Asia Mathematatika
Volume: 5 Issue: 3 , (2021) Pages: 125 – 131
Available online at **www.asiamath.org**

Restriction and induction functors for the cyclic group algebra modules

Ahmed A. Khammash[1]* **and Makkiah S. Makki**[2]
[1]Department of Mathematical Sciences, Umm Al-Qura University,
Makkah, Saudi Arabia. Orcid iD:0000-0001-9404-1732 0000-0001-9404-1732
[2]Department of Mathematical Sciences, Umm Al-Qura University,
Makkah, Saudi Arabia. ORCID iD:0000-0002-7319-2512 0000-0002-7319-2512

Received: 16 Sep 2021 • **Accepted:** 01 Sep 2021 • **Published Online:** 30 Dec 2021

Abstract: We investigate the effect of the restriction and induction functors on the indecomposable modules of finite cyclic p-group algebra over a field of characteristic. Such functors are significant in studying modules in blocks with cyclic defect groups.

Key words: Cyclic p-groups, Restriction Functor, Induction Functor, Vertex, Source, Cyclic defect.

1. Introduction

It is known that the cyclic p-group $G = C_q =< x : x^q = 1 >$; $q = p^a$ and p is a prime number has q non-isomorphism classes of indecomposable modules over a field of characteristic p (Higman [6]). The Green ring of G has been a subject of studies by many authors such as Green [5], Srinivasan [12] and T. Ralley [11]. The aim of this paper is to determine the effect of the induction and restriction functors on the indecomposable kG-modules. The induction functor is straightforward (Theorem 3.1); thanks to the Green indecomposable theorem, while the restriction functor turns out to be less obvious. We also determine the vertices and sources of the indecomposable kG-modules (Theorem 3.5). Restriction and induction of indecomposable modules for cyclic groups are essential in dealing with modules in blocks with cyclic defect groups (see for instance [9], [7]).

2. The indecomposable modules

Let $G = C_q =< x : x^q = 1 >$ be the cyclic group of order $q = p^a$ for some prime p and let k be a field of characteristic p. We then have $kG = \sum_{i=0}^{q-1} kx^i = kG = \sum_{i=0}^{q-1} k(x-1)^i$. On the other hand, since $(x-1)^q = \sum_{0\leq i\leq q}(-1)^i x^{q-i} = x^q - \binom{q}{1}x^{q-1} + \cdots - (1)^q$. we conclude, by taking $u = (x-1)$. that $kG = k - alg < u/u^q = 0 >$. It is known, by Higmann criterion (see [2], Theorem 62.21), that such a group algebra is of finite representation type and has q classes of indecomposable kG-modules $V_i = (V_i^a); i = 1, 2, \ldots .q$. where $V_i = kG/(x-1)^i kG.dim_k V_i = i$.

Theorem 2.1. *Every indecomposable kG-module is isomorphic to one of $V_i = (V_i^a); i = 1, 2, \ldots .q$.*

Proof. 1. V_i has a k-basis

$$\bar{1}, \bar{u}, \bar{u}^2, \ldots .\bar{u}^{i-1} \tag{1}$$

*Correspondence: prof.khammash@gmail.com

where $\bar{u} = u + (x-1)^i kG$. We notice that $x\bar{u} = x((x-1) + u^i kG) = (x-1) + u^i kG + (x-1)^2 + u^i kG = \bar{u} + \bar{u}^2$ Hence $x\bar{u}^\sigma = \bar{u}^\sigma + \bar{u}^{\sigma+1}$. therefore relative to the basis (1) we get a matrix representation

$$x \mapsto \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & . & & \\ & & . & . & \\ & & & 1 & 1 \end{pmatrix}_{i\times i} = \rho(x)$$

Note that $x\bar{u}^{i-1} = \bar{u}^{i-1} + \bar{u}^i = \bar{u}^{i-1}$ since $\bar{u}^i = u^i + u^i kG = \bar{0}$. So if you look at the commutant (endomorphism) algebra of V_i

$$E(\rho) = \{T \in M_q(k)/T\rho(x) = \rho(x)T\} = \begin{pmatrix} a & & & & & \\ b & a & & & & \\ . & . & . & & & \\ . & . & . & . & & \\ c & . & . & . & & \\ d & c & . & . & b & a \end{pmatrix} /a, b, \ldots .c, d \in k\}$$

we notice at once that it is a local algebra, since $F^2 = F; F \in E(\rho) \Rightarrow F = 0 \vee F = I_q$ Hence V_i is indecomposable.

2. It is clear that if $i \neq j$ then $i = dim_k V_i \neq dim_k V_j = j$ and so $V_i \ncong V_j$.

3. To show that Every indecomposable kG-module is isomorphic to one of $V_i = (V_i^a); i = 1, 2, \ldots, q$, we note that $kG = k - alg < u/u^q = 0 >\cong k[t]/t^q k[t]$, where k[t] is the polynomial ring in t with coefficients in k. Therefore any kG -module U can be considered a left $k[t]$ -module by the action $(k[t] \times U \to U; [(f, u) \mapsto (f + t^q k[t])u])$ and $U = V_{i_1} \oplus V_{i_2} \oplus V_{i_k}$, where $i_j \in \{1, 2, \ldots .q\}$. Since the polynomial ring $k[t]$ is a principle ideal domain, we have $U = k[t]^s \oplus k[t]/f_1 k[t] \oplus \ldots k[t]/f_r k[t]$ with $s = 0$; since kG is torsion, also f_i is non-unit and $f_i/f_{i+1}; \forall i = 1, 2, \ldots, r-1$ (See [3],theorem2(10.6)). It follows that $U = k[t]/f_1 k[t] \oplus \ldots k[t]/f_r k[t]$. Now $t^q U = 0 \Rightarrow t^q(k[t]/f_i k[t]), \forall i \Rightarrow t^q k[t] \subset f_i k[t] \Rightarrow t^q = f_i a.a \in k[t]$. But since $k[t]$ is a unique factorization domain, we have $f_i = \lambda t^s, \lambda \in k^x, 1 \leq s \leq q$. Hence $f_i k[t] = t^s k[t]$ and $k[t]/f_i k[t] = k[t]/t^s k[t] \cong V_s$. This completes the proof. □

. It is easy to deduce that V_i is projective kG-module if and only if $i = w$,in which case $V_i = V_q = kG$. The first notion need to be investigated is the projective cover of nonprojective indecomposable kG-modules.

Theorem 2.2. *([2], Exercise 6 page 829) For all $1 \leq i \leq q$ we have:*

1. $\Omega(V_i) \cong V_{q-i}$
2. $\Omega^2(V_i) \cong V_i$

Proof. 1. Define $\epsilon_i : V_q(\cong kC_q) \to V_i$ as follows: $\epsilon_i(\bar{1}) = 1 + u^i kG$.It is clear ϵ_i is a kG-epimormism with $Ker\epsilon = V_{q-i} \leq J(V_q)$. since $J(V_q) \lhd_{max} V_q$. Therefore ϵ_i is essential and so $V_q(\cong kC_q)$ is the projective cover of each V_i and $\Omega(V_i) = Ker\epsilon_i = V_{q-i}$.

2. By the same procedure as in (1), we see that $\Omega^2(V_i) = V_i$. □

It follows from 2.2(1) that the indecomposable modules V_i, V_{q-i}; being the two ends of the minimal projective presentation $0 \to V_{q-i} \to V_q(\cong kC_q) \to V_i \to 0$. have the same vertices (we shall determine Vertex (V_i) lately in this paper). It is known that all the subgroups of the cyclic group $G = C_q$ are cyclic and are indexed by the divisors of the number q (see [10], Theorem 4). We now study the effect of the induction and restriction functors on the modules $V_i = (V_i^a); i = 1, 2, \dots .q$.

3. The induction functor

Now let $H = G_\alpha =< x^{p^\alpha} >= \{1, x^{p^\alpha}, x^2p^\alpha, \dots, x^{(p^{(\alpha-\alpha)-1})p^\alpha}\}$. Then it is clear that $G_\alpha \leq G = C_q(= C_a)$, and $G = H \cup Hx \cup \cdots \cup Hx^{p^\alpha - 1}$ with $\mid G : G_\alpha \mid = p^\alpha$. We start by investigating the effect of the induction functor. According to theorem 1.1, the group $H = G_\alpha$ have indecomposable kH-modules $V_j^{(\alpha)}; j = 1, 2, \dots .p^{\alpha-\alpha}$. By the Green indecomposability theorem (see[1], Theorem 19.22] the induced kG-module $(V_j^{(a-\alpha)})^G$ (as G/G_α is p-group) is indecomposable of dimension $j \mid G : H \mid$. Hence, by 2.1 $(V_j^{(a-\alpha)}) \uparrow^G \cong V^a_{j|G:H|}$. Summarizing we have:

Theorem 3.1. $(V_j^{(a-\alpha)}) \uparrow^G \cong V^a_{j|G:G_\alpha|} = V^a_{jp^\alpha}$

The following theorem determines the subgroups G_α of $G = C_q$ for which the module $V_i = V_i^a$ is relative projective which is a step towards finding the vertices of the indecomposable modules $V_i = V_i^a$.

Theorem 3.2. *The module $V_i = V_i^a; 1 \leq i \leq p^\alpha$ is a G_α-projective, where $G_\alpha =< x^{p^\alpha} >$. if and only if $p^\alpha \mid i$.*

Proof. $V_i = V_i^a$ is G_α-projective if and only if $V_i \mid (V_j^\alpha) \uparrow^G$ for some $1 \leq j \leq p^{a-\alpha}$. But by Green's theorem $V_j^{(a-\alpha)}$ is indecomposable kG-module of dimension $j \mid G : G_\alpha \mid = jp^\alpha$. Hence $V_i = V_i^a$ is G_α -projective if and only if $V_i \cong V_j^{(a-\alpha)}$. Comparing dimensions, we then have $V_i = V_i^a$ is G_α-projective if and only if $i = jp^\alpha$, that is if and only if $p^\alpha \mid i$. □

We now consider the vertex $vx(V)$ and the source $sc(V)$ of indecomposable $K(g = C_q)$-modules. We shall use the following known general fact which relates the dimension of an indecomposable kG-module to the index of its vertex in a Sylow p-subgroup of G.

Theorem 3.3. *([1],Theorem.19.26) If $V \in indec(kG)$ and $S \in vx(V)$ then $\mid G_p : S \mid\mid dimV$. So if $p \nmid dimV$ then $G_p \in vx(V)$.*

The previous theorem implies at once the following:

Lemma 3.1. *If $p \nmid dimV_i = i$ then $vx(V_i) = G = C_q$.*

It remains then; in order to determine the vertex of the indecomposable kG-modules V_i, to consider $vx(V_i)$ in the case when $p \mid dimV_i = i$

Theorem 3.4. *For $V_i = V_i^{(\alpha)} \in indec(kG)$ we have*

1. $vx(V_i) =< x^{q-i} >$

2. $sc(V_i) = V^{a-\alpha}_{i/p^\alpha}$. *(Note:* $i/p^\alpha \leq p^{a-\alpha} = p^a/p^\alpha$ *since* $i \leq p^a$ *)*

Proof. (1) We need to distinguish two cases:

(I) If $p \nmid dimV_i = i$ then $vx(V_i) = G = C_q =< x^{q-i} >$, since $hcf(q, q-i) = 1$ which implies that $O(x^{q-i}) = O(x)$.

(II) If $p \mid dimV_i = i$; say $i = p^m t$, m is maximal, then $hcf(q, q-i) = p^m \neq 1$ and so $O(x^{q-i}) = q/hcf(q, q-i) = p^{a-m}$. Hence $< x^{q-i} >= G_m$. On the other hand $V_i = (V_t^{a-m}) \uparrow^G$ and $< x^{q-i} >= G_m$ is minimal subgroup with this property by the choice of m. this completes the proof.

(2) Clear since $V^{a-\alpha}_{i/p^\alpha} = V^a_{i/p^\alpha . p^\alpha} = V_i^a$; by 2.1. □

As a corollary to the previous theorem we have the following:

Corollary 3.1. $vx(V_i) = vx(V_{q-i})$

Proof. $vx(V_i) =< x^{q-i} >=< x^q x^{-i} >=< x^{-i} >=< x^i >=< x^{q-(q-i)} >= vx(V_{q-i})$. □

4. The restriction functor

We now take $H = G_\alpha =< x^{p^\alpha} >= \{1, x^{p^\alpha}, x^2 p^\alpha, \ldots, x^{(p^{(\alpha-\alpha)-1})P^\alpha}\}$. Then $G_\alpha \leq G$ with $\mid G_\alpha \mid= O(x)/p^\alpha = p^a/p^\alpha = p^{a-\alpha}$, $G = H \cup Hx \cup \cdots \cup Hx^{p^\alpha - 1}$ and $\mid G : G_\alpha \mid= p^\alpha$. Consider the restriction module $(V_i^a) \downarrow_H$ to the subgroup $H = G_\alpha$. Let $v = x^{p^\alpha} - 1 = (x-1)^{p^\alpha} = u^{p^\alpha} \in kH$, then $v^{p^{a-\alpha}} = 0$ and the element v generates the group algebra $kH = kG_\alpha$ of $H = G_\alpha$; i.e. $kG_\alpha =< v = u^{p^\alpha}/(u^{p^\alpha})^{p^{a-\alpha}} = 0 >$. We need to know how v acts on basis 1 of $V_i = V_i^a$. We have $v\bar{u}^\sigma = u^{p^\alpha}\bar{u}^\sigma = \bar{u}^{\sigma+p^\alpha}$ and so $\bar{1} \mapsto \bar{u}^{p^\alpha}, \bar{u} \mapsto \bar{u}^{1+p^\alpha}, \ldots .etc$. Write $i = i_0 + i_1 p + i_2 p^2 + \cdots + i_v p^\alpha, 0 \leq i_k \leq p-1$. Hence the generator $v = x^{p^\alpha} - 1 = (x-1)^{p^\alpha} = u^{p^\alpha} \in kG_\alpha$ of kG_α is represented on $V_i = V_i^a$ by the matrix:

$$\begin{pmatrix} & & & & & 1 & & \\ & & & & & & \cdot & \\ 1 & & & & & & & 1 \\ & 1 & & & & & & \\ & & \cdot & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \end{pmatrix}$$

Note that: $v\bar{1} = u^{p^\alpha}\bar{1} = \bar{u}^{p^\alpha}, v\bar{u} = u^{p^\alpha+1}, v\bar{u}^2 = u^{p^\alpha+2}, \ldots, v\bar{u}^{(i-1)-p^\alpha} = \bar{u}^{i-1}, v\bar{u}^{i-p^\alpha} = \bar{1}, v\bar{u}^{i-p^\alpha+1} = \bar{u}, v\bar{u}^{i-p^\alpha+2} = \bar{u}^2, \ldots, v\bar{u}^{i-1}(v\bar{u}^{i-p^\alpha+(p^\alpha-1)}) = \bar{u} = \bar{u}^{p^\alpha-1}$

Example 4.1. *(1)* $p = 2, q = 2^4, \mid H \mid= p^{a-\alpha} = p = 2 (\alpha = 3), H =< x^8 >, (x-1)^8 = u^8 \in kH$ *and ;* $kH = k - alg < u^8 >, V_9 (= V_9^{(4)}) =< \bar{1}, \bar{u}, \bar{u}^2, \ldots, \bar{u}^8 >; \bar{u} = u + u^9 kG$. *Consider the restriction* $(V_9^{(4)})_{<x^8>}$ *to* $C_3 =< x^8 >$*, (note that* $\mid C_3 \mid= 2^{4-3} = 2$ *hence the largest dimension of an indecomposable* kC_3*-module is 2). Since* $u^8\bar{u}^i = \bar{u}^{i+8}$*, and since* $\bar{u}^9 = 0$*, the generator* u^8 *of* kH *is represented on* $V_9^{(4)}$ *by the matrix:*

$$u^8 \mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ 1 & & & & & & 1 \end{pmatrix}$$

It is clear that $x^8\bar{1} = \bar{1} + \bar{u}^8$ *also* $x^8\bar{u} = \bar{u}$ *since* $\bar{u}^9 = 0$ *in* $V_9^{(4)}$. *The fact that* $x\bar{u}^\sigma = \bar{u}^\sigma + \bar{u}^{\sigma+1}$ *and* $char(k) = 2$ *implies that* $x^i\bar{u} = \bar{u} + \bar{u}^{i+1}$. *Therefore* $x^8\bar{u}^2 = \bar{u}^2, x^8\bar{u}^3 = \bar{u}^3, \ldots, x^8\bar{u}^7 = \bar{u}^7, x^8\bar{u}^8 = \bar{u}^8$ *which implies that* $(V_9^{(4)})_{<x^8>} = (k\bar{1} + k\bar{u}^8) \oplus k\bar{u} \oplus \bar{u}^2 \oplus k\bar{u}^3 \oplus \cdots \oplus k\bar{u}^3$.

$$x^8 \mapsto \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \; (1) \; (1) \; (1) \; \ldots \; (1)$$

Hence $(V_9^{(4)}) \downarrow_{<x^8>} = V_2^{(1)} \oplus V_1^{(1)} \oplus \cdots \oplus V_1^{(1)} = V_2^{(1)} \oplus 7V_1^{(1)}$.

2. *(2) Similarly we consider the restriction of* $(V_9^{(4)})$ *to* $C_2 =< x^4 >$, *(note that* $| C_2 |= 2^{4-2} = 4 = 2^2$, *hence the largest dimension of an indecomposable* kC_2*-module is* 4. *The following matrix represents the* $C_2 =< x^4 >$*-action on* $(V_9^{(4)})$.

$$u^4 \mapsto \begin{pmatrix} & & & & & 1 & & \\ & & & & & & 11 & \\ & & & & & & & 1 \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 11 & & & \end{pmatrix}$$

Hence $x^4\bar{1} = \bar{1} + \bar{u}^4, x^4\bar{u}^2 = \bar{u}^2 + \bar{u}^6, x^4\bar{u}^3 = \bar{u}^3 + \bar{u}^7, x^4\bar{u}^4 = \bar{u}^4 + \bar{u}^8, x^4\bar{u}^5 = \bar{u}^5$, *since* $\bar{u}^9 = 0$ *in* $(V_9^{(4)})$. *Also* $x^4\bar{u}^6 = \bar{u}^6$ *; since* $\bar{u}^{10} = \bar{u}^9\bar{u} = \bar{0}\bar{u} = \bar{0}$ *and* $x^4\bar{u}^7 = \bar{u}^7, x^4\bar{u}^8 = \bar{u}^8$. *Therefore* $(V_9^{(4)})_{<x^4>} = (k\bar{1} \oplus k\bar{u}^4 \oplus k\bar{u}^8) \oplus (k\bar{u} \oplus \bar{u}^5) \oplus (k\bar{u}^2 \oplus k\bar{u}^6) \oplus (k\bar{u}^3 \oplus k\bar{u}^7)$.

$$x^4 \mapsto \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$$

Hence $(V_9^{(4)})_{<x^4>} = V_3^{(2)} \oplus 3V_2^{(2)}$.

(3) *Now consider the restriction of* $(V_9^{(4)})$ *to* $G_1 =< x^2 >$. *It follows that* $\alpha = 1, p^\alpha = 2$ *and that* $| G_1 |= 8$. *We also have* $(V_9^{(4)})_{G_1} = V_8^{(1)} \oplus V_1^{(1)}$. *Note also that in the previous two cases, the number of indecomposable summands of* $(V_i^{(a)}) \downarrow_{G_\alpha}; 1 \leq i \leq p^\alpha = q$. *(with possible repetition) equals* p^α *provided that* $i > p^\alpha$.
What about the case when $i \leq p^\alpha$. *Consider the following case:*

(4) *Now we consider the restriction of* $(V_5^{(4)})$ *to the subgroup* $G_3 =< x^8 >$ *has k-basis* $\bar{1}, \bar{u}, \bar{u}^2, \bar{u}^3, \bar{u}^4$. *As we saw before we have:* $x^j\bar{u} = \bar{u} + \bar{u}^{j+1}$ *and so* $x^j\bar{u}^t = \bar{u}^t + \bar{u}^{j+t}$ $x^8\bar{1} = \bar{1} + \bar{u}^8 = \bar{1}$, *since* $\bar{u}^8 = \bar{0}$ *as* $\bar{u}^5 = \bar{0}$ $x^8\bar{u} = \bar{u}, x^8 + \bar{u}^2 = \bar{u}^2, x^8 + \bar{u}^3 = \bar{u}^3, x^8 + \bar{u}^4 = \bar{u}^4$. *Therefore* $(V_5^{(4)}) \downarrow_{G_3} = 5(V_1^{(3)})$

In order to study the restriction of the indecomposable kG-module V_i to a subgroup we need to investigate the action of the group generators on the basis elements of V_i. First we have the following lemma:

Lemma 4.1. *In* V_i, $x^j\bar{1} = \sum_{s=0}^{j} \binom{j}{s} \bar{u}^s$.

Proof. Note first that $x\bar{1} = x + (x-1)^i kG = 1 + (x-1) + (x-1)^i kG = \bar{1} + \bar{u}$, $x^2\bar{1} = x\bar{1} + x\bar{u} = \bar{1} + \bar{u} + \bar{u} + \bar{u}^2$, $x^3\bar{1} = x\bar{1} + 2x\bar{u} + x\bar{u}^2 = \bar{1} + \bar{u} + 2(\bar{u} + \bar{u}^2) + \bar{u}^2 + \bar{u}^3 = \bar{1} + 3\bar{u} + 3\bar{u}^2 + \bar{u}^3$, $x^4\bar{1} = x\bar{1} + 3x\bar{u} + 3x\bar{u}^2 + x\bar{u}^3 = \bar{1} + 4\bar{u} + 6\bar{u}^2 + 4\bar{u}^3 + \bar{u}^4, \ldots$. Hence $x^j\bar{1} = \sum_{s=0}^{j} \binom{j}{s} \bar{u}^s$. □

Also we have

Lemma 4.2. *In V_i, we have* $x^j\bar{u}^t = \sum_{s=0}^{j}\binom{j}{s}\bar{u}^{t+s}$.

Proof. We have: $x\bar{u}^t = x(x-1)^t + (x-1)^i kG = (x-1)(x-1)^t + (x-1)^t + u^i kG = \bar{u}^t + \bar{u}^{t+1}$, $x^2\bar{u}^t = xx\bar{u}^2 = x(\bar{u}^t + \bar{u}^{t+1}) = \bar{u}^t + \bar{u}^{t+1} + \bar{u}^{t+1} + \bar{u}^{t+2} = \bar{u}^t + 2\bar{u}^{t+1} + \bar{u}^{t+2}$, $x^3\bar{u}^t = xx^2\bar{u}^t = x(\bar{u}^t + 2\bar{u}^{t+1} + \bar{u}^{t+2}) = \bar{u}^t + \bar{u}^{t+1} + 2(\bar{u}^{t+1} + \bar{u}^{t+2}) + \bar{u}^{t+2} + \bar{u}^{t+3} = \bar{u}^t + 3\bar{u}^{t+1} + 3\bar{u}^{t+2} + \bar{u}^{t+3}$, $x^4\bar{u}^t = xx^3\bar{u}^t = x(\bar{u}^t + 3\bar{u}^{t+1} + 3\bar{u}^{t+2} + \bar{u}^{t+3}) = \bar{u}^t + \bar{u}^{t+1} + 4(\bar{u}^{t+1} + \bar{u}^{t+2}) + 6(\bar{u}^{t+2} + \bar{u}^{t+3}) + 4(\bar{u}^{t+3} + \bar{u}^{t+4}) + \bar{u}^{t+4} + \bar{u}^{t+5} = \bar{u}^t + 5\bar{u}^{t+1} + 10\bar{u}^{t+2} + 10\bar{u}^{t+3} + 5\bar{u}^{t+4} + \bar{u}^{t+5}$. Hence $x^j\bar{u}^t = \sum_{s=0}^{j}\binom{j}{s}\bar{u}^{t+s}$. □

Remark 4.1. *We can deduce the formula* $x^j\bar{1} = \sum_{s=0}^{j}\binom{j}{s}\bar{u}^s$ *of 4.1 from the formula* $x^j\bar{u}^t = \sum_{s=0}^{j}\binom{j}{s}\bar{u}^{t+s}$ *of 4.2 by letting* $\bar{1} = \bar{u}^0$.

Corollary 4.1. *In V_i, we have* $x^{p^\alpha}\bar{1} = \bar{1} + \bar{u}^{p^\alpha}$

Proof. Apply the formula in 4.1. □

Corollary 4.2. *In V_i, we have* $x^{p^\alpha}\bar{u}^t = \begin{cases} \bar{u}^t + \bar{u}^{t+p^\alpha} & : if \quad t + p^\alpha \le i-1 \\ \bar{u}^t & : if \quad t + p^\alpha > i-1 \end{cases}$

Proof. From the previous lemma we have: $x^{p^\alpha}\bar{u}^t = \sum_{s=0}^{p^\alpha}\binom{p^\alpha}{s}\bar{u}^{t+s} = \bar{u}^t + \bar{u}^{t+p^\alpha}$, since $\binom{p^\alpha}{s} = 0(mod p)$. Now we have two cases:

1. If $t + p^\alpha \le i-1$ then $x^{p^\alpha}\bar{u}^t = \bar{u}^t + \bar{u}^{t+p^\alpha}$
2. If $t + p^\alpha > i-1$ then $x^{p^\alpha}\bar{u}^t = \bar{u}^t$, since $\bar{u}^i = \bar{0}$, we have $\bar{u}^{t+p^\alpha} = \bar{0}$.

□

Lemma 4.3. *Suppose that $t \ne i-1$. Then* $x^{p^\alpha}\bar{u}^t = \bar{u}^t \Longleftrightarrow t + p^\alpha > i-1$

5. Main results

We now summarize the results in the previous section and present the main results in this paper which determines the number of indecomposable direct summands for the restriction of indecomposable $k(G = C_q)$ -modules to a subgroup of G. The following theorem determines the number of indecomposable summands in such decomposition.

Theorem 5.1. *The number of indecomposable $k(G_\alpha)$-summands of $(V_i^a)\downarrow_{G_\alpha}$ is:* $\begin{cases} p^\alpha & : if \quad i > p^\alpha \\ i & : if \quad i \le p^\alpha \end{cases}$

In the second case all the indecomposable summands are trivial.The following theorem determines basis for the indecomposable summands of the restriction module:

Theorem 5.2. $(V_i^a)\downarrow_{G_\alpha} = \sum_{j=0}^{p^\alpha-1}\bigoplus(V_{i_j}^{(\alpha)})$. *In fact:* $(V_i^a)_{G_\alpha} =< \bar{1}, \bar{u}^{p^\alpha}, \ldots, \bar{u}^{i_0 p^\alpha} > \oplus < \bar{u}, \bar{u}^{p^{1+\alpha}}, \ldots, \bar{u}^{1+i_1 p^\alpha} > \oplus \cdots \oplus < \bar{u}^{p^\alpha-1}, \ldots, \bar{u}^{2p^\alpha-1}, \ldots, \bar{u}^{2p^\alpha-1+i_{p^\alpha-1}p^\alpha} >$, *where i_j is the least number with $j + i_j p^{(\alpha)} < i$. Note that when $i \le p^\alpha$ this forces i_j to be 0 and hence every direct kG_α-summand in the above decomposition would be one dimensional in this case.*

References

[1] C. Curtis and R. Riener. Methods in representation theory I. John Wiley. New York 1981

[2] C. Curtis and R. Riener. Methods in representation theory II. John Wiley. New York 1985

[3] P. M. Cohn. Algebra Vol 2. John Wiley and Sons 1979.

[4] J.A. Green, On the indecomposable representation of a finite group, Math. Z. 70(1959), 430-445.

[5] J. A. Green, The modular representation algebra of a finite group, Illinoi J.Math. 6(1962), 607-619.

[6] D. Higman, Indecomposable representation at characteristic p, Duke Math. J. 21 (1954) 377-81.

[7] G. HISS and N. NAEHRIG, The classification of the indecomposable liftable modules in blocks with cyclic defect group, Bulletin of the LMS, Volume 44, Issue 5, 2012, 974–980.

[8] R. Knorr, Relative Projective Covers, in "Proc. Of symp. Mod. Representations of Finite Groups", Arhus University, 1978; Publication Series No.24, Math. Institute, Arhus University.

[9] S. KOSHITANI and C. LASSUEUR, Trivial source characters in blocks with cyclic defect groups, arXiv.org-math-arXiv:2003.05243v1.

[10] W. Ledermann, Introduction to group theory, Longman 1981

[11] T. Ralley, Decomposition of products of modular representations,

[12] B. Srinivasan, The modular representation ring of a cyclic p-group, Proc. LMS (3) 14(1964) 677-688.