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On Λ-fractional special relativity theory

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Abstract: Fractional relativity theory is developed in the Λ-fractional space, introduced by the recently proposed Λ-fractional derivative. Since the Λ-fractional derivative exhibits all the properties of a conventional derivative in the Λ-fractional space, conventional relativity theory, valid in Λ-fractional space, yields results that are pulled back to the initial space. Furthermore, Λ-fractional analysis is applied to relativistic electromagnetism.

Key words: Λ-fractional space, Λ-fractional derivative, fractional special relativity, fractional relativistic Maxwell's equations

1. Introduction

Fractional calculus is a robust and intensely active mathematical field. Fractional derivatives were originated in the works of Leibniz [1], Riemann [2] and Liouville [3], and thereafter further developed by many other scientists (cf. Machado et al. [4] for a presentation of the important events in the evolution of fractional calculus). The main advantage of the applied mathematical procedure is the global description of various phenomena with temporal or spatial dependence. A strong relationship between fractional calculus and fractal geometry also seems to exist [5]. Therefore, fractional calculus has been employed in several scientific fields, especially so in physics [6–9, 13, 20] and mechanics [11, 12, 14, 21].

In the field of mechanics, viscoelasticity is a popular subject for the application of fractional calculus since it involves global temporal dependence. Many researchers have worked on the subject and developed successful fractional mathematical models [15, 16]. Lazopoulos, in particular, introduced fractional derivatives in spatially dependent descriptions of materials [11].

The most well-known fractional derivatives are the Riemann-Liouville, Caputo and Grünwald-Letnikov ones [8]. However, these mathematical operators exhibit a serious shortcoming: they do not fulfil the necessary properties of a derivative, according to differential topology, in order to correspond to a fractional differential. There are three prerequisites for defining a derivative corresponding to a differential [10]:

(i) *Linearity*: $D(a\,f(x)+b\,g(x))=a\,Df(x)+b\,Dg(x)$

(ii) *Leibniz rule*: $D(f(x)\cdot g(x))=Df(x)\cdot g(x)+f(x)\cdot Dg(x)$

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(iii) *Chain rule*: $\quad D(g(f))(x) = Dg(f(x)) \cdot Df(x)$

Since the aforementioned derivatives do not meet all three necessary conditions for defining a differential corresponding to the derivative, they are not able to generate differential geometry. Hence, their use in the mathematical description of various phenomena in, among other fields, physics, biology, and economics is questionable. Researchers tried either to overcome this problem or prove that those derivatives cannot fulfil the requirements in any way [17]. For example, Lazopoulos et al. proposed the Leibniz fractional derivative (L-derivative) in order to formulate a fractional differential [12]. Unfortunately, the L-derivative did not comply with all requirements of differential topology as well, thus a more efficient version was introduced, termed the Λ-fractional derivative. The latter exhibits all properties of a proper derivative, in accordance to differential topology, and is therefore suitable for mathematical analysis [18]. The fractional Λ-derivative is not restricted to fractional calculus but introduces fractional mathematical analysis. Among other topics, it has been applied to formulate fractional Taylor series [24] and fractional differential geometry [25]. Furthermore, on the basis of the Λ-derivative, many problems in mechanics [19, 22, 23] and beyond [26, 27] have already been addressed.

Fractional relativistic analyses have already appeared in the literature. For example, Gonzalez has presented relativistic particle motion with linear dissipation [28, 29]. Since dissipation is more appropriately formulated in the context of fractional calculus, Vacaru [30], Munkhammar [31], and Wang [32], used conventional fractional derivatives for presenting fractional relativistic mechanics. Nevertheless, as pointed out previously, conventional fractional derivatives do not satisfy the properties demanded by differential topology in order to formulate differentials. The fractional Λ-derivative, on the other hand, can generate differential geometry in the Λ-space, wherein fractional derivatives are transformed into conventional (local) ones. Therefore, geometry can be generated in the Λ-space, where everything works conventionally, and subsequently the results can be transferred back into the initial space as functions. No fractional derivative is valid in the initial space. Since classical electromagnetism and special relativity have already been exhaustively discussed [33], the fractional relativistic electromagnetism is presented in the context of the present analysis.

2. The Λ-Fractional Derivative

A brief outline of fractional calculus is provided in the present section, while the interested reader is referred to the literature [10–12] for further information. For a real fractional order $0 < \gamma \leq 1$, the left and right fractional integrals are defined by:

$$ {}_aI_x^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(s)}{(x-s)^{1-\gamma}}\, ds, \tag{1} $$

$$ {}_xI_b^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_x^b \frac{f(s)}{(s-x)^{1-\gamma}}\, ds, \tag{2} $$

respectively, where $\Gamma(\gamma)$ is Euler's Gamma function. Further, the left Riemann-Liouville fractional derivative is defined by: Further, the left and right Riemann-Liouville (RL-FD) fractional derivatives are defined as

$$ {}_a^{RL}D_x^\gamma f(x) = \frac{d}{dx}\left({}_aI_x^{1-\gamma} f(x)\right) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_a^x \frac{f(s)}{(x-s)^\gamma}\, ds, \tag{3} $$

$$ {}_{x}^{RL}D_{b}^{\gamma}f(x) = \frac{d}{dx}\left({}_{x}I_{b}^{1-\gamma}f(x)\right) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int\limits_{x}^{b}\frac{f(s)}{(s-x)^{\gamma}}\,ds, \tag{4} $$

respectively. For the left fractional integral and derivative holds that

$$ {}_{a}^{RL}D_{x}^{\gamma}\left({}_{a}I_{x}^{\gamma}f(x)\right) = f(x). \tag{5} $$

A similar relation is true for the right fractional Riemann-Liouville derivative and integral.

The Λ-fractional derivative (Λ-FD) is defined as

$$ {}_{a}^{\Lambda}D_{x}^{\gamma}f(x) = \frac{{}_{a}^{RL}D_{x}^{\gamma}f(x)}{{}_{a}^{RL}D_{x}^{\gamma}x}. \tag{6} $$

Recalling the definition of the left RL-FD, Eq. (3), Λ-FD is expressed by

$$ {}_{a}^{\Lambda}D_{x}^{\gamma}f(x) = \frac{\frac{d\ {}_{a}I_{x}^{1-\gamma}f(x)}{dx}}{\frac{d\ {}_{a}I_{x}^{1-\gamma}x}{dx}} = \frac{d\ {}_{a}I_{x}^{1-\gamma}f(x)}{d\ {}_{a}I_{x}^{1-\gamma}x}. \tag{7} $$

Considering

$$ \begin{aligned} X &= {}_{a}I_{x}^{1-\gamma}x, \\ F(X) &= {}_{a}I_{x}^{1-\gamma}f(x(X)), \end{aligned} \tag{8} $$

the Λ-FD appears to behave as a conventional derivative in the fractional Λ-space $(X, F(X))$ with local properties. In fact, the Fractional Differential Geometry may be formulated as a conventional differential geometry in the fractional Λ-space $(X, F(X))$. Then the results may be transferred to the initial space invoking Eq. (5). Indeed, we may transfer the results from the Λ-fractional space to the initial one, using the relation

$$ f(x) = {}_{a}^{RL}D_{x}^{1-\gamma}F(X(x)) = {}_{a}^{RL}D_{x}^{1-\gamma}I^{1-\gamma}f(x). \tag{9} $$

Similar procedures may be followed for the right fractional derivatives.

3. Fractional Geometry in Λ-Space and its relation to the Initial Space

In order to clarify the ideas, let

$$ f(x) = x^{2}. \tag{10} $$

Then, for $a = 0$, the Λ-fractional plane $(X, F(X))$ is defined by

$$ X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)}, \tag{11} $$

$$ F(X) \;=\; {}_{0}I_{x}^{1-\gamma}f(x(X)) = \frac{1}{\Gamma(1-\gamma)}\int\limits_{0}^{x}\frac{s^{2}}{(x-s)^{1-\gamma}}\,ds = \frac{2x^{3-\gamma}}{\Gamma(4-\gamma)}. \tag{12} $$

Solving Eq. (11) for x yields

$$x = (\Gamma(3-\gamma)\ X)^{1/(2-\gamma)}, \tag{13}$$

and replacing to Eq. (12) results to

$$F(X) = \frac{2(\Gamma(3-\gamma)\ X)^{(3-\gamma)/(2-\gamma)}}{\Gamma(4-\gamma)}. \tag{14}$$

For $\gamma = 0.6$, the curve in the initial plane $(x, f(x))$, shown in Figure 1, corresponds to the fractional plane

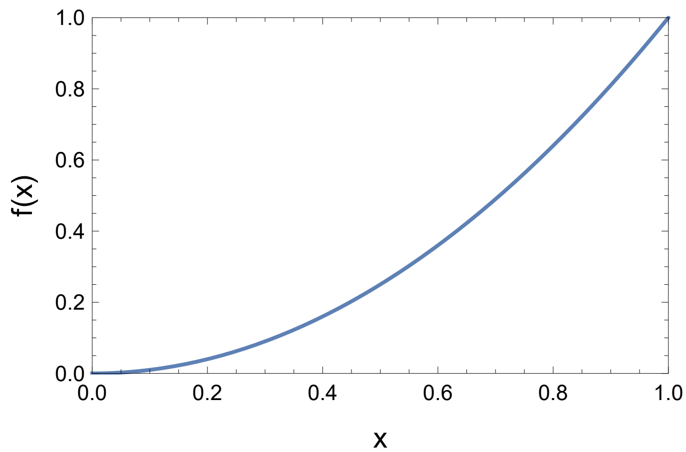

Figure 1. Initial plane $(x, f(x) = x^2)$.

(space) shown in Figure 2. Since the tangent space $Y(X)$ of the curve at a point X_0 is defined by the line

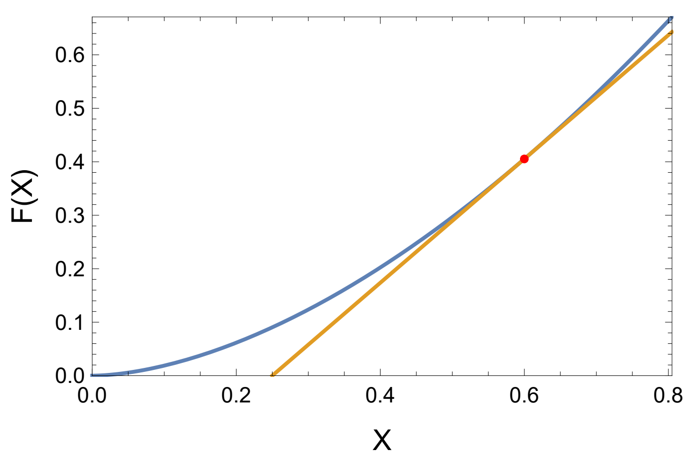

Figure 2. Curve with its tangent in fractional Λ-space $(X, F(X))$ for $\gamma = 0.6$.

$$Y(X) = F(X_0) + \frac{dF(X_0)}{dX}(X - X_0), \tag{15}$$

in the fractional plane the derivative

$$\frac{dF(X)}{dX} = \frac{24(\Gamma(3-\gamma)\ X)^{3/(2-\gamma)}}{(2-\gamma)\Gamma(6-\gamma)}, \tag{16}$$

for $\gamma = 0.6$ and $X_0 = 0.6$ is equal to $D(F(X_0)) = 1.34$ (Figure 2).

Let us point out that the geometry may not be transferred in the initial space, since formulation of the fractional differential is not possible in $(x, f(x))$. Functions may only be transferred from the fractional Λ-space to the initial space.

In addition, it is pointed out that for the derivation it is not necessary to express the various functions with respect to X. Indeed, since

$$\frac{dF(X)}{dX} = \frac{\frac{dF(X(x))}{dx}}{\frac{dX(x)}{dx}}, \tag{17}$$

there is no need to perform the substitution of x for the X variable. Calculus through the x variable is simpler but also effective.

4. Principles of Special Relativity Theory

The formulation of the theory of Special Relativity depends on the following two fundamental principles:

(i) all physical phenomena should be governed by the same physical laws in all inertial frames;

(ii) the velocity of light in vacuum is constant for all inertial observers and equal to $c \simeq 2.99792458\ m/s$.

Let $\Sigma(x, y, z, t)$, $\Sigma'(x', y', z', t')$ be two inertial systems, with the latter moving along the x-axis with constant relative velocity V. According to Special Relativity, the following two properties emerge:

($P1$) dilation of time

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{V^2}{c^2}}}, \tag{18}$$

($P2$) contraction of length along the x-axis

$$\Delta l' = \Delta l \sqrt{1 - \frac{V^2}{c^2}}. \tag{19}$$

Then the Lorentz transformation between the two inertial systems is expressed by

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \mathit{\Lambda}(V) \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}, \tag{20}$$

with

$$\begin{gathered} \mathit{\Lambda}(V) = \begin{pmatrix} \gamma(V) & -\beta(V)\gamma(V) & 0 & 0 \\ -\beta(V)\gamma(V) & \gamma(V) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \beta(V) = \frac{V}{c}, \\ \gamma(V) = \frac{1}{\sqrt{1 - \beta^2(V)}}, \\ \mathit{\Lambda}(V)\mathit{\Lambda}(-V) = \mathbf{1}, \end{gathered} \tag{21}$$

where $\mathbf{1}$ is the identity of 4×4 matrices.

If (u_x, u_y, u_z), (u'_x, u'_y, u'_z) are the components of velocity in Σ and Σ', respectively, then by means of the Lorentz transformation it follows that

$$\begin{aligned} u'_x &= \frac{u_x - V}{1-\frac{Vu_x}{c^2}}, \\ u'_y &= \sqrt{1-\frac{V^2}{c^2}}\frac{u_y}{1-\frac{Vu_x}{c^2}}, \\ u'_z &= \sqrt{1-\frac{V^2}{c^2}}\frac{u_z}{1-\frac{Vu_x}{c^2}}. \end{aligned} \tag{22}$$

In turn, for the corresponding components of acceleration (a_x, a_y, a_z), (a'_x, a'_y, a'_z) in Σ and Σ', respectively, we obtain

$$\begin{aligned} a'_x &= a_x \frac{(1-\frac{V^2}{c^2})^{3/2}}{(1-\frac{Vu_x}{c^2})^3}, \\ a'_y &= \frac{(1-\frac{V^2}{c^2})^{3/2}}{(1-\frac{Vu_x}{c^2})^3}\left(a_y + \frac{V}{c^2}(a_x u_y - a_y u_x)\right), \\ a'_z &= \frac{(1-\frac{V^2}{c^2})^{3/2}}{(1-\frac{Vu_x}{c^2})^3}\left(a_z + \frac{V}{c^2}(a_x u_z - a_z u_x)\right). \end{aligned} \tag{23}$$

The mass m of a body in not constant but depends upon its velocity $u = \|\mathbf{u}\|$, with $\mathbf{u}$ being the velocity vector, such that

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}, \tag{24}$$

where m_0 is the rest mass for the inertial observer. Subsequently, Newton's second law is defined by

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(\frac{m_0\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \frac{d}{dt}(m\mathbf{u}), \tag{25}$$

with $\mathbf{f}, \mathbf{p}$ the force and momentum vectors, respectively. Moreover, the kinetic energy of a point mass is defined as

$$K(u,m) = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2, \tag{26}$$

and the total energy as

$$E(u,m) = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} = mc^2 + K(u,m). \tag{27}$$

Further information regarding Special Relativity theory can be found in the literature [33].

5. Λ-Fractional Special Relativity Theory

According to Λ-fractional theory, a fractional derivative corresponding to a differential exists only in Λ-space, and only therein may a fractional differential geometry be established. Therefore, physical laws can be developed in Λ-space and thereafter the various results can be transferred as functions to the initial space.

Let V be the constant relative velocity of the inertial coordinate system ${}^{\Lambda}\Sigma'$ with respect to the inertial system ${}^{\Lambda}\Sigma$, in Λ-space, and c be the speed of light. Then, the relation between time intervals ΔT in ${}^{\Lambda}\Sigma$ and $\Delta T'$ in ${}^{\Lambda}\Sigma'$ is expressed by the property of the dilation of time, that is considered valid in Λ-space, such that

$$\Delta T' = \frac{\Delta T}{\sqrt{1-\frac{V^2}{c^2}}}, \tag{28}$$

where, according to Eqs. (8),

$$\begin{aligned} T &= {}_0I_t^{1-\gamma} t = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}, \\ F(T) &= {}_0I_t^{1-\gamma} f(t). \end{aligned} \tag{29}$$

The time dilation relation (28) is transferred to the initial space by means of

$${}_0^{RL}D_t^{1-\gamma}\Delta T' = \frac{{}_0^{RL}D_t^{1-\gamma}\Delta T}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{30}$$

In similar fashion, the contraction of length along the X-axis in Λ-space is expressed by

$$\Delta L' = \Delta L\sqrt{1-\frac{V^2}{c^2}}, \tag{31}$$

where

$$\begin{aligned} \Delta L' &= {}_0I_t^{1-\gamma}\Delta l', \\ \Delta L &= {}_0I_t^{1-\gamma}\Delta l. \end{aligned} \tag{32}$$

Transferring of the lengths (32) to the initial space is effected by

$$\begin{aligned} \Delta l &= {}_0^{RL}D_t^{1-\gamma}\Delta L, \\ \Delta l' &= {}_0^{RL}D_t^{1-\gamma}\Delta L'. \end{aligned} \tag{33}$$

The Lorentz transformation between ${}^{\Lambda}\Sigma$ and ${}^{\Lambda}\Sigma'$ is now formulated as

$$\begin{pmatrix} c\Delta T' \\ \Delta X' \\ \Delta Y' \\ \Delta Z' \end{pmatrix} = \mathit{\Lambda}(V) \begin{pmatrix} c\Delta T \\ \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix}, \tag{34}$$

with $\mathit{\Lambda}(V)$ provided by Eqs. (21), and then transferred to the initial space according to

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \mathit{\Lambda}(V)\, {}_0^{RL}D_t^{1-\gamma} \begin{pmatrix} c\Delta T \\ \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix}, \tag{35}$$

The last equation expresses the fractional Lorentz transformation. If (U_x, U_y, U_z), (U'_x, U'_y, U'_z) are the components of velocity in ${}^{\Lambda}\Sigma$, ${}^{\Lambda}\Sigma'$, respectively, then by means of the Lorentz transformation it follows that

$$
\begin{aligned}
U'_x &= \frac{U_x - V}{1 - \frac{VU_x}{c^2}},\\
U'_y &= \sqrt{1 - \frac{V^2}{c^2}}\frac{U_y}{1 - \frac{VU_x}{c^2}},\\
U'_z &= \sqrt{1 - \frac{V^2}{c^2}}\frac{U_z}{1 - \frac{VU_x}{c^2}}.
\end{aligned}
\tag{36}
$$

In the Λ-fractional space

$$
\begin{aligned}
U_x &= {}_0^{\Lambda}D_t^{\gamma}x(t) = \frac{dX(t)}{dT(t)},\\
U_y &= {}_0^{\Lambda}D_t^{\gamma}y(t) = \frac{dY(t)}{dT(t)},\\
U_z &= {}_0^{\Lambda}D_t^{\gamma}z(t) = \frac{dZ(t)}{dT(t)},
\end{aligned}
\tag{37}
$$

with

$$
\begin{gathered}
X(t) = {}_0I_t^{1-\gamma}x(t), \quad Y(t) = {}_0I_t^{1-\gamma}y(t), \quad Z(t) = {}_0I_t^{1-\gamma}z(t),\\
T(t) = {}_0I_t^{1-\gamma}t.
\end{gathered}
\tag{38}
$$

Therefore, Eqs. (36) can now be written in the form:

$$
\begin{aligned}
U'_x &= \frac{{}_0^{\Lambda}D_t^{\gamma}x(t) - V}{1 - \frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}},\\
U'_y &= \sqrt{1 - \frac{V^2}{c^2}}\frac{{}_0^{\Lambda}D_t^{\gamma}y(t)}{1 - \frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}},\\
U'_z &= \sqrt{1 - \frac{V^2}{c^2}}\frac{{}_0^{\Lambda}D_t^{\gamma}z(t)}{1 - \frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}}.
\end{aligned}
\tag{39}
$$

Transferring the components of velocity from Λ-space to the initial one, for the case of the Σ inertial frame we obtain:

$$
\begin{aligned}
u_x &= {}_0^{RL}D_t^{1-\gamma}U_x = {}_0^{RL}D_t^{1-\gamma}\left({}_0^{\Lambda}D_t^{\gamma}x(t)\right) = {}_0^{RL}D_t^{1-\gamma}\left(\frac{dX(t)}{dT(t)}\right),\\
u_y &= {}_0^{RL}D_t^{1-\gamma}U_y = {}_0^{RL}D_t^{1-\gamma}\left({}_0^{\Lambda}D_t^{\gamma}y(t)\right) = {}_0^{RL}D_t^{1-\gamma}\left(\frac{dY(t)}{dT(t)}\right),\\
u_z &= {}_0^{RL}D_t^{1-\gamma}U_z = {}_0^{RL}D_t^{1-\gamma}\left({}_0^{\Lambda}D_t^{\gamma}z(t)\right) = {}_0^{RL}D_t^{1-\gamma}\left(\frac{dZ(t)}{dT(t)}\right),
\end{aligned}
\tag{40}
$$

and, by virtue of Eqs. (39), for Σ':

$$\begin{aligned} u'_x &= {}_0^{RL}D_t^{1-\gamma}U'_x = {}_0^{RL}D_t^{1-\gamma}\left(\frac{{}_0^{\Lambda}D_t^{\gamma}x(t)-V}{1-\frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}}\right), \\ u'_y &= {}_0^{RL}D_t^{1-\gamma}U'_y = \sqrt{1-\frac{V^2}{c^2}}\,{}_0^{RL}D_t^{1-\gamma}\left(\frac{{}_0^{\Lambda}D_t^{\gamma}y(t)}{1-\frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}}\right), \\ u'_z &= {}_0^{RL}D_t^{1-\gamma}U'_z = \sqrt{1-\frac{V^2}{c^2}}\,{}_0^{RL}D_t^{1-\gamma}\left(\frac{{}_0^{\Lambda}D_t^{\gamma}z(t)}{1-\frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}}\right). \end{aligned} \tag{41}$$

Proceeding in similar fashion, if (A_x, A_y, A_z), (A'_x, A'_y, A'_z) are the components of acceleration in ${}^{\Lambda}\Sigma$, ${}^{\Lambda}\Sigma'$, respectively, it follows that

$$\begin{aligned} A'_x &= A_x\frac{(1-\frac{V^2}{c^2})^{3/2}}{(1-\frac{VU_x}{c^2})^3}, \\ A'_y &= \frac{(1-\frac{V^2}{c^2})^{3/2}}{(1-\frac{VU_x}{c^2})^3}\left(A_y+\frac{V}{c^2}(A_xU_y-A_yU_x)\right), \\ A'_z &= \frac{(1-\frac{V^2}{c^2})^{3/2}}{(1-\frac{VU_x}{c^2})^3}\left(A_z+\frac{V}{c^2}(A_xU_z-A_zU_x)\right). \end{aligned} \tag{42}$$

Analogously to Eqs. (37),

$$\begin{aligned} A_x &= {}_0^{\Lambda}D_t^{\gamma}(\,{}_0^{\gamma}D_t^{\gamma}x(t)) = \frac{d^2X(t)}{dT(t)^2}, \\ A_y &= {}_0^{\Lambda}D_t^{\gamma}(\,{}_0^{\gamma}D_t^{\gamma}y(t)) = \frac{d^2Y(t)}{dT(t)^2}, \\ A_z &= {}_0^{\Lambda}D_t^{\gamma}(\,{}_0^{\gamma}D_t^{\gamma}z(t)) = \frac{d^2Z(t)}{dT(t)^2}, \end{aligned} \tag{43}$$

with $X(t)$, $Y(t)$, $Z(t)$, $T(t)$ given by Eqs. (38). In turn, Eqs. (42) can now be written in the form:

$$\begin{aligned}
A'_x &= {}_0^{\Lambda}D_t^{\gamma}({}_0^{\Lambda}D_t^{\gamma}x(t))\frac{(1-\frac{V^2}{c^2})^{3/2}}{\left(1-\frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}\right)^3},\\
A'_y &= \frac{(1-\frac{V^2}{c^2})^{3/2}}{\left(1-\frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}\right)^3}\Bigg({}_0^{\Lambda}D_t^{\gamma}({}_0^{\Lambda}D_t^{\gamma}y(t)) + \frac{V}{c^2}\Big(\big({}_0^{\Lambda}D_t^{\gamma}({}_0^{\Lambda}D_t^{\gamma}x(t))\big)\ ({}_0^{\Lambda}D_t^{\gamma}y(t))\\
&\quad - \big({}_0^{\Lambda}D_t^{\gamma}({}_0^{\Lambda}D_t^{\gamma}y(t))\big)\ ({}_0^{\Lambda}D_t^{\gamma}x(t))\Big)\Bigg),\\
A'_z &= \frac{(1-\frac{V^2}{c^2})^{3/2}}{\left(1-\frac{V\,{}_0^{\Lambda}D_t^{\gamma}x(t)}{c^2}\right)^3}\Bigg({}_0^{\Lambda}D_t^{\gamma}({}_0^{\Lambda}D_t^{\gamma}z(t)) + \frac{V}{c^2}\Big(\big({}_0^{\Lambda}D_t^{\gamma}({}_0^{\Lambda}D_t^{\gamma}x(t))\big)\ ({}_0^{\Lambda}D_t^{\gamma}z(t))\\
&\quad - \big({}_0^{\Lambda}D_t^{\gamma}({}_0^{\Lambda}D_t^{\gamma}z(t))\big)\ ({}_0^{\Lambda}D_t^{\gamma}x(t))\Big)\Bigg).
\end{aligned} \tag{44}$$

Transferring the components of acceleration from Λ-space to the initial one, for the case of the Σ inertial frame we obtain:

$$\begin{aligned}
a_x &= {}_0^{RL}D_t^{1-\gamma}A_x = {}_0^{RL}D_t^{1-\gamma}\left({}_0^{\Lambda}D_t^{\gamma}\left({}_0^{\Lambda}D_t^{\gamma}x(t)\right)\right) = {}_0^{RL}D_t^{1-\gamma}\left(\frac{d^2X(t)}{dT(t)^2}\right),\\
a_y &= {}_0^{RL}D_t^{1-\gamma}A_y = {}_0^{RL}D_t^{1-\gamma}\left({}_0^{\Lambda}D_t^{\gamma}\left({}_0^{\Lambda}D_t^{\gamma}y(t)\right)\right) = {}_0^{RL}D_t^{1-\gamma}\left(\frac{d^2Y(t)}{dT(t)^2}\right),\\
a_z &= {}_0^{RL}D_t^{1-\gamma}A_z = {}_0^{RL}D_t^{1-\gamma}\left({}_0^{\Lambda}D_t^{\gamma}\left({}_0^{\Lambda}D_t^{\gamma}z(t)\right)\right) = {}_0^{RL}D_t^{1-\gamma}\left(\frac{d^2Z(t)}{dT(t)^2}\right),
\end{aligned} \tag{45}$$

and for Σ':

$$\begin{aligned}
a'_x &= {}_0^{RL}D_t^{\gamma-1}A'_x,\\
a'_y &= {}_0^{RL}D_t^{\gamma-1}A'_y,\\
a'_z &= {}_0^{RL}D_t^{\gamma-1}A'_z,
\end{aligned} \tag{46}$$

with A'_x, A'_y, A'_z defined by Eqs. (44).

In conventional Special Relativity theory the non-constant mass of a body depends upon the velocity $\mathbf{u}$, as expressed in Eq. (24). In Λ-space, the mass M corresponding to a rest mass M_0 is defined by

$$M = \frac{M_0}{\sqrt{1-\frac{({}_0^{\Lambda}D_t^{\gamma}x(t))^2+({}_0^{\Lambda}D_t^{\gamma}y(t))^2+({}_0^{\Lambda}D_t^{\gamma}z(t))^2}{c^2}}}. \tag{47}$$

It is further pointed out that

$$M_0 = {}_0I_t^{1-\gamma}m_0 = \frac{m_0t^{1-\gamma}}{\Gamma(2-\gamma)}, \tag{48}$$

where m_0 is the rest mass in the inertial system Σ of the initial space. Consequently, the current mass m in Σ is defined by

$$m = {}_{0}^{RL}D_t^{1-\gamma}M = {}_{0}^{RL}D_t^{1-\gamma}\left(\frac{m_0 t^{1-\gamma}}{\Gamma(2-\gamma)\sqrt{1-\frac{({}_{0}^{\Lambda}D_t^{\gamma}x(t))^2+({}_{0}^{\Lambda}D_t^{\gamma}y(t))^2+({}_{0}^{\Lambda}D_t^{\gamma}z(t))^2}{c^2}}}\right). \tag{49}$$

In Λ-space, Newton's second law is formulated as

$$\mathbf{F} = \frac{d\mathbf{P}}{dT} = \frac{d}{dT}\left(\frac{M_0\mathbf{U}}{\sqrt{1-\frac{U^2}{c^2}}}\right) = \frac{d}{dT}(M\mathbf{U}), \tag{50}$$

with $\mathbf{F}$ being the force, M, M_0 the current and rest masses, respectively, $\mathbf{P}$ the momentum, and $\mathbf{U}$ the velocity vector. Recalling Eqs. (37), Newton's second law can be further expressed as

$$\mathbf{F} = \frac{d}{dT}\left(\frac{M_0\Big(({}_{0}^{\Lambda}D_t^{\gamma}x(t))\ \mathbf{i} + ({}_{0}^{\Lambda}D_t^{\gamma}y(t))\ \mathbf{j} + ({}_{0}^{\Lambda}D_t^{\gamma}z(t))\ \mathbf{k}\Big)}{\sqrt{1-\frac{({}_{0}^{\Lambda}D_t^{\gamma}x(t))^2+({}_{0}^{\Lambda}D_t^{\gamma}y(t))^2+({}_{0}^{\Lambda}D_t^{\gamma}z(t))^2}{c^2}}}\right), \tag{51}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors of the coordinate space standard basis. The force is transferred to the initial space by

$$\mathbf{f} = {}_{0}^{RL}D_t^{1-\gamma}\mathbf{F}. \tag{52}$$

Moreover, the kinetic energy of a point with mass M and velocity magnitude $V = \|\mathbf{U}\|$ is expressed in Λ-space according to

$$K(V, M) = \frac{Mc^2}{\sqrt{1-\frac{V^2}{c^2}}} - Mc^2, \tag{53}$$

with

$$V^2 = \big({}_{0}^{\Lambda}D_t^{\gamma}x(t)\big)^2 + \big({}_{0}^{\Lambda}D_t^{\gamma}y(t)\big)^2 + \big({}_{0}^{\Lambda}D_t^{\gamma}z(t)\big)^2. \tag{54}$$

Transferring of the kinetic energy to the initial space reference frame is effected by

$$K(u, m) = {}_{0}I_t^{1-\gamma}K(V, M). \tag{55}$$

Finally, the total energy of the system in Λ-space is expressed as

$$E(V, M) = \frac{Mc^2}{\sqrt{1-\frac{V^2}{c^2}}} = Mc^2 + K(V, M), \tag{56}$$

and its transfer to the initial space reads

$$E(u, m) = {}_{0}I_t^{1-\gamma}E(V, M) = {}_{0}I_t^{1-\gamma}\big(Mc^2 + K(V, M)\big). \tag{57}$$

6. Fractional Special Relativity and Classical Electromagnetism

6.1. Special Relativity and Electromagnetism

The influence of Special Relativity theory on classical electromagnetism is well documented in literature (cf., for example, [33]). To outline that influence, let $\mathbf{E}$ be the electric field intensity, $\mathbf{D}$ the electric flux density, $\mathbf{H}$ the magnetic field strength, and $\mathbf{B}$ the magnetic flux density, of an inertial frame Σ, and let Σ' be another inertial frame moving relative to Σ with velocity $\mathbf{v}$. Denoting with $\hat{\mathbf{v}}$ the velocity unit vector, the various fields in the two inertial frames are related by

$$\begin{aligned}
\mathbf{E}' &= \gamma(\mathit{\Lambda})(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - (\gamma(\mathit{\Lambda}) - 1)(\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \\
\mathbf{B}' &= \gamma(\mathit{\Lambda})\left(\mathbf{B} - \mathbf{v} \times \frac{\mathbf{B}}{c^2}\right) - (\gamma(\mathit{\Lambda}) - 1)(\mathbf{B} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \\
\mathbf{D}' &= \gamma(\mathit{\Lambda})\left(\mathbf{D} + \mathbf{v} \times \frac{\mathbf{H}}{c^2}\right) + (1 - \gamma(\mathit{\Lambda}))(\mathbf{D} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \\
\mathbf{H}' &= \gamma(\mathit{\Lambda})(\mathbf{H} + \mathbf{v} \times \mathbf{D}) + (1 - \gamma(\mathit{\Lambda}))(\mathbf{H} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}},
\end{aligned} \tag{58}$$

with $\gamma(\mathit{\Lambda})$ defined by Eqs. (21). When a particle of charge q is moving with velocity $\mathbf{u}$ with respect to Σ, the Lorentz force is defined by

$$\mathbf{F} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B}. \tag{59}$$

Similarly, in Σ',

$$\mathbf{F}' = q\mathbf{E}' + q\mathbf{u} \times \mathbf{B}'. \tag{60}$$

Finally, the charge and current density in the two inertial frames are related by

$$\begin{aligned}
\rho' &= \gamma(\mathit{\Lambda})\left(\rho - \frac{\mathbf{J} \cdot \hat{\mathbf{v}}}{c^2}\right), \\
\mathbf{J}' &= \mathbf{J} - \gamma(\mathit{\Lambda})\rho\mathbf{v} + (\gamma(\mathit{\Lambda}) - 1)(\mathbf{J} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}},
\end{aligned} \tag{61}$$

respectively.

6.2. Fractional Relativity and Electromagnetism

Fractional analysis has already proved out to be a quite useful mathematical tool for the investigation of relativistic phenomena [28–32]. In the present section, a short introduction is provided for the application of the fractional Λ-derivative in the study of electromagnetic phenomena in the context of Special Relativity theory.

Recall that only the fractional Λ-derivative can generate differential geometry, whereas other fractional derivatives do not correspond to differential and, therefore, are not suitable for formulating equations for physical problems. It is further reminded that differential geometry along with the fractional Λ-derivative are only valid in Λ-space, wherein the various derivatives exhibit conventional behaviour and are local. Functions can be transferred from Λ-space to the initial one by invoking the relation

$$f(t) = {}_{0}^{RL}D_t^{1-\gamma}\left({}_{0}I_t^{1-\gamma}F(T(t))\right) = {}_{0}^{RL}D_t^{1-\gamma}\left({}_{0}I_t^{1-\gamma}f(t)\right), \tag{62}$$

where $F(T)$, in the present context, are the Λ-space expressions of the various functions presented in Eqs. (58)-(61). As Λ-space expressions, these functions should be expressed with respect to the T variable (recall Eqs. (29)). For example, the Lorentz force, Eq. (59), is expressed in Λ-space as

$$\mathbf{F}(T) = q\mathbf{E}(T) + q\mathbf{u}(T) \times \mathbf{B}(T), \tag{63}$$

where

$$\mathbf{u}(T) = \frac{d\mathbf{X}(T)}{dT}. \tag{64}$$

Taking into account Eqs. (29) and (62), the Lorentz force is expressed in the initial by means of the transformation

$$\mathbf{f}(t) = {}^{RL}_{\ 0}\boldsymbol{D}_t^{1-\gamma} \left({}_0\boldsymbol{I}_t^{1-\gamma} \mathbf{F}(T(t)) \right) = {}^{RL}_{\ 0}\boldsymbol{D}_t^{1-\gamma} \Big(q\mathbf{E}(T) + q\mathbf{u}(T) \times \mathbf{B}(T) \Big). \tag{65}$$

Moreover, combining Eq. (60) with (58a,b) we obtain the Lorentz force in Σ', which reads

$$\mathbf{F}' = q\Big(\gamma(\Lambda)(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - (\gamma(\Lambda) - 1)(\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\Big) + q\mathbf{u} \times \left(\gamma(\Lambda)\Big(\mathbf{B} - \mathbf{v} \times \frac{\mathbf{B}}{c^2}\Big) - (\gamma(\Lambda) - 1)(\mathbf{B} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\right). \tag{66}$$

Transferring $\mathbf{F}'$ to the initial space is effected by

$$\mathbf{f}'(t) = {}^{RL}_{\ 0}\boldsymbol{D}_t^{1-\gamma} \left({}_0\boldsymbol{I}_t^{1-\gamma} \mathbf{F}'(T(t)) \right). \tag{67}$$

The procedure employed above can be followed for the definition of any physical quantity pertaining to fractional relativistic electromagnetism.

7. Conclusion

Fractional relativity theory has been proposed in the context of the fractional Λ-derivative and the corresponding fractional Λ-space. Since the Λ-derivative behaves conventionally in Λ-space, the analysis concerning the results of Special Relativity theory is formulated in the Λ-space, where relativistic mechanics theorems are derived. The results are transferred from the fractional Λ-space back to the initial one through the Riemann-Liouville derivative. Furthermore, the proposed fractional special relativity theory is applied to classical electromagnetism, where the various fields are defined in Λ-space and subsequently they are appropriately pulled back to the initial space.

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