with} \quad 1\leq r\leq\infty, \tag{8}$$

and

$$(-1)^m D^m D^{\alpha}_{t|T}f = D^{m+\alpha}_{t|t}f \quad \text{for all} \quad f\in AC^{m+1}[0,T], \tag{9}$$

where

$$AC^{m+1}[0;T] = \left\{f:[0,T]\longrightarrow\mathbb{R}, \quad \text{such that} \quad D^m f\in AC[0,T]\right\},$$

and $D^m = \frac{d^m}{dt^m}$ is the usual m times derivative.

Lemma 2.1. *([9]) Let $T>0, 0<\alpha<1$ and $m\geq 0$. For all $t\in[0,T]$, we have*

$$D^{m+\alpha}_{t|T}\left(1-\frac{t}{T}\right)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m-\alpha)}T^{-(m+\alpha)}\left(1-\frac{t}{T}\right)^{\beta-\alpha-m}. \tag{10}$$

Lemma 2.2. *([4]) Let $T>0, 0<\alpha<1, m\geq 0$ and $p>1$. Then, we have*

$$\int_0^T \psi(t)^{-\frac{1}{p-1}}|D^{m+\alpha}_{t|T}\psi(t)|^{\frac{p}{p-1}}dt = CT^{1-(m+\alpha)\frac{p}{p-1}}. \tag{11}$$

3. Main results

Before stating our main result, we define weak solutions for (1)-(2).

Definition 3.1. A function $u \in \mathbb{L}^p\left((0,T), \mathbb{L}^p(\mathbb{R}^{2n+1})\right) \cap \mathbb{L}^1\left((0,T), \mathbb{L}^2(\mathbb{R}^{2n+1})\right)$ is said to be a local weak solution of (1)-(2) subject to the initial data $(u_0, u_1) \in H^\sigma(\mathbb{R}^{2n+1}) \times \mathbb{L}^2(\mathbb{R}^{2n+1})$ if the following formulation

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^{2n+1}} I^\alpha_{0|t}(|u(\eta,t)|^p)\varphi(\eta,t)d\eta dt + \int_{\mathbb{R}^{2n+1}} u_1(\eta)\varphi(\eta,0)d\eta \\
&-\int_{\mathbb{R}^{2n+1}} u_0(\eta)\varphi_t(\eta,0)d\eta + \int_{\mathbb{R}^{2n+1}} (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}} u_0(\eta)\varphi(\eta,0)d\eta \\
&= \int_0^T\int_{\mathbb{R}^{2n+1}} u(\eta,t)\varphi_{tt}(\eta,t)d\eta dt - \int_0^T\int_{\mathbb{R}^{2n+1}} u(\eta,t)\Delta_{\mathbb{H}}\varphi(\eta,t)d\eta dt \\
&-\int_0^T\int_{\mathbb{R}^{2n+1}} u(\eta,t)(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}\varphi_t(\eta,t)d\eta dt,
\end{aligned}$$

holds for any regular function

$$\varphi \in C^1\left((0,T]; H^\sigma(\mathbb{R}^{2n+1})\right) \cap C\left((0,T]; H^2(\mathbb{R}^{2n+1})\right) \cap C^2\left((0,T]; \mathbb{L}^2(\mathbb{R}^{2n+1})\right),$$

such that $\varphi(\eta,T) = 0$ and $\varphi_t(\eta,T) = 0$ for all $\eta \in \mathbb{R}^{2n+1}$. The solution is called global if $T = +\infty$.

Our main result reads as follows.

Theorem 3.1. *Let $0 < \sigma < 2, \tilde{\sigma} = \min(\sigma, 1)$ and $n \geq 1$. We assume that the initial data $(u_0, u_1) \in H^\sigma(\mathbb{R}^{2n+1}) \times \mathbb{L}^2(\mathbb{R}^{2n+1})$ satisfy the following relation*

$$\int_{\mathbb{R}^{2n+1}} (u_1(\eta) + (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}} u_0(\eta)d\eta > 0. \tag{12}$$

If

$$p \leq p_c = \frac{Q+2-\tilde{\sigma}}{(2-\tilde{\sigma})\gamma + Q - 2}, \tag{13}$$

or

$$p < \frac{1}{\gamma},$$

then no nontrivial global weak solutions exist for problem (1)-(2).

Proof. Throughout, C denotes a positive constant, whose value may change from line to line. The proof is based on a contradiction. Suppose that u is a local weak solution to (1)-(2), then u satisfies

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^{2n+1}} I^\alpha_{0|t}(|u(\eta,t)|^p)\varphi(\eta,t)d\eta dt + \int_{\mathbb{R}^{2n+1}} u_1(\eta)\varphi(\eta,0)d\eta \\
&-\int_{\mathbb{R}^{2n+1}} u_0(\eta)\varphi_t(\eta,0)d\eta + \int_{\mathbb{R}^{2n+1}} (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}} u_0(\eta)\varphi(\eta,0)d\eta \\
&= \int_0^T\int_{\mathbb{R}^{2n+1}} u(\eta,t)\varphi_{tt}(\eta,t)d\eta dt - \int_0^T\int_{\mathbb{R}^{2n+1}} u(\eta,t)\Delta_{\mathbb{H}}\varphi(\eta,t)d\eta dt \\
&-\int_0^T\int_{\mathbb{R}^{2n+1}} u(\eta,t)(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}\varphi_t(\eta,t)d\eta dt,
\end{aligned} \tag{14}$$

where $\alpha = 1-\gamma \in (0,1)$, for all test function φ such that $\varphi(\eta,T)=\varphi_t(\eta,T)=0$ for all $T>>1$. We define the following auxiliary functions:

$$\phi_R(\eta)=\phi\left(\frac{\eta}{R}\right), \quad \tilde{\varphi}(\eta,t)=\phi_R(\eta)\psi(t), \quad \psi(t)=\left(1-\frac{t}{T}\right)^{\beta},$$

where ϕ is a non-negative smooth function such that

$$\phi(x)=\phi(|x|), \quad \phi(0)=1, \quad 0<\phi(r)\leq 1, \quad \text{for} \quad r\geq 0. \tag{15}$$

Moreover, ϕ is decreasing and $\phi(r)\to 0$, as $r\to\infty$ sufficiently fast. Then, we define the test function

$$\varphi(\eta,t)=D^{\alpha}_{t|T}\tilde{\varphi}(\eta,t)=\phi_R(\eta)D^{\alpha}_{t|T}(\psi(t)).$$

From (14), using (9) and (10), we have

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^{2n+1}} I^{\alpha}_{0|t}(|u(\eta,t)|^p)D^{\alpha}_{t|T}\tilde{\varphi}(\eta,t)d\eta dt + CT^{-\alpha}\int_{\mathbb{R}^{2n+1}}(u_1(\eta)+(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}u_0(\eta))\phi_R(\eta)d\eta\\
&-CT^{-1-\alpha}\int_{\mathbb{R}^{2n+1}}u_0(\eta)\phi_R(\eta)d\eta=\int_0^T\int_{\mathbb{R}^{2n+1}}u(\eta,t)\phi_R(\eta)D^{2+\alpha}_{t|T}(\psi(t))d\eta dt\\
&-\int_0^T\int_{\mathbb{R}^{2n+1}}u(\eta,t)D^{\alpha}_{t|T}(\psi(t))\Delta_{\mathbb{H}}\phi_R(\eta)d\eta dt+\int_0^T\int_{\mathbb{R}^{2n+1}}u(\eta,t)D^{\alpha}_{t|T}(\psi(t))(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}(\phi_R(\eta))d\eta dt.
\end{aligned}\tag{16}$$

Using (7) and then (8), we arrive at

$$\begin{aligned}
&I_1+CT^{-\alpha}\int_{\mathbb{R}^{2n+1}}(u_1(\eta)+(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}u_0(\eta))\phi_R(\eta)d\eta-CT^{-1-\alpha}\int_{\mathbb{R}^{2n+1}}u_0(\eta)\phi_R(\eta)d\eta\\
&=C\int_0^T\int_{\mathbb{R}^{2n+1}}u(\eta,t)\phi_R(\eta)D^{2+\alpha}_{t|T}(\psi(t))d\eta dt-C\int_0^T\int_{\{|\eta|_{\mathbb{H}}\geq R\}}u(\eta,t)D^{\alpha}_{t|T}(\psi(t))\Delta_{\mathbb{H}}\phi_R(\eta)d\eta dt\\
&+C\int_0^T\int_{\mathbb{R}^{2n+1}}u(\eta,t)D^{\alpha}_{t|T}(\psi(t))(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}(\phi_R(\eta))d\eta dt=\mathcal{A}(\psi)+\mathcal{B}(\psi)+\mathcal{C}(\psi),
\end{aligned}\tag{17}$$

where

$$I_1=\int_0^T\int_{\mathbb{R}^{2n+1}}|u(\eta,t)|^p\tilde{\varphi}(\eta,t)d\eta dt.$$

On the other hand, using Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$, we can proceed the estimate for $A(\psi)$ as follows:

$$\begin{aligned}
|\mathcal{A}(\psi)| &\leq C\int_0^T\int_{\mathbb{R}^{2n+1}}|u(\eta,t)|\phi_R(\eta)|D^{2+\alpha}_{t|T}(\psi(t))|d\eta dt\\
&=C\int_0^T\int_{\mathbb{R}^{2n+1}}|u(\eta,t)|\tilde{\varphi}^{\frac{1}{p}}(\eta,t)\tilde{\varphi}^{-\frac{1}{p}}(\eta,t)\phi_R(\eta)|D^{2+\alpha}_{t|T}(\psi(t))|d\eta dt\\
&\leq CI_1^{\frac{1}{p}}\left(\int_0^T\int_{\mathbb{R}^{2n+1}}\phi_R(\eta)(\psi(t))^{-\frac{p'}{p}}|D^{2+\alpha}_{t|T}(\psi(t))|^{p'}d\eta dt\right)^{\frac{1}{p'}}.
\end{aligned}$$

At this stage, we pass to the scaled variables

$$\tilde{t} = \frac{t}{T} \quad \text{and} \quad \tilde{\eta} = (\tilde{x},\ \tilde{y}, \tilde{\tau}) \quad \text{such that} \quad \tilde{\tau} = \frac{\tau}{R^2},\ \tilde{x} = \frac{x}{R}, \tilde{y} = \frac{y}{R}\cdot$$

Using Lemma 2.2, one has

$$|\mathcal{A}(\psi)| \leq C I_1^{\frac{1}{p}} R^{\frac{Q}{p'}} T^{\frac{1}{p'}-2-\alpha}. \tag{18}$$

Similarly, one obtains

$$\begin{aligned} |\mathcal{B}(\psi)| &\leq C I_2^{\frac{1}{p}} \left(\int_0^T \int_{\{|\eta|_{\mathbb{H}} \geq R\}} (\phi_R(\eta))^{-\frac{p'}{p}} (\psi(t))^{-\frac{p'}{p}} |D_{t|T}^{\alpha}(\psi(t))|^{p'} |\Delta_{\mathbb{H}} \phi_R(\eta)|^{p'} d\eta dt \right)^{\frac{1}{p'}} \\ &\leq C I_2^{\frac{1}{p}} R^{\frac{Q}{p'}-2} T^{\frac{1}{p'}-\alpha}, \end{aligned} \tag{19}$$

where

$$I_2 = \int_0^T \int_{\{|\eta|_{\mathbb{H}} \geq R\}} |u(\eta, t)|^p \tilde{\varphi}(\eta, t) d\eta dt.$$

Moreover, we can proceed the estimate for $C(\psi)$ as follows:

$$\begin{aligned} |\mathcal{C}(\psi)| &\leq C I_1^{\frac{1}{p}} \left(\int_0^T \int_{\mathbb{R}^{2n+1}} (\phi_R(\eta))^{-\frac{p'}{p}} (\psi(t))^{-\frac{p'}{p}} |D_{t|T}^{1+\alpha}(\psi(t))|^{p'} |(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}} \phi_R(\eta)|^{p'} d\eta dt \right)^{\frac{1}{p'}} \\ &\leq C I_1^{\frac{1}{p}} R^{\frac{Q}{p'}-\sigma} T^{\frac{1}{p'}-\alpha-1}. \end{aligned} \tag{20}$$

Combining the estimates from (18) to (20) into (17), one deduces that

$$\begin{aligned} & I_1 + C T^{-\alpha} \int_{\mathbb{R}^{2n+1}} (u_1(\eta) + (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}} u_0(\eta)) \phi_R(\eta) d\eta - C T^{-1-\alpha} \int_{\mathbb{R}^{2n+1}} u_0(\eta) \phi_R(\eta) d\eta \\ & \leq C I_1^{\frac{1}{p}} \left(R^{\frac{Q}{p'}-\sigma} T^{\frac{1}{p'}-\alpha-1} + R^{\frac{Q}{p'}} T^{\frac{1}{p'}-2-\alpha} \right) + C I_2^{\frac{1}{p}} R^{\frac{Q}{p'}-2} T^{\frac{1}{p'}-\alpha}. \end{aligned} \tag{21}$$

Hence, one has

$$\begin{aligned} & I_1 + C T^{-\alpha} \int_{\mathbb{R}^{2n+1}} (u_1(\eta) + (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}} u_0(\eta)) \phi_R(\eta) d\eta \\ & \leq C I_1^{\frac{1}{p}} R^{\frac{Q}{p'}} T^{\frac{1}{p'}-\alpha} \left(T^{-2} + T^{-1} R^{-\sigma} \right) + C I_2^{\frac{1}{p}} R^{\frac{Q}{p'}-2} T^{\frac{1}{p'}-\alpha} + C T^{-1-\alpha} \int_{\mathbb{R}^{2n+1}} u_0(\eta) \phi_R(\eta) d\eta. \end{aligned} \tag{22}$$

Because of the assumption (12) and the fact that $\phi_R(\eta) \to 1$, as $R \to \infty$, there exists a sufficiently large constant $R_0 > 0$ such that it holds

$$\int_{\mathbb{R}^{2n+1}} (u_1(\eta) + (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}} u_0(\eta)) \phi_R(\eta) d\eta > 0, \tag{23}$$

for all $R > R_0$. It is easy to see that the inequality (13) is equivalent to $1 - \alpha p' + \frac{Q}{2-\tilde{\sigma}} - \frac{2p'}{2-\tilde{\sigma}} \leq 0$. So, we have to consider two cases:

1. **Case 1**. If $1-\alpha p'+\frac{Q}{2-\tilde{\sigma}}-\frac{2p'}{2-\tilde{\sigma}}<0$, then we take $R=T^{\frac{1}{2-\tilde{\sigma}}}$. Therefore, we clearly have from (22) and (23) that

$$I_1\le CI_1^{\frac{1}{p}}T^{\frac{1}{p'}-\alpha+\frac{Q}{(2-\tilde{\sigma})p'}-\frac{2}{2-\tilde{\sigma}}}+CT^{-1-\alpha}\int_{\mathbb{R}^{2n+1}}u_0(\eta)\phi_R(\eta)d\eta.$$

Thanks to the following Young inequality:

$$ab\le\frac{a^p}{p}+\frac{b^{p'}}{p'}\quad\text{for all}\quad a,b>0,$$

with

$$a=I_1^{\frac{1}{p}},\quad b=T^{\frac{1}{p'}-\alpha+\frac{Q}{(2-\tilde{\sigma})p'}-\frac{2}{2-\tilde{\sigma}}},$$

we conclude that

$$\frac{1}{p'}I_1\le\frac{C}{p'}T^{1-\alpha p'+\frac{Q}{2-\tilde{\sigma}}-\frac{2p'}{2-\tilde{\sigma}}}+CT^{-1-\alpha}\int_{\mathbb{R}^{2n+1}}u_0(\eta)\phi_R(\eta)d\eta. \tag{24}$$

It is clear that if $1-\alpha p'+\frac{Q}{2-\tilde{\sigma}}-\frac{2p'}{2-\tilde{\sigma}}<0$, then by letting $T\to+\infty$, we deduce that $u\equiv 0$. By invoking (22), we easily obtains

$$\int_{\mathbb{R}^{2n+1}}(u_1(\eta)+(-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}u_0(\eta))\phi_R(\eta)d\eta\le CT^{-1}\int_{\mathbb{R}^{2n+1}}u_0(\eta)\phi_R(\eta)d\eta.$$

Hence, passing to the limit in the above inequality as T goes to $+\infty$, one obtains a contradiction to (12).

2. **Case 2**. If $1-\alpha p'+\frac{Q}{2-\tilde{\sigma}}-\frac{2p'}{2-\tilde{\sigma}}=0$, then we can see the following estimate from (24)

$$\frac{1}{p'}I_1\le\frac{C}{p'}+CT^{-1-\alpha}\int_{\mathbb{R}^{2n+1}}u_0(\eta)\phi_R(\eta)d\eta.$$

Hence, it follows that $I_1\le C$, as $T\to+\infty$. By the dominated convergence theorem, one has

$$\int_0^{+\infty}\int_{\mathbb{R}^{2n+1}}|u(\eta,t)|^p d\eta dt=\lim_{T\to+\infty}\int_0^T\int_{\mathbb{R}^{2n+1}}|u(\eta,t)|^p\tilde{\varphi}(\eta,t)d\eta dt=\lim_{T\to+\infty}I_1\le C,$$

which yields, $u\in\mathbb{L}^p\left((0,+\infty)\times\mathbb{R}^{2n+1}\right)$. On the other hand, repeating the same calculations as above, with $R=T^{\frac{1}{2-\tilde{\sigma}}}L^{-\frac{1}{2-\tilde{\sigma}}}$, where $1\le L<R$ is large enough such that when $R\to+\infty$ do not have $L\to+\infty$ at the same time, we arrive at

$$\begin{aligned}I_1\le{}& CI_1^{\frac{1}{p}}\left(T^{-\frac{2(1-\tilde{\sigma})}{2-\tilde{\sigma}}}L^{-\frac{Q}{(2-\tilde{\sigma})p'}}+T^{-\frac{\sigma-\tilde{\sigma}}{2-\tilde{\sigma}}}L^{-\frac{Q}{(2-\tilde{\sigma})p'}+\frac{2}{2-\tilde{\sigma}}}\right)+I_2^{\frac{1}{p}}L^{-\frac{Q}{(2-\tilde{\sigma})p'}+\frac{2}{2-\tilde{\sigma}}}\\&+CT^{-1-\alpha}\int_{\mathbb{R}^{2n+1}}u_0(\eta)\phi_R(\eta)d\eta.\end{aligned}$$

Therefore, using Young's inequality with

$$\begin{cases}a=I_1^{\frac{1}{p}},\quad b=T^{-\frac{2(1-\tilde{\sigma})}{2-\tilde{\sigma}}}L^{-\frac{Q}{(2-\tilde{\sigma})p'}},\\[2mm] a=I_1^{\frac{1}{p}},\quad b=T^{-\frac{\sigma-\tilde{\sigma}}{2-\tilde{\sigma}}}L^{-\frac{Q}{(2-\tilde{\sigma})p'}+\frac{2}{2-\tilde{\sigma}}},\end{cases}$$

one concludes that

$$\frac{1}{p'}I_1 \le \frac{C}{p'}\left(T^{-\frac{2(1-\tilde{\sigma})p'}{2-\tilde{\sigma}}}L^{-\frac{Q}{2-\tilde{\sigma}}} + T^{-\frac{(\sigma-\tilde{\sigma})p'}{2-\tilde{\sigma}}}L^{-\frac{Q-\sigma p'}{2-\tilde{\sigma}}}\right) + I_2^{\frac{1}{p}}L^{-\frac{Q}{(2-\tilde{\sigma})p'}+\frac{2}{2-\tilde{\sigma}}}$$
$$+ CT^{-1-\alpha}\int_{\mathbb{R}^{2n+1}} u_0(\eta)\phi_R(\eta)d\eta. \tag{25}$$

We have to distinguish two cases:

- If $\sigma \in (0,1]$, then $\sigma = \tilde{\sigma}$. Consequently, using the fact that $u \in \mathbb{L}^p\left((0,+\infty)\times \mathbb{R}^{2n+1}\right)$, one has

$$\lim_{T\to+\infty} I_2 = \lim_{T\to+\infty}\int_{\{|\eta|_{\mathbb{H}} \ge T^{\frac{1}{2-\sigma}}L^{-\frac{1}{2-\sigma}}\}} |u(\eta,t)|^p\tilde{\varphi}(\eta,t)d\eta dt = 0.$$

Taking into account the inequality (25), it follows by letting $T\to+\infty$ that

$$\int_0^{+\infty}\int_{\mathbb{R}^{2n+1}} |u(\eta,t)|^p d\eta dt \le C\left(L^{-\frac{Q}{2-\sigma}} + L^{-\frac{Q-\sigma p'}{2-\sigma}}\right).$$

Applying similar arguments as in Case 1, one concludes the desired result.

- If $\sigma \in (1,2]$, then $\tilde{\sigma} = 1$. Due to the fact that $u \in \mathbb{L}^p\left((0,+\infty)\times\mathbb{R}^{2n+1}\right)$, we get

$$\lim_{T\to+\infty} I_2 = \lim_{T\to+\infty}\int_0^T\int_{\{|\eta|_{\mathbb{H}}\ge TL^{-1}\}} |u(\eta,t)|^p\tilde{\varphi}(\eta,t)d\eta dt = 0,$$

which implies, as $T\to+\infty$, that

$$\int_0^{+\infty}\int_{\mathbb{R}^{2n+1}} |u(\eta,t)|^p d\eta dt \le CL^{-Q}.$$

Employing similar arguments as in Case 1, one obtains a contradiction with the fact that

$$\int_{\mathbb{R}^{2n+1}}\left(u_1(\eta) + (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}\right)u_0(\eta)d\eta > 0.$$

3. **Case 3**. If $p < \dfrac{1}{\gamma}\cdot$ Substituting $R = Ln(T)$ in (22), we may immediately derive

$$I_1 + CT^{-\alpha}\int_{\mathbb{R}^{2n+1}}\left(u_1(\eta) + (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}u_0(\eta)\right)\phi_R(\eta)d\eta$$
$$\le CI_1^{\frac{1}{p}}(Ln(T))^{\frac{Q}{p'}}T^{\frac{1}{p'}-\alpha}\left(T^{-2} + T^{-1}R^{-\sigma}\right) \tag{26}$$
$$+ CI_2^{\frac{1}{p}}(Ln(T))^{\frac{Q}{p'}-2}T^{\frac{1}{p'}-\alpha} + CT^{-1-\alpha}\int_{\mathbb{R}^{2n+1}} u_0(\eta)\phi_R(\eta)d\eta.$$

Letting $R\to+\infty$ in the above inequality, one obtains

$$\int_{\mathbb{R}^{2n+1}}\left(u_1(\eta) + (-\Delta_{\mathbb{H}})^{\frac{\sigma}{2}}u_0(\eta)\right)d\eta \le 0,$$

where we have used the fact that $\dfrac{1}{p'} - \alpha < 0$. This is the desired contradiction. Summarizing, the proof of Theorem 3.1 is completed.

□

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