On $^b$-open sets and $^b$-sets in nano topological spaces

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Abstract: The aim of this paper is to introduce certain novel sets, $^b$-set and $^b$-set. Moreover, we study about some of their properties.

Key words: $n$-open sets, $ns$-open sets, $nb$-open sets and $n^b$-open sets and $n^b$-sets.

1. Introduction
M. Lellis Thivagar et al [1] introduced the concept of nano topological spaces which was defined in terms of approximations and boundary region of a subset of a universe $U$ using an equivalence relation on it.

2. Preliminaries

Definition 2.1. [3] Let $U$ be a non-empty finite set of objects called the universe and $R$ be an equivalence relation on $U$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U, R)$ is said to be the approximation space. Let $X \subseteq U$.

1. The lower approximation of $X$ with respect to $R$ is the set of all objects, which can be for certain classified as $X$ with respect to $R$ and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U} \{ R(x) : R(x) \subseteq X \}$, where $\tau_R(x)$ denotes the equivalence class determined by $x$.

2. The upper approximation of $X$ with respect to $R$ is the set of all objects, which can be possibly classified as $X$ with respect to $R$ and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U} \{ R(x) : R(x) \cap X \neq \phi \}$.

3. The boundary region of $X$ with respect to $R$ is the set of all objects, which can be classified neither as $X$ nor as not $X$ with respect to $R$ and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [1] Let $U$ be the universe, $R$ be an equivalence relation on $U$ and $\tau_R(X) = \{ U, \phi, L_R(X), U_R(X), B_R(X) \}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:

1. $U$ and $\phi \in \tau_R(X)$,
2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,
3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

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Thus $\tau_R(X)$ is a topology on $U$ called the nano topology with respect to $X$ and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a $n$-open set is called $n$-closed.

Throughout this paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset $H$ of $U$ are denoted by $I_n(H)$ and $C_n(H)$, respectively.

**Definition 2.3.** A subset $H$ of a space $(U, \mathcal{N})$ is called

1. nano semi-open set (resp. ns-open) [1] if $H \subseteq C_n(I_n(H))$.
2. nano pre-open (resp. np-open) [1] if $H \subseteq I_n(C_n(H))$.
3. nano regular-open (resp. nr-open) [1] if $H = I_n(C_n(H))$.
4. nano b-open (resp. nb-open) [2] if $H \subseteq I_n(C_n(H)) \cup C_n(I_n(H))$.

The complements of the above mentioned sets are called their respective closed sets.

3. On nano $b$-open sets and nano $b$-sets

**Definition 3.1.** Let $(U, \mathcal{N})$ be a nanotopological space and $A \subseteq U$. Then,

1. $H$ is said to be nano $b$-open (resp. $n^b$-open) if $H \subseteq C_n(I_n(H)) \cap I_n(C_n(H))$.
   
   The complement of a $b$-open set is called $b$-closed set.
2. An element $x \in H$ is said to be $n^b$-interior point of $H$ if there exits a $n^b$-open set $K$ such that $x \in K \subseteq H$.
   
   The set of all $n^b$-interior points of $H$ is said to be $n^b$-interior of $H$ and it is denoted by $b-I_n(H)$.

**Proposition 3.1.** In a nanotopological spaces $(U, \mathcal{N})$, every open set is $n^b$-open.

**Proof.**

Let $H$ be a $n$-open set of $U$. Then, $H \subseteq C_n(I_n(H)) \cap I_n(C_n(H))$, since $H$ is $n$-open set, then $I_n(H) = H$ and $C_n(H) = U$ or $C_n(H) = Q$ where $Q$ is $n$-closed set of $U$. Thus, $H \subseteq C_n(H) \cap I_n(Q)$. Now, $C_n(H) = U$ or $C_n(Q)$ where $Q$ is $n$-closed set of $U$ and $H \subseteq I_n(Q)$, therefore $I_n(Q) = H$ or $I_n(Q) = S$ where $H \subseteq S \subseteq I_n(Q)$. In consequence, $H \subseteq Q \cap S$, but $H \subseteq Q$ and $H \subseteq S$, this implies that $H \subseteq Q \cap S$.

Hence $H$ is a $n^b$-open set of $U$. \(\square\)

**Remark 3.1.** The next Example shows that the converse of the Proposition 3.1, it is not always true.

**Example 3.1.**

Let $U = \{\alpha, \beta, \gamma, \eta, \delta\}$ with $U/R = \{\{\alpha\}, \{\beta, \gamma\}, \{\eta, \delta\}\}$ and $X = \{\alpha, \beta\}$. Then the nano topology $\mathcal{N} = \{\phi, \{\alpha\}, \{\beta, \gamma\}, \{\alpha, \beta, \gamma\}, U\}$. Clear the set $\{\alpha, \beta\}$ is $n^b$-open set but $n$-open.

**Theorem 3.1.** Let $(U, \mathcal{N})$ be a nanotopological space and $H \subseteq U$. Then, $b-I_n(H) = \bigcup\{K|K$ is $n^b$-open and $K \subseteq H\}$.

**Proof.**

Let $x \in b-I_n(H)$. Then, there exits $n^b$-open set $K$ such that $x \in K \subseteq H$. Hence, $x \in \bigcup\{K|K$ is $n^b$-open and $K \subseteq H\}$. Then, $y \subseteq K_0$ for some $n^b$-open set $K_0 \subseteq H$. Therefore, $y \in b-I_n(H)$.

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Theorem 3.2. Let \((U, \mathcal{N})\) be a nanotopological space and \(H \subseteq U\). Then, \(H\) is \(n^*b\)-open \(\iff H = ^*b-I_n(H)\).

Proof.
Let \(H\) be a \(n^*b\)-open set. Then, \(H \subseteq H\) and this implies that \(H \in \{K | K\) is \(n^*b\)-open and \(K \subseteq H\}\). Since union of this collection is in \(H\). Therefore, \(H = ^*b-I_n(H)\). Conversely, suppose that \(H = ^*b-I_n(H)\). Hence, \(H\) is \(n^*b\)-open.

Theorem 3.3. Let \((U, \mathcal{N})\) be a nanotopological space and \(H \subseteq U\). If \(n^*b\)-open set is \(ns\)-open.

Proof.
Let \(H\) be a \(n^*b\)-open set, then \(H \subseteq C_n(I_n(H)) \cap I_n(C_n(H))\); \(H \subseteq C_n(I_n(H))\). Therefore, \(H\) is a \(ns\)-open set.

Remark 3.2. The next Example shows that the converse of the Theorem 3.3, it is not always true.

Example 3.2. In Example 3.1, the set \(\{\alpha, \eta\}\) is \(ns\)-open set but not \(n^*b\)-open set.

Theorem 3.4. Let \((U, \mathcal{N})\) be a nanotopological space and \(H \subseteq U\). If \(n^*b\)-open set is \(np\)-open.

Proof.
Let \(H\) be a \(n^*b\)-open set, then \(H \subseteq C_n(I_n(H)) \cap I_n(C_n(H))\); \(H \subseteq I_n(C_n(H))\). Therefore, \(H\) is a \(np\)-open set.

Remark 3.3. The converse of the Theorem 3.4 is not always true when \(C_n(I_n(H)) = \phi\) and \(I_n(C_n(H)) = Q\), where \(H \subseteq Q\).

Proposition 3.2. In a nanotopological spaces, every \(n^*b\)-open set is \(nb\)-open set.

Proof.
The proof is followed by the Theorem 3.3 and 3.4.

Theorem 3.5. The Intersection of a \(nb\)-open set and a \(n^*b\)-open set is a \(nb\)-open set.

Proof.
Let \(H\) be a \(n^*b\)-open set and \(Q\) be a \(nb\)-open set.
Then \(H \cap Q \subseteq [C_n(I_n(H)) \cap C_n(I_n(Q))] \cap [C_n(I_n(H)) \cap C_n(I_n(Q))] = [C_n(I_n(H)) \cap C_n(I_n(H)) \cap C_n(I_n(Q))] \cup [I_n(C_n(H)) \cap C_n(I_n(H)) \cap I_n(C_n(Q))] \subseteq [I_n(C_n(H)) \cap C_n(I_n(H \cap Q))] \cup [C_n(I_n(H)) \cap C_n(I_n(H \cap Q))].

By The Proposition 3.2, \(H \cap Q\) is a \(nb\)-open set and let \(H \cap Q = S\).
Then \([C_n(I_n(H)) \cap C_n(I_n(S))] \cup [C_n(I_n(H)) \cap I_n(C_n(S))].
Now, \(I_n(C_n(H)) \cap C_n(I_n(S)) = E\), where \(E \neq \phi\) and \(C_n(I_n(H)) \cap I_n(C_n(S)) = F\), where \(F \neq \phi\) and so, \(H \cap Q \subseteq E \cup F\), where \(E \cup F\) is a \(nb\)-open set.

\(\square\)
4. On nano *b*-sets

**Definition 4.1.** Let \((U, \mathcal{N})\) be a nanotopological space and \(H \subseteq U\). Then, \(H\) is said to be nano *b*-set (resp. \(n^*\)-set) if \(H = C_n(I_n(H)) \cap I_n(C_n(H))\).

**Remark 4.1.** The next example shows that the union of two \(n^*\)-sets not need be a \(n^*\)-set.

**Theorem 4.1.** In a nanotopological spaces, the intersection of two \(n^*\)-sets is a \(n^*\)-set.

**Proof.**
Let \(H\) and \(Q\) two \(n^*\)-sets, then \(H = C_n(I_n(H)) \cap I_n(C_n(H))\) and \(Q = C_n(I_n(Q)) \cap I_n(C_n(Q))\). Now, \(H \cap Q = [C_n(I_n(H)) \cap I_n(C_n(H))] \cap [C_n(I_n(Q)) \cap I_n(C_n(Q))] = [C_n(I_n(H)) \cap C_n(I_n(Q))] \cap [I_n(C_n(H)) \cap I_n(C_n(Q))] \subseteq [C_n(I_n(H)) \cap I_n(C_n(H)) \cap C_n(C_n(Q))] \subseteq C_n(I_n(H)) \cap C_n(H) \cap C_n(Q)\).

Therefore, \(H \cap Q\) is a \(n^*\)-set.

**Theorem 4.2.** In nanotopological spaces, If \(H\) is a \(n^*\)-set then \(H\) is a \(n^*\)-open set.

**Proof.**
Let \(H\) be a \(n^*\)-set, then \(H = C_n(I_n(H)) \cap I_n(C_n(H))\); \(H \subseteq C_n(I_n(H)) \cap I_n(C_n(H))\). Therefore, \(H\) is a \(n^*\)-open set.

**Remark 4.2.** The next example shows that the converse of the Theorem 4.2, it is not always true.

**Example 4.1.** In Example 3.1, the set \(\{\alpha, \beta\}\) is \(n^*\)-open set but \(n^*\)-set.

**Proposition 4.1.** Let \((U, \mathcal{N})\) be a nanotopological space and \(H \subseteq U\). Then, the following statements hold:

1. If \(H\) is a \(n^*\)-set, then \(H\) is a \(ns\)-open set.
2. If \(H\) is a \(n^*\)-set, then \(H\) is a \(np\)-open set.
3. If \(H\) is a \(n^*\)-set, then \(H\) is a \(nb\)-open set.

**Proof.**

1. Let \(H\) be a \(n^*\)-set, then \(H = C_n(I_n(H)) \cap I_n(C_n(H)) \subseteq C_n(I_n(H)) \cap (I_n(C_n(H)) \subseteq C_n(I_n(H)))\). Therefore, \(H\) is \(ns\)-open.
2. Let \(H\) be a \(n^*\)-set, then \(H = C_n(I_n(H)) \cap I_n(C_n(H)) \subseteq C_n(I_n(H)) \cap (I_n(C_n(H)) \subseteq I_n(C_n(H)))\). Therefore, \(H\) is \(np\)-open.
3. Let \(H\) be a \(n^*\)-set, then \(H = C_n(I_n(H)) \cap I_n(C_n(H)) \subseteq C_n(I_n(H)) \cap (I_n(C_n(H)) \subseteq C_n(I_n(H)) \cup I_n(C_n(H)))\). Therefore, \(H\) is \(nb\)-open.

**Theorem 4.3.** Let \((U, \mathcal{N})\) be a nanotopological space and \(H \subseteq U\). If \(H\) is a \(n^*\)-set and \(C_n(I_n(H)) \subsetneq I_n(C_n(H))\), then \(H\) is a \(nr\)-open set.

**Proof.**
Let \(H\) be a \(n^*\)-set, then \(H = C_n(I_n(H)) \cap I_n(C_n(H))\), since \(C_n(I_n(H)) \subseteq I_n(C_n(H))\), then \(H \in I_n(C_n(H))\). Therefore, \(H\) is a \(nr\)-open set.
Remark 4.3. These relations are shown in the diagram.

Example 4.2. In Example 3.1, the set \( \{ \beta \} \) is np-open set but not \( n^*b \)-open.

References


