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Local existence for class of nonlinear higher-order wave equation with logarithmic source term

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Abstract: The purpose of this work is to establish the local existence for a class of higher-order logarithmic wave equation with memory term. The local existence result was established by means of Faedo-Galerkin technique and Logarithmic Sobolev inequality.

Key words: Existence, Nolinear higher-order, Logarithmic nonlinearity.

1. Introduction

Let $\Omega \subset R^3$ be an open bounded subset in with smooth boundary $\partial\Omega$. In this article, we investigate the local existence of solutions the following nonlinear initial boundary value equation for $x \in \Omega \times (0,T)$

$$|u_t|^{\rho} u_{tt} + \Delta^2 u_{tt} + M\left(\int_0^t \left|A^{\frac{1}{2}}u\right|^2\right) Au - \int_0^t h(t-s)\, Au\, ds = u \ln |u|, \tag{1}$$

and initial-boundary conditions

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0,1,2,...,m-1, & x \in \partial\Omega, \end{cases}$$

$\rho > 0$ is a exponent and $A = (-\triangle)^m u,\ m \geq 1$ is a positive integer. And we take $M(s) = 1 + s^{\gamma}$. Convenient hypotheses on h to be specified later.

It is clear from the researches that the logarithmic wave equations are contra distinguished from several interesting physical properties. They appeared in many branches of physics such as inflation cosmology, super symmetric field theories, quantum mechanics, nuclear physics (see [7, 8, 14]). In past years, the hyperbolic equations with logarithmic nonlinearity have captured lots of attention. Hereby, logarithmic wave equations have been analyzed and several results concerning mathematical behavior have been established by many mathematicians, we refer to the studies [10, 13, 17, 20, 22, 23].

In [4], for case $m = 2$, problem (1) was studied. The authors obtained well posedness and asymptotic stability of solutions for the problem. Later, different authors obtained properties of mathematical behavior for

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hyperbolic type equations with viscoelastic term (see [2, 3, 5, 15]). The same authors of this paper, in their work [19], investigated the following equation

$$u_{tt}+\Delta^{2}u_{tt}+(-\triangle)^{m}u-\int_{0}^{t}g(t-s)(-\triangle)^{m}u\,ds+u=u\ln|u|^{k}. \tag{2}$$

They showed that energy functional of the problem (2) grow thing exponentially to infinity as the time goes to infinity growth of the solution. Later, they add the damping and strong damping terms to the problem (2). In [18], they established the existence and asymptotic behavior of solution for the problem.

On the other hand we can say that it is qualitative theory of higher order and fractional hyperbolic type equations without logarithmic source term still often studied (see [12, 21]).

Motivated by the above studies, our purpose in the present paper is to proved the local existence of a weak solution for the problem (1).

2. Preliminaries

We consider Sobolev Space $H_0^m(\Omega)$ as the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we state by $\|.\|_q$ and $\|.\|_2$ $L^q(\Omega)$ norm, $\|.\|_2$ Lebesgue space $L^2(\Omega)$ norm and we write equivalent norm $\left\|A^{\frac{1}{2}}u\right\|$ instead of $H_0^m(\Omega)$ norm (see [1, 16], for details). Now we give some Lemma which will be used for the proof of the Theorem 7.

Lemma 2.1. *[11] Suppose that u is any function $u\in H_0^1(\Omega)$, and $a>0$ is a constant. Then,*

$$\int_{\Omega}u^{2}\ln|u|\,dx<\left(\frac{3}{4}\ln\frac{4a}{e}\right)\|u\|^{2}+\frac{a}{4}\|\nabla u\|^{2}+\|u\|\ln|u|.$$

Corollary 2.1. *Suppose that s satisfies $2\le s<\infty$ for $n\le 2m$ and $2\le s<\frac{2n}{n-2m}$ for $n>2m$. Then, c_p is small enough positive constant fulfills*

$$\|u\|_{s}\le c_{p}\left\|A^{\frac{1}{2}}u\right\|,\quad \forall u\in H_0^m(\Omega).$$

. Then, we obtain

$$\int_{\Omega}u^{2}\ln|u|\,dx<\left(\frac{3}{4}\ln\frac{4a}{e}\right)\|u\|^{2}+\frac{c_{p}a}{4}\left\|A^{\frac{1}{2}}u\right\|^{2}+\|u\|\ln|u|. \tag{3}$$

Lemma 2.2. *[9]. Suppose that $y(t)$ is element of the bounded function space a.e. everywhere in the $(0,T)$ region, $y(t)\ge 0$ and $y(0)\ge 0$, and the function satisfies that for $t\in[0,T]$*

$$y(t)\le y(0)+\mu\int_{0}^{t}y(s)\left[\ln\mu+y(s)\right]ds,$$

where $\alpha>1$. Therefore, we conclude that

$$y(t)\le(\mu+y(0))^{e^{\mu t}},\ t\in[0,T]. \tag{4}$$

Lemma 2.3. *[4] Let* $\epsilon_0 \in (0,1)$ *hold. Then there is* $d_{\epsilon_0} > 0$ *such that for* $\forall x > 0$

$$x\left|\ln x\right| \leq x^2 + d_{\epsilon_0} x^{1-\epsilon_0}. \tag{5}$$

$(A1)$ $h : R^+ \to R^+$ is a C^1 non increasing function

$$h(s) \geq 0,\ h'(s) \leq 0,\ \int_0^\infty h(s)\,ds < \infty,\ 1 - \int_0^\infty h(s)\,ds = l_0 > 0, \tag{6}$$

Lemma 2.4. *We define the energy of equation (1) such that*

$$\begin{aligned} E(t) &= \frac{1}{\rho+2}\left\|u_t\right\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left\|\Delta u_t\right\|_2^2 \\ &+\frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)\left\|A^{\frac{1}{2}}u\right\|^2 \\ &+\frac{1}{2(\gamma+1)}\left\|A^{\frac{1}{2}}u\right\|^{2(\gamma+1)} \\ &+\frac{1}{4}\left\|u\right\|^2 + \frac{1}{2}h \circ A^{\frac{1}{2}}u - \frac{1}{2}\int_\Omega \ln|u|\,u^2 dx. \end{aligned} \tag{7}$$

The energy functional defined by (7) is decreasing with respect to t.

Proof. The both sides of the equation (1) was multiplied by u_t and integrated over Ω, we obtain

$$\begin{aligned} &\int_\Omega |u_t|^{\rho+1} u_{tt} dx + \int_\Omega \Delta^2 u_t u_t dx \\ &+\int_\Omega M\left(\int_0^t \left|(-\triangle)^{\frac{m}{2}} u\right|^2\right)(-\triangle)^m u u_t dx \\ &-\int_\Omega \int_0^t h(t-s)(-\triangle)^m u u_t\, ds dx \\ = &\int_\Omega u \ln|u|\, u_t dx, \end{aligned}$$

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{\rho+2}\left\|u_t\right\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left\|\Delta u_t\right\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)\left\|A^{\frac{1}{2}}u\right\|^2\right. \\ &\left.+\frac{1}{2(\gamma+1)}\left\|A^{\frac{1}{2}}u\right\|^{2(\gamma+1)} + \frac{1}{4}\left\|u\right\|^2 + \frac{1}{2}h \circ A^{\frac{1}{2}}u - \frac{1}{2}\int_\Omega u^2 \ln|u|\,dx\right] \\ = &\ \frac{1}{2}\left[h' \circ P^{\frac{1}{2}}u - \int_0^t h(s)\,ds\left\|A^{\frac{1}{2}}u\right\|^2\right] \leq 0, \end{aligned} \tag{8}$$

$$E'(t) = \frac{1}{2}\left[h' \circ A^{\frac{1}{2}} - g(t)\left\|A^{\frac{1}{2}}u\right\|^{2}\right] \leq 0. \tag{9}$$

□

3. Local existence

In this section we establish the proof of local existence solution for our problem (1). We used Faedo-Galerkin technique and Logarithmic Sobolev inequality for our proof.

Definition 3.1. A function u defined as weak solution of problem (1) on $[0,T]$ if

$$u \in C\left([0,T); H_0^m(\Omega) \cap H^{2m}(\Omega)\right), \ u_t \in C\left([0,T); H_0^2(\Omega)\right),$$

and u satisfies

$$\begin{aligned}
&\int_\Omega |\, u_t|^\rho\, u_{tt}(x,t)\, w(x)\, dx + \int_\Omega \Delta u_{tt}(x,t)\, \Delta w(x)\, dx \\
&\int_\Omega M\left[\left(\int_\Omega \left|A^{\frac{1}{2}}u\right|^2 dx\right)\right] A^{\frac{1}{2}}uA^{\frac{1}{2}}w(x)\, dx \\
&-\int_\Omega \int_0^t h(t-s)\left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w\right) dx \\
= &\int_\Omega u(x,t)\ln|u(x,t)|\, w(x)\, dx,
\end{aligned}$$

for $w \in H_0^m(\Omega)$.

Theorem 3.1. *Let (A1) and initial conditions* $(u_0,u_1) \in \left(H_0^m(\Omega) \cap H^{2m}(\Omega)\right) \times H_0^2(\Omega)$ *hold. Morever there is a weak solution for problem (1) such that*

$$u \in L\left(0,T,H_0^m(\Omega) \cap H^{2m}(\Omega)\right), \ u_t \in L\left(0,T,H_0^2(\Omega)\right), u_{tt} \in L(0,T,H_0^{-2}(\Omega).$$

Proof. Our aim is to establish approximate solutions according to Faedo-Galerkin method. Assume that $\{w_j\}_{j=1}^\infty$ is an eigenfunctions of the $A = (-\triangle)^m$ with the initial data on V_n which is defined the finite dimensional subspace be given by

$$\begin{aligned}
u_0^n(x) &= \sum_{j=1}^n a_j w_j(x) \to u_0 \text{ in } H_0^m(\Omega), \\
u_1^n(x) &= \sum_{j=1}^n b_j w_j(x) \to u_1 \text{ in } H_0^2(\Omega),
\end{aligned} \tag{10}$$

for $j = 1,2,...,n$. It is well known $\{w_j\}_{j=1}^\infty$ is an orthogonal system of a base function in space $H_0^m(\Omega)$ which is orthonormal in $H_0^2(\Omega)$

$$V_n = \text{span}\{w_1, w_2, ..., w_n\}.$$

Let

$$u^{n}(x,t)=\sum_{j=1}^{n}k_{j}^{n}(t)\,w_{j}(x),$$

be approximate solution. Then $k_j^n(t)$ verifies a system of ordinary differantial equations in V_n such that

$$\begin{cases} \int\limits_{\Omega}\Big[|u_t^n|^{\rho}u_{tt}^n w+\ \Delta u_{tt}^n\Delta w+M\left[\left(\int_{\Omega}\left|A^{\frac{1}{2}}u^n\right|^2dx\right)\right]A^{\frac{1}{2}}(u^n)\,A^{\frac{1}{2}}w(x) \\ -\int\limits_{0}^{t}h(t-s)\left(A^{\frac{1}{2}}u^n,A^{\frac{1}{2}}w\right)dx\Big] \\ =\int\limits_{\Omega}\ln|u^n|\,u^n w dx,\ \ w\in V_n, \\ u^n(0)=u_0^n=\ \sum\limits_{j=1}^{n}(u_0,w_j)\,w_j, \\ u_t^n(0)=u_1^n=\ \sum\limits_{j=1}^{n}(u_1,w_j)\,w_j. \end{cases} \tag{11}$$

Based on standard existence theory (Peano's theorem) for ordinary differantial equation, there is a maximal interval $[0,t_n)$ such that $k_j^n(t)\in C^2[0,t_n)$. Morever we conclude that $u^n\in C^2([0,t_n),H_0^m(\Omega))$. Next, our aim is to show that

i) $t_n=T$

ii) u^n is uniformly indboundedependent of t and n.

Firstly, we take $w=u_t^n$ in the equation (11), by direct calculation, we have

$$\begin{aligned} &\frac{d}{dt}\Bigg[\frac{1}{\rho+2}\|u_t^n\|_{\rho+2}^{\rho+2}+\frac{1}{2}\ \|\Delta u_t^n\|^2+\frac{1}{2}\left(1-\int\limits_0^t h(s)\,ds\right)\left\|A^{\frac{1}{2}}u^n\right\|^2 \\ &\frac{1}{2(\gamma+1)}\left\|A^{\frac{1}{2}}u^n\right\|^{2(\gamma+1)}+\frac{1}{2}h\circ A^{\frac{1}{2}}u^n+\frac{1}{4}\|u^n\|^2-\frac{1}{2}\int\limits_{\Omega}\ln|u^n|\,(u^n)^2\,dx\Bigg] \\ =\ &\frac{1}{2}\left[h'\circ A^{\frac{1}{2}}u^n-\int\limits_0^t h(s)\,ds\left\|A^{\frac{1}{2}}u^n\right\|^2\right]. \end{aligned} \tag{12}$$

So that, from (12) we can write

$$\frac{d}{dt}E^n(t)=\frac{1}{2}\left(h'\circ A^{\frac{1}{2}}u^n\right)-\frac{1}{2}h(t)\left\|A^{\frac{1}{2}}u^n\right\|^2\leq\frac{1}{2}\left(h'\circ A^{\frac{1}{2}}u^n\right)\leq 0, \tag{13}$$

where

$$\begin{aligned} E^n(t)\ =\ &\frac{1}{\rho+2}\|u_t^n\|_{\rho+2}^{\rho+2}+\frac{1}{2}\ \|\Delta u_t^n\|^2+\frac{1}{2}\left(1-\int\limits_0^t h(s)\,ds\right)\left\|A^{\frac{1}{2}}u^n\right\|^2 \\ &\frac{1}{2(\gamma+1)}\left\|A^{\frac{1}{2}}u^n\right\|^{2(\gamma+1)}+\frac{1}{2}h\circ A^{\frac{1}{2}}u^n+\frac{1}{4}\|u^n\|^2-\frac{1}{2}\int\limits_{\Omega}\ln|u^n|\,(u^n)^2\,dx. \end{aligned}$$

Consequently if we integrate (13) over $(0,t)$, we have the following inequality

$$E^n(t) \leq E^n(0). \tag{14}$$

If we use the Logarithmic Sobolev Inequality, we leads to

$$\begin{aligned} E^n(t) \geq\ & \frac{1}{\rho+2}\|u_t^n\|_{\rho+2}^{\rho+2} + \frac{1}{2}\ \|\Delta u_t^n\|^2 + \frac{1}{2(\gamma+1)}\left\|A^{\frac{1}{2}}u^n\right\|^{2(\gamma+1)} \\ & + \frac{1}{2}\left[\left(1-\int_0^t h(s)\,ds\right)\left\|A^{\frac{1}{2}}(u^n)\right\|^2 + h\circ A^{\frac{1}{2}}u^n + \frac{1}{2}\|u^n\|^2\right] \\ & - \frac{1}{2}\left[\left(\frac{3}{4}\ln\frac{4a}{e}\right)\|u^n\|_2^2 + \frac{c_p a}{4}\left\|A^{\frac{1}{2}}u^n\right\|^2 + \|u^n\|^2\ln\|u^n\|\right], \\ =\ & \frac{1}{\rho+2}\|u_t^n\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(l_0-\frac{c_p a}{4}\right)\left\|A^{\frac{1}{2}}u^n\right\|^2 + \frac{1}{2}h\circ A^{\frac{1}{2}}u^n \\ & + \frac{1}{2(\gamma+1)}\left\|A^{\frac{1}{2}}u^n\right\|^{2(\gamma+1)} + \left(\frac{1}{4}-\left(\frac{3}{8}\ln\frac{4a}{e}\right)\right)\|u^n\|^2 \\ & - \frac{1}{2}\|u^n\|^2\ln\|u^n\|, \end{aligned} \tag{15}$$

By combining (14) and (15), we obtain

$$\begin{aligned} & \frac{2}{\rho+2}\|u_t^n\|_{\rho+2}^{\rho+2} + \|\Delta u_t^n\|^2 + \left(l_0-\frac{c_p a}{4}\right)\left\|A^{\frac{1}{2}}u^n\right\|^2 \\ & + \frac{1}{(\gamma+1)}\left\|A^{\frac{1}{2}}u^n\right\|^{2(\gamma+1)} + h\circ P^{\frac{1}{2}}(u^n) \\ & + \left(\frac{1}{2}-\left(\frac{3}{4}\ln\frac{4a}{e}\right)\right)\|u^n\|^2 \\ \leq\ & C + \|u^n\|^2\ln\|u^n\|, \end{aligned} \tag{16}$$

where $C = 2E^n(0)$.

By taking $\alpha = \min\left\{\frac{4l_0}{kc_p}, \frac{e^{\frac{5}{3}}}{4}\right\}$ we guarantee

$$\left(l_0-\frac{c_p a}{4}\right) > 0$$

and

$$\left(\frac{1}{2}-\left(\frac{3}{4}\ln\frac{4a}{e}\right)\right) > 0.$$

By this choice we have

$$\begin{aligned} & \|u_t^n\|_{\rho+2}^{\rho+2} + \|\Delta u_t^n\|^2 + \left\|A^{\frac{1}{2}}u^n\right\|^2 \\ & + \left\|A^{\frac{1}{2}}u^n\right\|^{2(\gamma+1)} + h\circ A^{\frac{1}{2}}u^n + \|u^n\|^2 \\ \leq\ & C\left(1+\|u^n\|^2\ln\|u^n\|\right). \end{aligned} \tag{17}$$

We know that

$$u^n(.,t) = u^n(.,0) + \int_0^t \frac{\partial u^n}{\partial \upsilon}(.,\upsilon)\, d\upsilon.$$

Then, ıf we use the following inequality

$$(a+b)^p \leq 2^{p-1}(a^p + b^p),$$

for $p = 2,$ we obtain

$$\begin{aligned} \|u^n(t)\|^2 &= \left\| u^n(.,0) + \int_0^t \frac{\partial u^n}{\partial \upsilon}(.,\upsilon)\, d\upsilon \right\|^2 \\ &\leq 2\|u^n(0)\|^2 + 2\left\| \int_0^t \frac{\partial u^n}{\partial \upsilon}(.,\upsilon)\, d\upsilon \right\|^2 \\ &\leq 2\|u^n(0)\|^2 + \max\{1, 2T\} \frac{C_1+1}{C_1} \int_0^t \|u_t^n(\upsilon)\|^2\, d\upsilon. \end{aligned} \tag{18}$$

Because of inequality (17), inequality (18) leads to

$$\|u^n(t)\|^2 \leq 2\|u^n(0)\|^2 + \max\{1, 2T\} \frac{C_1+1}{C_1} C\left(1 + \|u^n\|^2 \ln\|u^n\|\right). \tag{19}$$

Then by (19), we obtain

$$\|u^n\|^2 \leq M + N \int_0^t \|u^n\|^2 \ln\|u^n\|\, d\tau, \tag{20}$$

where

$$M = 2\|u^n(0)\|^2 + \max\{1, 2T\}(1+C_1)T,\ N = \max\{1, 2T\}(1+C_1).$$

Because of knowing $N \geq 1,$ then using Logarithmic Gronwall inequality, we obtain

$$\|u^n\|^2 \leq (M+N)^{e^{Nt}} \leq C_2. \tag{21}$$

Hence, from inequality (21) and (17)

$$\|u_t^n\|_{\rho+2}^{\rho+2} + \|\Delta u_t^n\|^2 + \left\|A^{\frac{1}{2}} u^n\right\|^2 + h \circ A^{\frac{1}{2}} u^n + \|u^n\|^2 \leq C_3, \tag{22}$$

where $C_3 > 0$ and independent of n and t. So, we show that u^n is uniformly bounded independent of n and t. Morever, we can take $t_n = T$.

Substituting $u_{tt}^n = w$ in (11) and by thanks to Young's, Cauchy-Schwarz and embedding's inequalities, we get

$$
\begin{aligned}
&\int_\Omega |u_t^n|^\rho |u_{tt}^n|^2 dx + \|\Delta u_{tt}^n\|^2 \\
= & -\int_\Omega M\left(\left\|A^{\frac{1}{2}}u\right\|^2\right) A^{\frac{1}{2}}u^n A^{\frac{1}{2}}u_{tt}^n dx + \int_\Omega u^n \ln|u^n| u_{tt}^n dx \\
& + \int_\Omega \int_0^t h(t-s) A^{\frac{1}{2}}u^n A^{\frac{1}{2}}u_{tt}^n \, ds dx \\
\leq & -\int_\Omega A^{\frac{1}{2}}uA^{\frac{1}{2}}u_{tt} dx - \left\|A^{\frac{1}{2}}u\right\|^{2\gamma} \int_\Omega A^{\frac{1}{2}}uA^{\frac{1}{2}}u_{tt} dx \\
& + \int_\Omega \int_0^t h(t-s) A^{\frac{1}{2}}u^n A^{\frac{1}{2}}u_{tt}^n \, ds dx + \int_\Omega u^n \ln|u^n| u_{tt}^n dx \\
\leq & \ \delta\left(1 + \left\|A^{\frac{1}{2}}u\right\|^{2\gamma}\right)\left\|A^{\frac{1}{2}}u_{tt}^n\right\|^2 + \frac{1}{4\delta}\left(\int_0^t h(t-s)\left\|A^{\frac{1}{2}}u^n\right\| ds\right)^2 \\
& + \delta\left\|A^{\frac{1}{2}}u_{tt}^n\right\| + \frac{1}{4\delta}\left(1 + \left\|A^{\frac{1}{2}}u\right\|^{2\gamma}\right)\left\|A^{\frac{1}{2}}u^n\right\|^2 + \int_\Omega \ln|u^n| u^n u_{tt}^n dx. \qquad (23)
\end{aligned}
$$

Now, we try to have estimation for last term of (23). For this reason Lemma 4 with $\epsilon_0 = \frac{1}{2}$ and some based inequalities are used. So that, (23) becomes

$$
\begin{aligned}
\int_\Omega \ln|u^n| u^n u_{tt}^n dx &\leq c\int_\Omega \left(|u^n|^2 + d_2\sqrt{u^n}\right) u_{tt}^n dx \\
&\leq c\left(\delta\int_\Omega u_{tt}^n dx + \frac{1}{4\delta}\int_\Omega \left(|u^n|^2 + d_2\sqrt{u^n}\right)^2 dx\right) \\
&\leq c\delta\|\Delta u_{tt}^n\|^2 + \frac{c}{4\delta}\left(\int_\Omega |u^n|^4 dx + \int_\Omega |u^n| dx\right) \\
&\leq c\delta\|\Delta u_{tt}^n\|^2 + \frac{c}{4\delta}\left(\|\Delta u^n\|_2^4 + \|u^n\|_2\right). \qquad (24)
\end{aligned}
$$

Combining (24) and (23) to have

$$\int_{\Omega} |u_t^n|^{\rho} |u_{tt}^n|^2 dx + \left(1 - c\delta - \delta\left(2 + \left\|A^{\frac{1}{2}}u\right\|^{2\gamma}\right)\right) \|\Delta u_{tt}^n\|^2$$
$$\leq \frac{1}{4\delta}\left(\int_0^t h(t-s)\left\|A^{\frac{1}{2}}u^n\right\|_2 ds\right)^2$$
$$+\frac{1}{4\delta}\left\|A^{\frac{1}{2}}u^n\right\|^2 + \frac{c}{4\delta}\left(\|\Delta u^n\|_2^4 + \|u^n\|_2\right).$$

Integrate the last inequality on $(0,T)$ and use (22) and $(A2)$ leads to

$$\int_0^T \int_{\Omega} |u_t^n|^{\rho} |u_{tt}^n|^2 dxdt + \left(1 - c\delta - \delta\left(2 + \left\|A^{\frac{1}{2}}u\right\|^{2\gamma}\right)\right) \int_0^T \|\Delta u_{tt}^n\|^2 dt$$
$$\leq \frac{c}{\delta}\int_0^T \left[h \circ A^{\frac{1}{2}}u^n + \left\|A^{\frac{1}{2}}u^n\right\|^2 + \|\Delta u^n\|_2^4 + \|u^n\|_2\right] dt. \tag{25}$$

That is to say, if we take $\delta > 0$ and using (22), we have the following inequality,

$$\int_0^T \|\Delta u_{tt}^n\|^2 dt \leq C_3. \tag{26}$$

where $C_3 > 0$ constant which is independent n or t.

Therefore the estimations (22) and (26) satisfies that

$$\begin{cases} u^n, \text{ is uniformly bounded in } L^{\infty}\left(0,T; H_0^m(\Omega) \cap H^{2m}(\Omega)\right), \\ u_t^n, \text{ is uniformly bounded in } L^{\infty}\left(0,T; H_0^2(\Omega)\right), \\ u_{tt}^n, \text{ is uniformly bounded in } L^2\left(0,T; H_0^2(\Omega)\right). \end{cases} \tag{27}$$

We deduce that there is a subsequence of (u^n) (still denoted by (u^n)), such that

$$\begin{cases} u^n \xrightarrow{w^*} u, \; L^{\infty}\left(0,T; H_0^m(\Omega) \cap H^{2m}(\Omega)\right), \\ u_t^n \xrightarrow{w^*} u_t, \quad L^{\infty}\left(0,T; H_0^2(\Omega)\right), \\ u_{tt}^n \xrightarrow{w^*} u_{tt}, \; L^2\left(0,T; H_0^2(\Omega)\right), \\ u_t^n \longrightarrow u_t, \text{ in } L^2\left(0,T; H_0^2(\Omega)\right) \text{ weakly}, \\ u_{tt}^n \xrightarrow{w^*} u_{tt}, \text{ in } L^2\left(0,T; H_0^2(\Omega)\right) \text{ weakly}, \end{cases} \tag{28}$$

where $\xrightarrow{w^*}$ is defined as the weakly star convergence.

By using (27), we have the solution (u^n) is bounded in $L^{\infty}(0,T; H_0^m(\Omega))$ by using $H_0^m(\Omega) \hookrightarrow L^{\infty}(\Omega)\left(\Omega \subset R^3\right)$ the boundedness of (u^n) in $L^2(\Omega \times (0,T))$. Similary, we have (u_t^n) is bounded in $L^2(\Omega \times (0,T))$. Then, thanks to Aubin–Lions–Simon Lemma and (27), we obtain

$$u^n \to u, \text{ in } L^2(\Omega \times (0,T)) \text{ strongly}$$

which satisfies

$$u^n \to u, \quad \Omega \times (0,T).$$

Similary we conclude that

$$u_t^n \to u_t, \text{ in } L^2\left(0,T;L^2(\Omega)\right) \text{ strongly,}$$

and

$$u_t^n \to u_t, \Omega \times (0,T). \tag{29}$$

Using (22) and the embedding theorems, we have

$$\begin{aligned} \left\| |u_t^n|^\rho u_t^n \right\|_{L^2(0,T;L^2(\Omega))}^2 &= \int_0^T \|u_t^n\|_{2(\rho+1)}^{2(\rho+1)} dt \\ &\leq c\int_0^T \|\Delta u_t^n\|_2^{2(\rho+1)} dt \leq cTC_2, \end{aligned} \tag{30}$$

which implies that $(|u_t^n|^\rho u_t^n)$ is bounded in $L^2(\Omega \times (0,T))$. Combining (29) and (30) and using Aubin–Lions' lemma, we have

$$|u_t^n|^\rho u_t^n \to |u_t|^\rho u_t \text{ in } L^2\left(0,T;L^2(\Omega)\right) \text{ weakly.} \tag{31}$$

We obtain the following convergence

$$u^n \ln|u^n| \to u\ln|u|, \quad \Omega \times (0,T). \tag{32}$$

because of $s \to s\ln|s|$ is continuous.

It is clear that $|u^n \ln|u^n| - u\ln|u||$ is bounded in $L^\infty(\Omega \times (0,T))$ thanks to $H_0^m(\Omega) \hookrightarrow L^\infty(\Omega)$. Morever, this satisfies that

$$u^n \ln|u^n| \to u\ln|u|, \text{ in } L^2\left(0,T;L^2(\Omega)\right) \text{ strongly.} \tag{33}$$

On the other hand, integrating the equation of (11) over $(0,t)$, we have

$$\begin{aligned} &\frac{1}{\rho+1}\int_\Omega |u_t^n|^\rho u_t^n w dxds - \frac{1}{\rho+1}\int_\Omega |u_1^n|^\rho u_1^n w dx \\ &+ \int_\Omega \Delta u_t^n \Delta w dxds - \int_\Omega \Delta u_1^n \Delta w dx \\ &+ \int_0^t \int_\Omega M\left(\left\|A^{\frac12}u\right\|^2\right) A^{\frac12}uA^{\frac12}w dxds \\ &- \int_0^t \int_\Omega \left(\int_0^\tau h(t-s)A^{\frac12}u^m\right) A^{\frac12}w dsd\tau dx \\ = &\int_0^t \int_\Omega u^n \ln|u^n| w dxds, \end{aligned} \tag{34}$$

for $\forall w \in V_m$.

Consequently by using convergences (10), (28) ,(33) and (31) , passing to the limit in (34) as $n \to \infty$ will be possible. Thus (34) becomes such that

$$\begin{aligned}
&\frac{1}{\rho+1}\int_\Omega |\,u_t|^\rho\, u_t w dx ds \\
=\;& \frac{1}{\rho+1}\int_\Omega |\,u_1|^\rho\, u_1 w dx - \int_\Omega \Delta u_t \Delta w dx ds \\
&+\int_\Omega \Delta u_1 \Delta w dx - \int_0^t \int_\Omega M\left(\left\|A^{\frac{1}{2}}u\right\|^2\right) A^{\frac{1}{2}} u A^{\frac{1}{2}} w dx ds \\
&+\int_0^t \int_\Omega \left(\int_0^\upsilon h\,(t-s)\, A^{\frac{1}{2}} u^m\right) A^{\frac{1}{2}} w ds d\upsilon dx \\
&+\int_0^t \int_\Omega u \ln|u|\, w dx ds, \qquad (35)
\end{aligned}$$

which implies for $\forall w \in H_0^m(\Omega)$. We can see clearly the terms of right-hand side of equation (35) are differentiable for a.e. $t \in R^+$ .Therefore, taking derivative of (35) over $t \in (0,T)$, we have,

$$\begin{aligned}
&\int_\Omega |\,u_t|^\rho\, u_{tt}(x,t)\, w(x)\, dx + \int_\Omega \Delta u_{tt}(x,t)\, \Delta w(x)\, dx \\
&+\int_\Omega M\left(\left\|A^{\frac{1}{2}}u\right\|^2\right) A^{\frac{1}{2}} u A^{\frac{1}{2}} w dx ds \\
&-\int_\Omega \int_0^t h\,(t-s)\left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w\right) dx \\
=\;& \int_\Omega \ln|u(x,t)|\, u(x,t)\, w(x)\, dx,
\end{aligned}$$

for any $w \in H_0^m(\Omega)$. This completes proof. □

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