Asia Mathematika
Volume: 5 Issue: 3 , (2021) Pages: 69 – 83
Available online at **www.asiamath.org**

Generalized Hyers-Ulam-Rassias type stability of the with $2k$-variable quadratic functional inequalities in non-Archimedean Banach spaces and Banach spaces

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Received: 16 Oct 2021 • **Accepted:** 01 Dec 2021 • **Published Online:** 30 Dec 2021

Abstract: In this paper, we use the direct method to prove two generalized quadratic functional inequalities with 2k-variables and their Hyers-Ulam-Rassias stability. The first is investigated in Banach spaces and the last are investigated in non-Archimedean Banach spaces. Then I will show that the solutions of the equation are quadratic mapping. These are the main results of this paper.

Key words: Functional equation, quadratic functional inequality, quadratic $\beta-$functional inequalities, Banach space, non-Archimedian Banach space, Hyers-Ulam-Rassias stability.

1. Introduction

Let $\mathbf{X_1}$ and $\mathbf{X_2}$ be a normed spaces on the same field $\mathbb{K}$, and $F : \mathbf{X_1} \to \mathbf{X_2}$. We use the notation $\|\cdot\|$ for all the norm on both $\mathbf{X_1}$ and $\mathbf{X_2}$. In this paper, we investisgate some quadratic functional inequality when $\mathbf{X_1}$ and $\mathbf{X_2}$ is a Banach spaces or $\mathbf{X_1}$ is a non-Archimedean normed space and $\mathbf{X_2}$ is a non-Archimedean Banach space.

In fact, when $\mathbf{X_1}$ and $\mathbf{X_2}$ is Banach spaces we solve and prove the Hyers-Ulam-Rassias type stability of forllowing quadratic functional inequality.

$$\begin{aligned}&\Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\Big)+F\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\Big)-2\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-2\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X_2}}\\&\leq\Big\|F\Big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\Big)+F\Big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\Big)-\frac{2}{k}\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\Big)\Big\|_{\mathbf{X_2}}\end{aligned}\tag{1}$$

and when $\mathbf{X_1}$ is a non-Archimedean normed space and $\mathbf{X_2}$ is a non-Archimedean Banach spaces we solve and prove the Hyers-Ulam stability of forllowing quadratic functional inequality.

$$\begin{aligned}&\Big\|F\Big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\Big)+F\Big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\Big)-\frac{2}{k}\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X_2}}\\&\leq\Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\Big)+F\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\Big)-2\sum_{j=1}^{k}F\Big(\frac{x_{k+1}}{k}\Big)-2\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X_2}},\end{aligned}\tag{2}$$

 DOI: 10.5281/zenodo.5809214
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The study of the functional equation stability originated from a question of S.M. Ulam [24], concerning the stability of group homomorphisms. Let $(\mathbf{G}, *)$ be a group and let $(\mathbf{G}', \circ, d)$ be a metric group with metric $d(\cdot,\cdot)$. Geven $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : \mathbf{G} \to \mathbf{G}'$ satisfies

$$d\big(f(x * y), f(x) \circ f(y)\big) < \delta$$

for all $x, y \in \mathbf{G}$ then there is a homomorphism $h : \mathbf{G} \to \mathbf{G}'$ with

$$d\Big(f(x), h(x)\Big) < \epsilon$$

for all $x \in \mathbf{G}$, if the answer, is affirmative, we would say that equation of homomophism $h(x * y) = h(y) \circ h(y)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation.

The stability of quadratic functional equation was proved by Skof [21] for mappings $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation:

$$f(x+y) + f(x+y) - 2f(x) - 2f(y)$$

is called the quadratic functional equation.
The functional equation:

$$f(\frac{x+y}{2}) + f(\frac{x-y}{2}) - \frac{1}{2}f(x) - \frac{1}{2}f(y) = 0$$

is called a Jensen type the quadratic functional equation.

The first work on the stability problem for functional equations in non-Archimedean spaces was started by Moslehian and Rassias [16]. Moslehian and Sadeghi [15] investigated the stability of cubi functional equations in non-Archimedean normed space

In [10] Gil$\acute{a}$ny showed that is if satisfies the functional inequality

$$\Big\|2f(x) + 2f(y) - f(xy^{-1})\Big\| \le \big\|f(xy)\big\|$$

then f satisfies the Jordan-von Newman functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}). \tag{3}$$

Seen [20]. Gil$\acute{a}$nyi [11] and Fechner [8] proved the Hyers-Ulam-Rassia stability of the functional inequality. Choonkil $Park$ [18] obtained the solutions of the quadratic functional inequality. Recently, in [2, 14, 18] the authors studied the Hyers-Ulam-Rassia stability for the following functional inequalities in Banach space and non-Archimedean Banach space:

$$\begin{aligned} &\Big\|f(x+y) + f(x+y) - 2f(x) - 2f(y)\Big\| \\ &\qquad \le \Big\|f(\frac{x+y}{2}) + f(\frac{x-y}{2}) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\Big\| \end{aligned} \tag{4}$$

and

$$\Big\|f\big(\frac{x+y}{2}\big)+f\big(\frac{x-y}{2}\big)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\Big\| \\ \leq \Big\|f(x+y)+f(x-y)-2f(x)-2f(y)\Big\|. \tag{5}$$

Next

$$\Big\|f\big(\frac{x+y}{2}+z\big)+f\big(\frac{x+y}{2}-z\big)-2f\big(\frac{x+y}{2}\big)-2f(z)\Big\| \\ \leq \Big\|f\big(\frac{x+y}{2^2}+\frac{z}{2}\big)+f\big(\frac{x+y}{2^2}-\frac{z}{2}\big)-f\big(\frac{x+y}{2}\big)-f(z)\Big\| \tag{6}$$

and

$$\Big\|f\big(\frac{x+y}{2^2}+\frac{z}{2}\big)+f\big(\frac{x+y}{2^2}-\frac{z}{2}\big)-f\big(\frac{x+y}{2}\big)-f(z)\Big\| \\ \leq \Big\|f\big(\frac{x+y}{2}+z\big)+f\big(\frac{x+y}{2}-z\big)-2f\big(\frac{x+y}{2}\big)-2f(z)\Big\|. \tag{7}$$

In this paper, we solve and proved the Hyers-Ulam-Rassias type stability for two quadratic functional inequalities (1)-(2), ie the quadratic functional inequalities with $2k-variables$. Under suitable assumptions on spaces $\mathbf{X_1}$ and $\mathbf{X_2}$, we will prove that the mappings satisfying the quadratic functional inequatilies (1) or (2). Thus, the results in this paper are generalization of those in [2, 14, 18] for functional inequatilies with $2k-variables$.
The paper is organized as followns:
In section preliminaries we remind some basic notations in [15, 16] such as We only redefine the solution definition of the quadratic equation function.
Section 3: is devoted to prove the Hyers-Ulam stability of the quadratic functional inequalities (1) when we assume that $\mathbf{X_1}$ and $\mathbf{X_2}$ is a Banach spaces.
Section 4: is devoted to prove the Hyers-Ulam stability of the quadratic functional inequalities (2) when $\mathbf{X_1}$ is a non-Archimedean normed space and $\mathbf{X_2}$ is a non-Archimedean Banach space.

2. preliminaries

2.1. non-Archimedean normed spaces.

In this subsection we recall some basic notations from [14–16] such as non-Archimedean fields, non-Archimedean normed spaces and non-Archimedean Banach spaces. A valuation is a function $\big|\cdot\big|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation,

$$\Big|r\Big| = 0 \Leftrightarrow r = 0$$

$$\big|r\cdot s\big| := \big|r\big|\,|s|, \forall r,s\in\mathbb{K}$$

and the triangle inequality holds, i.e.,

$$\big|r+s\big| \leq \big|r\big| + \big|s\big|, \forall r,s\in\mathbb{K}.$$

A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the strong triangle inequality is replaced by

$$\left|r+s\right| \leq max\left\{\left|r\right|,\left|s\right|\right\}, \forall r,s \in \mathbb{K},$$

then the function $\left|\cdot\right|$ is called a non-Archimedean valuation. Clearly, $\left|1\right| = \left|-1\right| = 1$ and $\left|n\right| \leq 1, \forall n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $\left|\cdot\right|$ talking everything except for 0 into 1 and $\left|0\right| = 0$. In this paper, we assume that the base field is a non-Archimedean field with $\left|2\right| \neq 1$, hence call it simply a field. .

Definition 2.1. Let be a vecter space over a filed K with a non -Archimedean $|\cdot|$. A function $\|\cdot\| : X \to \left[0,\infty\right)$ is said a non -Archimedean norm if it satisfies the follwing conditions:

1. $\left\|x\right\| = 0$ if and only if $x = 0$;
2. $\left\|rx\right\| = \left|r\right|\left\|x\right\|(r \in \mathbb{K}, x \in X)$;
3. $\left\|x+y\right\| \leq max\left\{\left\|x\right\|,\left\|y\right\|\right\}x,y \in X$ hold.

Then $(X, \|\cdot\|)$ is called a norm -Archimedean norm space.

Definition 2.2. A sequence $\left\{x_n\right\}$ in a norm -Archimedean normed space **X** is a Cauchy sequence if and only if $\left\{x_n - x_m\right\} \to 0$.

Definition 2.3. Let $\left\{x_n\right\}$ be a sequence in a norm -Archimedean normed space X.

1. A sequence $\left\{x_n\right\}_{n=1}^{\infty}$ in a non -Archimedean space is a Cauchy sequence iff the sequence $\left\{x_{n+1} - x_n\right\}_{n=1}^{\infty}$ converges to zero.
2. The sequence $\left\{x_n\right\}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer N and $x \in X$ such that
$$\left\|x_n - x\right\| \leq \epsilon. \forall n \geq N,$$
for all n,m $\geq N$. The we call x$\in X$ a limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.
3. If every sequence Cauchy in X converger, then the norm -Archimedean normed space X is called a norm -Archimedean Bnanch space.

2.2. Solutions of the equation.

The functional equation

$$f\left(x+y\right) + f\left(x-y\right) = 2f\left(x\right) + 2f\left(y\right)$$

is called the quadratic equation. In particular, every solution of the quadratic equation is said to be an *quadratic mapping*.

3. Quadratic functional inequality in Banach space

Now, we study the solutions of (1). Note that for these inequalitie, $\mathbf{X_1}$ and $\mathbf{X_2}$ is a Banach spaces. Under this setting, we can show that the mapping satisfying (1) is qudratic. These results are give in the following.

Lemma 3.1. *A mapping* $F:\mathbf{X}_1\to\mathbf{X}_2$ *satilies*

$$\Big\|F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)-2\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-2\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}$$
$$\leq\Big\|F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(x_j\big)\big)\Big\|_{\mathbf{X_2}} \tag{8}$$

for all $x_j,x_{k+j}\in\mathbf{X}_1$ *for all* $j=1\to k$ *if and only if* $F:\mathbf{X}_1\to\mathbf{X}_2$ *is quadratic.*

Proof. Assume that $F:\mathbf{X}_1\to\mathbf{X}_2$ satisfies (8).
Letting $x_j=x_{k+j}=0, j=1\to k$ in (8), we get

$$\big(\big|2k-1\big|-1\big)\big\|F\big(0\big)\big\|_{\mathbf{X_2}}\leq 0.$$

So $F\Big(0\Big)=0$.
Letting $x_{k+j}=0$ and $x_j=x$ for all $j=1\to k$ in (8), we get

$$\Big\|f\big(kx\big)-kf\big(x\big)\big\|_{\mathbf{X_2}}\leq 0 \tag{9}$$

and so $f(kx)=kf(x)$ for all $x\in X_1$.
Thus

$$f\big(\frac{x}{k}\big)=\frac{1}{k}f(x) \tag{10}$$

for all $x\in\mathbf{X_1}$ It follows from (3.1) and (10) that:

$$\Big\|f\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+f\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)-2\sum_{j=1}^{k}f\big(\frac{x_{k+j}}{k}\big)-2\sum_{j=1}^{k}f\big(x_j\big)\Big\|_{\mathbf{X_2}}$$
$$\leq\Big\|\frac{1}{k}f\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+\frac{1}{k}f\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)-\frac{2}{k}\sum_{j=1}^{k}f\big(\frac{x_{k+1}}{k}\big)-\frac{2}{k}\sum_{j=1}^{k}f\big(x_j\big)\big)\Big\|_{\mathbf{X_2}}$$
$$=\frac{1}{k}\Big\|f\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+1}+\sum_{j=1}^{k}x_j\big)+f\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+1}-\sum_{j=1}^{k}x_j\big)-2\sum_{j=1}^{k}f\big(\frac{x_{k+j}}{k}\big)-2\sum_{j=1}^{k}f\big(x_j\big)\big)\Big\|_{\mathbf{X_2}} \tag{11}$$

and so

$$f\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+f\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)=2\sum_{j=1}^{k}f\big(\frac{x_{k+1}}{k}\big)+2\sum_{j=1}^{k}f\big(x_j\big)$$

for all $x_j,x_{k+j}\in\mathbf{X_1}$ for all $j=1\to k$. Hence $F:\mathbf{X_1}\to\mathbf{X_2}$ is quadratic.
The coverse is obviously true. □

Theorem 3.1. *Let* $\varphi : \mathbf{X_1^{2k}} \to [0,\infty)$ *be a function and let* $F : \mathbf{X}_1 \to \mathbf{X}_2$ *be mapping such that*

$$\varphi\big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\big) = \sum_{j=1}^{\infty} k^j \psi\big(\frac{x_1}{k^j}, \frac{x_2}{k^j}, ..., \frac{x_k}{k^j}, \frac{x_{k+1}}{k^j}, \frac{x_{k+2}}{k^j}, ..., \frac{x_{2k}}{k^j}\big) < \infty \tag{12}$$

$$\begin{aligned}&\Big\|F\big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\big) + F\big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\big) - 2\sum_{j=1}^{k} F\big(\frac{x_{k+j}}{k}\big) - 2\sum_{j=1}^{k} F\big(x_j\big)\Big\|_{\mathbf{X_2}}\\ &\le \Big\|F\big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\big) + F\big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j\big) - \frac{2}{k}\sum_{j=1}^{k} F\big(\frac{x_{k+j}}{k}\big) - \frac{2}{k}\sum_{j=1}^{k} F\big(x_j\big)\Big\|_{\mathbf{X_2}}\\ &+ \psi\big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\big)\end{aligned} \tag{13}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, *for all* $j = 1 \to k$. *Then there exists a unique quadratic mapping* $Q : \mathbf{X_1} \to \mathbf{X_2}$ *such that*

$$\Big\|F(x) - Q(x)\Big\|_{\mathbf{X_2}} \le \frac{1}{k}\varphi\big(x, x, ..., x, 0, 0..., 0\big) \tag{14}$$

for all $x \in \mathbf{X_1}$.

Proof. Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$ in (13), we get

$$\big(|2k-1| - 1\big)\Big\|F(0)\Big\|_{\mathbf{X_2}} \le 0. \tag{15}$$

So

$$F(0) = 0.$$

Letting $x_{k+j} = 0$, $x_j = x$ for all $j = 1 \to k$ in (13), we get

$$\Big\|F(kx) - kF(x)\Big\|_{\mathbf{X_2}} \le \frac{1}{k}\psi\big(x, x, ..., x, 0, 0, ..., 0\big) \tag{16}$$

$$\Big\|F(x) - kF\big(\frac{x}{k}\big)\Big\|_{\mathbf{X_2}} \le \frac{1}{k}\psi\big(\frac{x}{k}, \frac{x}{k}, ..., \frac{x}{k}, 0, 0, ..., 0\big)$$

Hence

$$\begin{aligned}&\Big\|k^l F\big(\frac{x}{k^l}\big) - k^m F\big(\frac{x}{k^m}\big)\Big\|_{\mathbf{X_2}}\\ &\le \sum_{j=l}^{m-1}\Big\|k^j F\big(\frac{x}{k^j}\big) - k^{j+1} F\big(\frac{x}{k^{j+1}}\big)\Big\|_{\mathbf{X_2}}\\ &\le \frac{1}{k}\sum_{j=l+1}^{m} k^j \psi\big(\frac{x}{k^j}, \frac{x}{k^j}, ..., \frac{x}{k^j}, 0, 0, ..., 0\big)\end{aligned} \tag{17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X_1}$. It follows from (17) that the sequence $\Big\{k^n F\big(\frac{x}{k^n}\big)\Big\}$ is a cauchy sequence for all $x \in \mathbf{X_1}$. Since $\mathbf{X_2}$ is complete space, the sequence $\Big\{k^n F\big(\frac{x}{k^n}\big)\Big\}$

coverges.
So one can define the mapping $Q : \mathbf{X}_1 \to \mathbf{X}_2$ by

$$Q(x) := \lim_{n\to\infty} k^n F(\frac{x}{k^n})$$

for all $x \in \mathbf{X_1}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (17), we get (14).
Now, It follows from (12) and (13) that

$$\begin{aligned}
&\Big\|Q(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j)+Q(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j)-2\sum_{j=1}^{k}Q(\frac{x_{k+j}}{k})-2\sum_{j=1}^{k}Q(x_j)\Big\|_{\mathbf{X_2}}\\
&=\lim_{n\to\infty}k^n\Big\|F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}}+\frac{1}{k^n}\sum_{j=1}^{k}x_j)+F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}}-\frac{1}{k^n}\sum_{j=1}^{k}x_j)-2\sum_{j=1}^{k}F(\frac{x_{k+j}}{k^{n+1}})-2\sum_{j=1}^{k}F(\frac{x_j}{k^n})\Big\|_{\mathbf{X_2}}\\
&\le\lim_{n\to\infty}k^n\Big\|F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}}+\frac{1}{k^n}\sum_{j=1}^{k}x_j)+F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}}-\frac{1}{k^n}\sum_{j=1}^{k}x_j)-\frac{2}{k}\sum_{j=1}^{k}F(\frac{x_{k+j}}{k^{n+1}})-\frac{2}{k}\sum_{j=1}^{k}F(\frac{x_j}{k^n})\Big\|_{\mathbf{X_2}}\\
&+\lim_{n\to\infty}k^n\psi(\frac{x_1}{k^j},\frac{x_2}{k^j},...,\frac{x_k}{k^j},\frac{x_{k+1}}{k^j},\frac{x_{k+2}}{k^j},...,\frac{x_{2k}}{k^j})\\
&=\Big\|F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j)+F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j)-\frac{2}{k}\sum_{j=1}^{k}F(\frac{x_{k+j}}{k})-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X_2}} \qquad (18)
\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$.
So

$$\begin{aligned}
&\Big\|Q(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j)+Q(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j)-2\sum_{j=1}^{k}Q(\frac{x_{k+j}}{k})-2\sum_{j=1}^{k}Q(x_j)\Big\|_{\mathbf{X_2}}\\
&\le\|Q(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j)+Q(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j)-\frac{2}{k}\sum_{j=1}^{k}Q(\frac{x_{k+j}}{k})-\frac{2}{k}\sum_{j=1}^{k}Q(x_j)\Big\|_{\mathbf{X_2}} \qquad (19)
\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$. By Lemma (3.1), the mapping $Q : \mathbf{X_1} \to \mathbf{X_2}$ is quadratic.
Next, suppose that $T : \mathbf{X_1} \to \mathbf{X_2}$ be another quadratic mapping satisfying (14). Then we have

$$\begin{aligned}
\Big\|Q(x)-T(x)\Big\|_{\mathbb{Y}}&=k^n\Big\|Q(\frac{x}{k^n})-T(\frac{x}{k^n})\Big\|_{\mathbb{Y}}\\
&\le k^n(\|Q(\frac{x}{k^n})-F(\frac{x}{k^n})\Big\|_{\mathbb{Y}}+\Big\|T(\frac{x}{k^n})-F(\frac{x}{k^n})\Big\|_{\mathbb{Y}})\\
&\le k^n\Big(\frac{1}{k^n}\varphi(\frac{x}{k^n},\frac{x}{k^n},...,\frac{x}{k^n},0,0,...,0)+\frac{1}{k^n}\varphi(\frac{x}{k^n},\frac{x}{k^n},...,\frac{x}{k^n},0,0,...,0)\Big)\\
&=k^n\cdot\frac{2}{k}\varphi(\frac{x}{k^n},\frac{x}{k^n},...,\frac{x}{k^n},0,0,...,0)\\
&\le k^n\varphi(\frac{x}{k^n},\frac{x}{k^n},...,\frac{x}{k^n},0,0,...,0) \qquad (20)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbf{X_1}$. So we can conclude that $Q(x) = T(x)$ for all $x \in \mathbf{X_1}$. This proves the uniqueness of Q. Thus the mapping $Q : \mathbf{X_1} \to \mathbf{X_2}$ is a unique quadratic mapping satisfying (14). □

Corollary 3.1. *Let* $r>1$ *and* θ *be nonnegative real numbers and* $F:\mathbf{X_1}\to\mathbf{X_2}$ *be a mapping satisfying*

$$\begin{aligned}&\Big\|F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)-2\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-2\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}\\&\leq\Big\|F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}\\&+\theta\big(\sum_{j=1}^{k}\|x_i\|_{\mathbf{X}_1}^r+\sum_{j=1}^{k}\|x_{k+i}\|_{\mathbf{X}_1}^r\big)\end{aligned}\tag{21}$$

for all $x_j,x_{k+j}\in\mathbf{X_1}$, *for all* $j=1\to k$. *Then there exists a unique quadratic mapping* $Q:\mathbf{X}_1\to\mathbf{X}_2$ *such that*

$$\Big\|F\big(x\big)-Q\big(x\big)\Big\|_{\mathbf{X_2}}\leq\frac{2k\theta}{k^r-k}\|x\|^r\tag{22}$$

for all $x\in\mathbf{X_1}$

Theorem 3.2. *Let* $\varphi:\mathbf{X_1^{2k}}\to[0,\infty\big)$ *be a function and let* $F:\mathbf{X}_1\to\mathbf{X}_2$ *be mapping such that*

$$\begin{aligned}&\varphi\big(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\big)\\&\qquad=\sum_{j=1}^{\infty}\frac{1}{k^j}\psi\big(k^jx_1,k^jx_2,...,k^jx_k,k^jx_{k+1},k^jx_{k+2},...,k^jx_{2k}\big)<\infty\end{aligned}\tag{23}$$

$$\begin{aligned}&\Big\|F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)-2\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-2\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}\\&\leq\Big\|F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}\\&+\psi\big(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\big)\end{aligned}\tag{24}$$

for all $x_j,x_{k+j}\in\mathbf{X_1}$, *for all* $j=1\to k$. *Then there exists a unique additive mapping* $Q:\mathbf{X}_1\to\mathbf{X}_2$ *such that*

$$\Big\|F\big(x\big)-Q\big(x\big)\Big\|_{\mathbf{X_2}}\leq\frac{1}{k}\varphi\big(x,x,...,x,0,0,...,0\big)\tag{25}$$

, *for all* $x\in\mathbf{X_1}$.

Proof. Letting $x_j=x_{k+j}=0$ for all $j=1\to k$ in (24), we get

$$\big(|2k-1|-1\big)\Big\|F\big(0\big)\Big\|_{\mathbf{X_2}}\leq 0.\tag{26}$$

So

$$F\big(0\big)=0.$$

Letting $x_{k+j}=0,\ x_j=x$ for all $j=1\to k$ in (24), we get

$$\Big\|F\big(kx\big)-kF\big(x\big)\Big\|_{\mathbf{X_2}}\le\psi\big(x,x,...,x,0,0,...,0\big) \tag{27}$$

thus

$$\Big\|F\big(x\big)-\frac{1}{k}F\big(kx\big)\Big\|_{\mathbf{X_2}}\le\frac{1}{k}\psi\big(x,x,...,x,0,0,...,0\big)$$

Hence

$$\begin{aligned}\Big\|\frac{1}{k^l}F\big(k^lx\big)&-\frac{1}{k^m}F\big(k^mx\big)\Big\|_{\mathbf{X_2}}\\&\le\sum_{j=l}^{m-1}\Big\|\frac{1}{k^j}F\big(k^jx\big)-\frac{1}{k^{j+1}}F\big(k^{j+1}x\big)\Big\|_{\mathbf{X_2}}\\&\le\frac{1}{k}\sum_{j=l}^{m}\frac{1}{k^{j+1}}\psi\big(k^jx,k^jx,...,k^jx,0,0,...,0\big)\end{aligned} \tag{28}$$

for all nonnegative integers m and l with $m>l$ and all $x\in\mathbf{X_1}$. It follows from (28) that the sequence $\Big\{\frac{1}{k^n}F\big(k^nx\big)\Big\}$ is a cauchy sequence for all $x\in\mathbf{X_1}$. Since $\mathbf{X_2}$ is complete space, the sequence $\Big\{\frac{1}{k^n}F\big(k^nx\big)\Big\}$ coverges.
So one can define the mapping $Q:\mathbf{X}_1\to\mathbf{X}_2$ by

$$Q\big(x\big):=\lim_{n\to\infty}\frac{1}{k^n}f\big(k^nx\big)$$

for all $x\in\mathbb{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (28), we get (25).
We use the similar manner to the proof of Theorem 3.2 for the rest of the proof.

□

Corollary 3.2. *Let $r<1$ and θ be nonnegative real numbers and $F:\mathbf{X_1}\to\mathbf{X_2}$ be a mapping satisfying*

$$\begin{aligned}&\Big\|F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)-\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}\\&\le\Big\|F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}\\&+\theta\big(\sum_{j=1}^{k}\|x_i\|^r+\sum_{j=1}^{k}\|x_{k+i}\|^r\big)\end{aligned} \tag{29}$$

for all $x_j,x_{k+j}\in\mathbf{X_1}$, for all $j=1\to k$. Then there exists a unique additive mapping $Q:\mathbf{X}_1\to\mathbf{X}_2$ such that

$$\Big\|F\big(x\big)-Q\big(x\big)\Big\|_{\mathbf{X_2}}\le\frac{2k\theta}{k-k^r}\|x\|^r \tag{30}$$

for all $x\in\mathbf{X_1}$

4. quadratic functional inequality in non-Archimedean Banach space

Now, we study the solutions of (2). Note that for these inequalitie, $\mathbf{X_1}$ is a non-Archimedean normed space and $\mathbf{X_2}$ is a non-Archimedean Banach spaces. Under this setting, we can show that the mapping satisfying (2) is quadratic. These results are give in the following. Assume that where k is a fixed positive integer with $|k| \neq 1$.

Lemma 4.1. *A mapping $F : \mathbf{X}_1 \to \mathbf{X}_2$ satilies*

$$\begin{aligned}&\Big\|F\Big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j}+\frac{1}{k}\sum_{j=1}^{k} x_j\Big)+F\Big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j}-\frac{1}{k}\sum_{j=1}^{k} x_j\Big)-\frac{2}{k}\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\Big)\Big\|_{\mathbf{X_2}}\\ &\leq \Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}+\sum_{j=1}^{k} x_j\Big)+F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}-\sum_{j=1}^{k} x_j\Big)-2\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-2\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X_2}}\end{aligned}\tag{31}$$

for all $x_j, x_{k+j} \in \mathbf{X}_1$ for all $j = 1 \to k$ if and only if $F : \mathbf{X}_1 \to \mathbf{X}_2$ is quadratic.

Proof. Assume that $F : \mathbf{X_1} \to \mathbf{X_2}$ satisfies (31).

Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$ in (31), we get

$$\big(|2k-1|-1\big)\big\|F(0)\big\|_{\mathbf{X_2}} \leq 0.$$

So

$$F(0) = 0.$$

Other face

Letting $x_1 = x,\ x_{j+1} = x_{k+j} = 0, j = 1 \to k$ in (31), we obtain

$$\Big\|F\Big(\frac{x}{k}\Big)-\frac{1}{k}F(x)\Big\|_{\mathbf{X_2}} \leq 0$$

and so $F\big(\frac{x}{k}\big) = \frac{1}{k}F(x)$ Thus

$$\begin{aligned}&\Big\|F\Big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j}+\frac{1}{k}\sum_{j=1}^{k} x_j\Big)+F\Big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j}-\frac{1}{k}\sum_{j=1}^{k} x_j\Big)-\frac{2}{k}\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\Big)\Big\|_{\mathbf{X_2}}\\ &=\Big\|F\Big(\frac{1}{k}\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}+\sum_{j=1}^{k} x_j\Big)\Big)+F\Big(\frac{1}{k}\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}-\sum_{j=1}^{k} x_j\Big)\Big)-\frac{2}{k}\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\Big)\Big\|_{\mathbf{X_2}}\\ &=\Big|\frac{1}{k}\Big|\Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}+\sum_{j=1}^{k} x_j\Big)+F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}-\sum_{j=1}^{k} x_j\Big)-2\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-2\sum_{j=1}^{k}F(x_j)\Big)\Big\|_{\mathbf{X_2}}\\ &\leq \Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}+\sum_{j=1}^{k} x_j\Big)+F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}-\sum_{j=1}^{k} x_j\Big)-2\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-2\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X_2}}\end{aligned}\tag{32}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$ for all $j = 1 \to k$. Since $\Big|k\Big| < 1$

$$F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\Big) + F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\Big) - 2\sum_{j=1}^{k} F\Big(\frac{x_{k+j}}{k}\Big) - 2\sum_{j=1}^{k} F\big(x_j\big) = 0$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$ for all $j = 1 \to k$. On the other hand the converse is obviously true. □

Theorem 4.1. *Let $\varphi : \mathbf{X_1^{2k}} \to [0, \infty)$ be a function and let $F : \mathbf{X_1} \to \mathbf{X_2}$ be mapping with $\varphi(0, ..., 0, 0, ..., 0) = 0$ satisfying*

$$\varphi\big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\big) = \sum_{j=1}^{\infty} |k^j| \psi\Big(\frac{x_1}{k^j}, \frac{x_2}{k^j}, ..., \frac{x_k}{k^j}, \frac{x_{k+1}}{k^j}, \frac{x_{k+2}}{k^j}, ..., \frac{x_{2k}}{k^j}\Big) < \infty \tag{33}$$

and

$$\begin{aligned}
&\Big\|F\Big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\Big) + F\Big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j\Big) - \frac{2}{k}\sum_{j=1}^{k} F\Big(\frac{x_{k+j}}{k}\Big) - \frac{2}{k}\sum_{j=1}^{k} F\big(x_j\big)\Big\|_{\mathbf{X_2}} \\
&\le \Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\Big) + F\|\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\Big) - 2\sum_{j=1}^{k} F\Big(\frac{x_{k+j}}{k}\Big) - 2\sum_{j=1}^{k} F\big(x_j\big)\Big\|_{\mathbf{X_2}} \\
&+ \psi\big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\big)
\end{aligned} \tag{34}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : \mathbf{X_1} \to \mathbf{X_2}$ such that

$$\Big\|F\big(x\big) - H\big(x\big)\Big\|_{\mathbf{X_2}} \le |k|\varphi\big(x, 0, ..., 0, 0, .., 0\big) \tag{35}$$

for all $x \in \mathbf{X_1}$.

Proof. Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$ in (34), we get

$$\big(|2k-1| - 1\big)\Big\|F\big(0\big)\Big\|_{\mathbf{X_2}} \le 0.$$

So

$$F\big(0\big) = 0.$$

Other face

Letting $x_1 = x,\ x_{j+1} = x_{k+j} = 0$ for all $j = 1 \to k$ in (34), we get

$$\Big\|F\big(x\big) - kF\Big(\frac{x}{k}\Big)\Big\|_{\mathbf{X_2}} \le |k|\psi\big(x, 0, ..., 0, 0, ..., 0\big) \tag{36}$$

for all $x \in \mathbf{X_1}$ Hence

$$
\begin{aligned}
&\Big\|k^l F\Big(\frac{x}{k^l}\Big) - k^m F\Big(\frac{x}{k^m}\Big)\Big\|_{\mathbf{X_2}}\\
&\le max\Big\{\big\|k^l F(\frac{x}{k^l}) - k^{l+1}F(\frac{x}{k^{l+1}})\big\|_{\mathbf{X_2}}, \cdots, \big\|k^{m-1}F(\frac{x}{k^{m-1}}) - k^m F(\frac{x}{k^m})\big\|_{\mathbf{X_2}}\\
&\le max\Big\{\big|k\big|^l \Big\|F(\frac{x}{k^j}) - kF(\frac{x}{k^{j+1}})\Big\|_{\mathbf{X_2}}, \cdots, \big|k\big|^{m-1}\big\|F(\frac{x}{k^{m-1}}) - kF(\frac{x}{k^m})\big\|_{\mathbf{X_2}}\\
&\le \sum_{j=l}^{\infty}\big|k\big|^{j+1}\psi(\frac{x}{k^j}, 0, ..., 0, 0, 0, ..., 0)
\end{aligned}
\tag{37}
$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X_1}$. It follows from (37) that the sequence $\Big\{k^n F\Big(\frac{x}{k^n}\Big)\Big\}$ is a cauchy sequence for all $x \in \mathbf{X_1}$. Since $\mathbf{X_2}$ is complete space, the sequence $\Big\{k^n F\Big(\frac{x}{k^n}\Big)\Big\}$ coverges.

So one can define the mapping $H : \mathbf{X}_1 \to \mathbf{X}_2$ by

$$
H(x) := \lim_{n\to\infty} k^n F\Big(\frac{x}{k^n}\Big)
$$

for all $x \in \mathbf{X_1}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (37), we get (35).

Now, It follows from (33) and (34) that

$$
\begin{aligned}
&\Big\|H(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j) + H(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j) - \frac{2}{k}\sum_{j=1}^{k}H(\frac{x_{k+j}}{k}) - \frac{2}{k}\sum_{j=1}^{k}H(x_j)\Big\|_{\mathbf{X_2}}\\
&= \lim_{n\to\infty}\big|k\big|^n\Big\|F\Bigg(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}} + \frac{1}{k^{n+1}}\sum_{j=1}^{k}x_j) + F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}} - \frac{1}{k^{n+1}}\sum_{j=1}^{k}x_j) - \frac{2}{k}\sum_{j=1}^{k}F(\frac{x_{k+j}}{k^{n+1}}) - \frac{2}{k}\sum_{j=1}^{k}F(\frac{x_j}{k^n})\Big\|_{\mathbf{X_2}}\\
&\le \lim_{n\to\infty}\big|k\big|^n\Big\|F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}} + \frac{1}{k^n}\sum_{j=1}^{k}x_j) + F(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}} - \frac{1}{k^n}\sum_{j=1}^{k}x_j) - 2\sum_{j=1}^{k}F(\frac{x_{k+j}}{k^{n+1}}) - 2\sum_{j=1}^{k}F(\frac{x_j}{k^n})\Big\|_{\mathbf{X_2}}\\
&+ \big|\lim_{n\to\infty}\big|k\big|^n\psi(\frac{x_1}{k^n}, \frac{x_2}{k^n}, ..., \frac{x_k}{k^n}, \frac{x_{k+1}}{k^n}, \frac{x_{k+2}}{k^n}, ..., \frac{x_{2k}}{k^n})\\
&= \Big\|F(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j) + F(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j) - 2\sum_{j=1}^{k}F(\frac{x_{k+j}}{k}) - 2\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X_2}}
\end{aligned}
\tag{38}
$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$.

So

$$
\begin{aligned}
&\Big\|H(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j) + H(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j) - \frac{2}{k}\sum_{j=1}^{k}H(\frac{x_{k+j}}{k}) - \frac{2}{k}\sum_{j=1}^{k}H(x_j)\Big\|_{\mathbf{X_2}}\\
&\le \Bigg\|H(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j) + H(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j) - 2\sum_{j=1}^{k}H(\frac{x_{k+j}}{k}) - 2\sum_{j=1}^{k}H(x_j)\Bigg\|_{\mathbf{X_2}}
\end{aligned}
\tag{39}
$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$. By Lemma 4.1, the mapping $H : \mathbf{X_1} \to \mathbf{X_2}$ is quadratic. Next, suppose that $T : \mathbf{X_1} \to \mathbf{X_2}$ be another quadratic mapping satisfying (35). Then we have

$$\begin{aligned}\Big\|H(x) - T(x)\Big\|_{\mathbf{X_2}} &= \Big\|k^n H(\frac{x}{k^n}) - k^n T(\frac{x}{k^n})\Big\|_{\mathbf{X_2}}\\ &\leq max\{\Big\|k^n H(\frac{x}{k^n}) - k^n F(\frac{x}{k^n})\Big\|_{\mathbf{X_2}}, \Big\|k^n T(\frac{x}{k^n}) - k^n F(\frac{x}{k^n})\Big\|_{\mathbf{X_2}}\}\\ &\leq \frac{1}{|k|}|k|^n \varphi(\frac{x}{k^n}, 0, ..., 0, 0, ..., 0)\end{aligned} \tag{40}$$

which tends to zero as$n \to \infty$ for all $x \in \mathbf{X_1}$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbf{X_1}$. This proves the uniqueness of H. Thus the mapping $H : \mathbf{X_1} \to \mathbf{X_2}$ is a unique quadratic mapping satisfying (35). □

Corollary 4.1. *Let $r < 1$ and θ be nonnegative real numbers and $F : \mathbf{X_1} \to \mathbf{X_2}$ be a mapping satisfying*

$$\begin{aligned}&\Big\|F(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j) + F(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j) - \frac{2}{k}\sum_{j=1}^{k} F(\frac{x_{k+j}}{k}) - \frac{2}{k}\sum_{j=1}^{k} F(x_j)\Big\|_{\mathbf{X_2}}\\ &\leq \Big\|F(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j) + F(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j) - 2\sum_{j=1}^{k} F(\frac{x_{k+j}}{k}) - 2\sum_{j=1}^{k} F(x_j)\Big\|_{\mathbf{X_2}}\\ &+ \theta(\sum_{j=1}^{k} \|x_i\|_{\mathbf{X_1}}^r + \sum_{j=1}^{k} \|x_{k+i}\|_{\mathbf{X_1}}^r)\end{aligned} \tag{41}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : \mathbf{X}_1 \to \mathbf{X}_2$ such that

$$\Big\|F(x) - H(x)\Big\|_{\mathbf{X_2}} \leq \frac{|k|^{r+1}\theta}{|k|^r - |k|}\|x\|^r \tag{42}$$

for all $x \in \mathbf{X_1}$

Theorem 4.2. *Let $\varphi : \mathbf{X_1^{2k}} \to [0, \infty)$ be a function and let $F : \mathbf{X}_1 \to \mathbf{X}_2$ be mapping with with $\varphi(0, ..., 0, 0, ..., 0) = 0$ satisfying*

$$\begin{aligned}&\varphi\Big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\Big)\\ &= \sum_{j=1}^{\infty} \frac{1}{|k^j|}\psi\big(k^j x_1, k^j x_2, ..., k^j x_k, k^j x_{k+1}, k^j x_{k+2}, ..., k^j x_{2k}\big) < \infty\end{aligned} \tag{43}$$

$$\Big\|F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}$$
$$\leq\Big\|F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)-2\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-2\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}$$
$$+\psi\big(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\big) \tag{44}$$

for all $x_j, x_{k+j}\in\mathbf{X_1}$, *for all* $j=1\to k$. *Then there exists a unique quadratic mapping* $H:\mathbf{X}_1\to\mathbf{X}_2$ *such that*

$$\Big\|F\big(x\big)-H\big(x\big)\Big\|_{\mathbf{X_2}}\leq\big|k\big|\psi\big(x,0,...,0,0,..,0\big) \tag{45}$$

for all $x\in\mathbf{X_1}$.

Proof. Letting $x_1=x,\ x_{j+1}=x_{k+j}=0$ for all $j=1\to k$ in (44), we get

$$\Big\|F\big(x\big)-\frac{1}{k}F\big(kx\big)\Big\|_{\mathbf{X_2}}\leq\psi\big(kx,0,...,0,0,0,...,0\big) \tag{46}$$

for all $x\in\mathbf{X_1}$ the rest of the proof is similar to the proof of theorem 4.2. □

Corollary 4.2. *Let* $r>1$ *and* θ *be nonnegative real numbers and* $F:\mathbf{X_1}\to\mathbf{X_2}$ *be a mapping with* $F(0)=0$ *satisfying*

$$\Big\|F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-\frac{2}{k}\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}$$
$$\leq\Big\|F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\big)+F\big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\big)-2\sum_{j=1}^{k}F\big(\frac{x_{k+j}}{k}\big)-2\sum_{j=1}^{k}F\big(x_j\big)\Big\|_{\mathbf{X_2}}$$
$$+\theta\big(\sum_{j=1}^{k}\Big\|x_i\|^r_{\mathbf{X}_1}+\sum_{j=1}^{k}\|x_{k+i}\|^r_{\mathbf{X}_1}\big) \tag{47}$$

for all $x_j, x_{k+j}\in\mathbf{X_1}$, *for all* $j=1\to k$. *Then there exists a unique qudaratic mapping* $H:\mathbf{X}_1\to\mathbf{X}_2$ *such that*

$$\Big\|F\big(x\big)-H\big(x\big)\Big\|_{\mathbf{X}_2}\leq\frac{\big|k\big|^{r+1}\theta}{\big|k\big|-\big|k\big|^r}\big\|x\big\|^r \tag{48}$$

for all $x\in\mathbf{X_1}$

References

[1] Tosio Aoki. On the stability of the linear transformation in banach spaces. Journal of the Mathematical Society of Japan, 1950, 2(1-2), 64–66.

[2] Hyers-Ulam stability of functional inequalities with three variables in Banach spaces and non-Archimedean Banach spaces, International Journal of Mathematical Analysis, 2019, 13(11), 519–537. https://doi.org/10.12988/ijma.2019.9954

[3] Ly Van An, Generalized Hyers-Ulam type stability of the additive functional equation inequalities with 2n-variables on an approximate group and ring homomorphism, Asia Mathematika, 2020, 4(2), 161 – 175.

[4] Ly Van An, Hyers-Ulam stability quadratic β-functional inequalities with three variable in γ-homogeneous normed space International Journal of Mathematical Research, 2020, 12(1), 47–64. https://www.irphouse.com

[5] Cholewa, P W, Remarks on the stability of functional equations. Aequationes Math, 1984, 27, 76–86.

[6] Cieplinski, K, Applications of fixed point theorems to the Hyers-Ulam stability of functional equations - a survey, Ann. Funct. Anal., 2012, 3, 151–164.

[7] Eshaghi Gordji, M., Khodaei, H., Khodabakhsh, R. et al. Fixed points and quadratic equations connected with homomorphisms and derivations on non-Archimedean algebras. Adv Differ Equ 2012, 128 (2012). https://doi.org/10.1186/1687-1847-2012-128

[8] Włodzimierz Fechner, Stability of a functional inequality associated with the jordan–von neumann functional equation, Aequationes Mathematicae, 2006, 71(1), 149–161.

[9] P. Gavruta, A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings, Journal of Mathematical Analysis and Applications, 1994, 184(3), 431–436. https://doi.org/10.1006/jmaa.1994.1211.

[10] Gilányi, A. Eine zur Parallelogrammgleichung äquivalente Ungleichung. Aequat. Math. 2001, 62, 303–309. https://doi.org/10.1007/PL00000156

[11] Attila Gilányi, On a problem by k. nikodem. Mathematical Inequalities and Applications, 2002, 5, 707–710.

[12] Donald H Hyers, On the stability of the linear functional equation, Proceedings of the National Academy of Sciences of the United States of America, 1941, 27(4), 222.

[13] K. Hensel, Uber eine neue Begrundung der Theorie der algebraischen Zahlen, Jahres-ber Deutsch. Math. Verein, 6(1897), 83–88

[14] Jung Rye Lee, Choonkil Park, and Dong Yun Shin, Additive and quadratic functional in equalities in non-archimedean normed spaces, 2014.

[15] Moslehian M. S. , Sadeghi, Gh: Stability of two types of cubi functional equations in non-Archimedean spaces. Real Anal Exchange, 2008, 33, 375–383. .

[16] Moslehian and M.Th. Rassias, Stability of functional equations in non-Archimedean spaces, Appl. Anal. Discrete Math, 2007, 1, 325-–334.

[17] W. P and J. Schwaiger, A system of two inhomogeneous linear functional equations, Acta Math. Hungar, 2013, 140, 377–406

[18] Choonkil Park. Functional inequalities in non-archimedean normed spaces. Acta Mathematica Sinica, English Series, 2015, 31(3), 353–366.

[19] Themistocles M Rassias, On the stability of the linear mapping in banach spaces. Proceedings of the American Mathematical Society, 1978, 72(2), 297–300.

[20] I. Rajasekaran, Weak forms of strongly nano open sets in ideal nanotopological spaces. 2021, 5(2),96–102.

[21] Jürg Rätz, On inequalities associated with the jordan-von neumann functional equation. Aequationes mathematicae, 2003, 66(1), 191–200.

[22] Skof, F. Proprie locali e approssimazione di operatori. Rend. Sen. Mat. Fis. Milano, 1983, 53, 113–129.

[23] R. Sayed, S. A. Abd El-Baki and N. H. Sayed, Some properties and continuity of transitive binary relational sets, Asia Mathematika, 2021, 5(2), 49–59.

[24] Stanislaw M Ulam. A collection of mathematical problems, Interscience Publishers, 1960.