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# Delayed flexible structure - Existence, uniqueness and exponential stability

**Mohamed Houasni[1*] and Salah Zitouni[2]**
[1]Faculty of Sciences and Technology, University Djilali Bounaama de Khemis Miliana, Algeria. ORCID: 0000-0002-1966-9201
[2]Department of Mathematics and Informatics, University of Souk Ahras, Souk Ahras, Algeria. ORCID: 0000-0002-9949-7939

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**Abstract:** In this paper, we concerned with a delayed flexible structure system, where the heat flux is given by Cattaneo's law. We prove the wellposed of the system as well as its exponential stability under suitable hypotheses on the weights of the delay, heating effect and material damping.

**Key words:** Semigroups theory, flexible structure, second sound, decay, exponential stability, distributed delay.

## 1. Introduction

In this work, we study the following delayed flexible structure system:

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \int_{\tau_1}^{\tau_2} \mu(s)u_t\,(x, t-s)\,ds + \eta\theta_x = 0, \\ \theta_t + \eta u_{tx} + kq_x = 0, \\ \tau q_t + \beta q + k\theta_x = 0, \end{cases} \tag{1}$$

where $u(x,t)$ is the displacement of a particle at position $x \in (0, L)$ and time $t > 0$. $\eta > 0$ is the coupling constant, that accounts for the heating effect, and $\beta, k > 0$. $\theta$ is the temperature of the body, $q = q(x,t)$ is the heat flux and the parameter $\tau > 0$ is the relaxation time describing the time lag in the response for the temperature. $s > 0$ is a real number represents the time delay. $m(x)$, $\delta(x)$ and $p(x)$ are specific functions denote mass per unit length of structure, coefficient of internal material damping and a positive function related to the stress acting on the body at a point $x$, respectively, and for $\tau_1$, $\tau_2$ two real numbers satisfying $0 \leq \tau_1 < \tau_2$, $\mu : [\tau_1; \tau_2] \longrightarrow \mathbb{R}$ is a bounded function. The model of heat condition, originally due to Cattaneo, is of hyperbolic type. We consider the following initial and boundary conditions:

$$\begin{array}{l} u\,(.,0) = u_0\,(x)\,,\ u_t\,(.,0) = u_1\,(x)\,,\ \theta\,(.,0) = \theta_0\,(x)\,,\ q\,(.,0) = q_0\,(x)\,, \quad \forall x \in [0, L] \\ u\,(0,t) = u\,(L,t) = \theta\,(0,t) = \theta\,(L,t) = 0,\ \forall t \geq 0, \\ u_t\,(x, -t) = f_0(x,t), \quad 0 < t \leq \tau_2, \end{array} \tag{2}$$

where $f_0$ is the history function.

The issue of existence and stability of solutions of dynamical systems continues to attract a great deal of attention in the recent years (e.g. [2, 9, 13–15, 20, 28]). S. Misra et al. [22] considered the vibrations of a cantilever structure modeled by the standard linear flexible model of viscoelasticity coupled to an expectedly

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*Correspondence: m.houasni@yahoo.fr

dissipative effect through heat conduction

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x - k\theta_x = f, \\ \theta_t - \theta_{xx} - ku_{tx} = 0. \end{cases}$$

By using semigroups theory and multiplier technique, they established the well-posedness and an exponential stability of the system when the disturbing force is insignificant. Alves et al. [2] concerned with the system (1)-(2) in the absence of delay term

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x = 0, \\ \theta_t + kq_x + \eta u_{tx} = 0, \\ \tau q_t + \beta q + k\theta_x = 0, \end{cases} \tag{3}$$

with the initial and boundary conditions

$$\begin{array}{l} u\,(.,0) = u_0\,(x)\,,\ u_t\,(.,0) = u_1\,(x)\,,\ \theta\,(.,0) = \theta_0\,(x)\,,\ q\,(.,0) = q_0\,(x)\,,\ \forall x \in [0, L] \\ u\,(0,t) = u\,(L,t) = \theta\,(0,t) = \theta\,(L,t) = 0,\ \forall t \geq 0. \end{array}$$

They established the well-posedness of the system and proved its stability exponential and polynomial under suitable boundary conditions. Houasni et al. [17] studied the following inhomogeneous flexible structure system:

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + dw_x + \eta\theta_x = 0, \\ c\theta_t - k\theta_{xx} + \eta u_{tx} + k_1 w_x = 0, \\ \tau w_t - k_3 w_{xx} + k_2 w + k_1\theta_x + du_{tx} = 0, \end{cases} \tag{4}$$

They proved the wellposed of the problem using semi-group theory, as well as an exponential stability using the multiplier method without any restriction or relation on the coefficients of the system.

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research. We refer the interested readers to [3, 4, 6, 7, 11, 12, 16, 18, 21, 29] for details discussion on the subject. The original motivation of this type of problem was first introduced by Datko et al. [10] in 1986 when they showed that the presence of the delay may not only destabilize a system which is asymptotically stable in the absence of it but may also lead to ill-posedness (see also [24] and [27]). On the other hand, it has been established that voluntary introduction of delay can benefit the control (see [1]). Our purpose in the present manuscript is to obtain an exponential decay rate estimates of the energy, for this end we consider (3) with an internal distributed delay term on the first equation, under a suitable assumption on the weights of the delay, heating effect and material damping, we establish a well-posed result of the system using semigroups theory and an exponential stability using the multiplier method. We should mention here that, to the best of our knowledge, there is no result concerning flexible structure system with the presence of distributed delay term.

This paper is organized as follows; In the second section, we introduce some assumptions needed in our work then prove the well-posedness of the system (1)-(2). In the last section we state and prove our stability result.

## 2. Preliminaries and well-posedness

In this section, we present some hypotheses and prove the well-posedness of system (1)-(2). Throughout this paper, $c$ represents a generic positive constant and is different in various occurrences.

Taking the following new variable

$$z(x,\rho,s,t)=u_t\left(x,t-\rho s\right),\ \text{in}\ (0,L)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty),$$

then we obtain

$$\begin{cases} sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0,\\ z(x,0,s,t)=u_t(x,t).\end{cases}$$

Consequently, problem (1)-(2) is equivalent to

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+\eta\theta_x+\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s,t)ds=0,\\ \theta_t+kq_x+\eta u_{tx}=0,\\ \tau q_t+\beta q+k\theta_x=0,\\ sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0,\end{cases}\tag{5}$$

where $(x,\rho,s,t)\in(0,L)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty)$, with the following initial and boundary conditions:

$$\begin{array}{l} u\left(.,0\right)=\varphi_0\left(x\right),\ u_t\left(.,0\right)=\varphi_1\left(x\right),\ \theta\left(.,0\right)=\theta_0\left(x\right),\ q\left(.,0\right)=q_0\left(x\right),\ \forall x\in[0,L],\\ u\left(0,t\right)=u\left(L,t\right)=\theta\left(0,t\right)=\theta\left(L,t\right)=0,\ \forall t>0,\\ z(x,\rho,s,0)=f_0(x,\rho s)\ \text{ in }\ (0,L)\times(0,1)\times(\tau_1,\tau_2).\end{array}\tag{6}$$

We shall use the following assumptions:
**(H1)** $\mu:\ [\tau_1;\tau_2]\longrightarrow\mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|ds<\eta.\tag{7}$$

**(H2)** The functions $m(x),\ \delta(x)$ and $p(x)$ will be supposed such that:

$$m,\delta,p\in W^{1,\infty}(0,L),\ m(x),p(x)>0\text{ and }2\delta(x)>l\eta,\quad\forall x\in[0,L],\ l=L^2/\pi^2.\tag{8}$$

From Equation $(5)_3$ and (6), we infer that

$$\frac{d}{dt}\int_0^L q\left(x,t\right)dx+\frac{\beta}{\tau}\int_0^L q\left(x,t\right)dx=0,$$

thus

$$\int_0^L q\left(x,t\right)=\left(\int_0^L q_0\left(x\right)dx\right)\exp\left(-\frac{\beta t}{\tau}\right).$$

So, if we set

$$\widetilde{q}\left(x,t\right)=q\left(x,t\right)-\frac{1}{L}\left(\int_0^L q_0\left(x\right)dx\right)\exp\left(-\frac{\beta t}{\tau}\right),$$

then, $(u, u_t, \theta, \widetilde{q})$ is a solution of (1), and

$$\int_0^L \widetilde{q}\,(x,t)\,dx = 0,$$

for all $t \geq 0$. in what follows, we shall use $q$ instead of $\widetilde{q}$.

Let us introducing the vector function $U = (u, v, \theta, q, z)^T$, where $v = u_t$, using $L^2(0,L)$ and $H_0^1(0,L)$ with their usual scalar products and norms for define the spaces:

$$\mathcal{H} := H_0^1(0,L) \times \left[L^2(0,L)\right]^2 \times L_*^2(0,L) \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)).$$

and

$$H_*^1(0,L) = H^1(0,L) \cap L_*^2(0,L).$$

where

$$L_*^2(0,L) = \left\{ w \in L^2(0,L) : \int_0^L w\,(s)\,ds = 0 \right\}.$$

We equip $\mathcal{H}$ with the inner product

$$\begin{aligned}(U,\widetilde{U})_{\mathcal{H}} &= \int_0^L p\,(x)\,u_x\widetilde{u}_x dx + \int_0^L m\,(x)\,v\widetilde{v} dx + \int_0^L \theta\widetilde{\theta} dx + \tau \int_0^L q\widetilde{q} dx \\ &\quad + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu(s)|\,z(x,\rho,s)\widetilde{z}(x,\rho,s) ds d\rho dx.\end{aligned}$$

Next, the system (5)-(6) can be written as the following Cauchy problem:

$$\begin{cases} U'(t) + (\mathcal{A}+\mathcal{B})\,U(t) = 0, & t > 0 \\ U(0) = U_0 = (u_0, u_1, \theta_0, q_0, f_0)^T, \end{cases} \tag{9}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} -v \\ -\frac{1}{m(x)}\left((p(x)u_x + 2\delta(x)v_x - \eta\theta)_x - \int_{\tau_1}^{\tau_2} \mu(s) z(1,s) ds - v\int_{\tau_1}^{\tau_2} |\mu(s)|\,ds\right) \\ kq_x + \eta v_x \\ \frac{1}{\tau}(k\theta_x + \beta q) \\ \frac{1}{s} z_\rho(x,\rho,s) \end{pmatrix},$$

and

$$\mathcal{B}U = \begin{pmatrix} 0 \\ -v\frac{\int_{\tau_1}^{\tau_2}|\mu(s)|ds}{m(x)} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Where

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} \mid u \in H^2(0,L) \cap H_0^1(0,L), \quad v \in H_0^1(0,L), \\ \theta \in H_0^1(0,L),\ q \in H_*^1(0,L), \\ z, z_\rho \in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)),\ z(x,0,s) = v \end{array} \right\}.$$

One can easily prove that $D(\mathcal{A})$ is dense in $\mathcal{H}$ (see [3, 5, 8]).

Before state an existence and uniqueness result, we refer the reader to [19] (from page 90), [25] and the references therein, for more details discussion about solutions of (9), then we have

**Proposition 2.1.** *Let $U_0 \in \mathcal{H}$ be given. Assume that **(H1)**− **(H2)** are satisfied, Problem (9) possesses then a unique solution satisfying $U \in C(\mathbb{R}^+; \mathcal{H})$. If $U_0 \in D(\mathcal{A})$, then $U \in C^1(\mathbb{R}^+; \mathcal{H}) \cap C(\mathbb{R}^+; D(\mathcal{A}))$.*

*Proof.* First, we prove that $\mathcal{A}$ is monotone. For any $U \in D(\mathcal{A})$, we have

$$\begin{aligned}
(\mathcal{A}U, U)_{\mathcal{H}} &= -\int_0^L p(x) v_x u_x dx - \int_0^L [p(x) u_x]_x v dx - 2\int_0^L [\delta(x) v_x]_x v dx \\
&\quad + \int_0^L \eta\theta_x v dx + \int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds dx \\
&\quad + k\int_0^L \theta q_x dx + \int_0^L \eta\theta v_x dx + k\int_0^L \theta_x q dx + \int_0^L \beta q^2 dx \\
&\quad + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| z z_\rho ds d\rho dx + \int_{\tau_1}^{\tau_2} |\mu(s)| ds \int_0^L v^2 dx.
\end{aligned}$$

Integration by parts and by using the fact that

$$\begin{aligned}
\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| z z_\rho ds d\rho dx &= \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 |\mu(s)| \frac{\partial}{\partial\rho} z^2 d\rho ds dx \\
&= \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,1,s,t) ds dx \\
&\quad - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)| ds \int_0^L v^2 dx,
\end{aligned}$$

we get

$$\begin{aligned}
(\mathcal{A}U, U)_{\mathcal{H}} &= 2\int_0^L \delta(x) v_x^2 dx + \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)| ds \int_0^L v^2 dx + \int_0^L \beta q^2 dx \\
&\quad + \int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s) ds dx + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,1,s) ds dx.
\end{aligned} \tag{10}$$

By using Young's inequality (see [8] p.92), we get

$$\begin{aligned}
&- \int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s) ds dx \\
\leq\ & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu(s)| ds\right) \int_0^L v^2 dx + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,1,s) ds dx,
\end{aligned} \tag{11}$$

which implies that

$$\begin{aligned}
&\int_0^L v\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s)dsdx\\
\geq\ & -\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\right)\int_0^L v^2dx-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|\,z^2(x,1,s)dsdx,
\end{aligned}$$

from this last, the Equation (10) yields

$$(\mathcal{A}U,U)_{\mathcal{H}}\geq 2\int_0^L\delta\left(x\right)v_x^2dx+\int_0^L\beta q^2dx\geq 0.$$

Next, we prove that the operator $\mathcal{A}+\mathcal{I}$ is surjective.

Given $G=(g_1,g_2,g_3,g_4,g_5)^T\ \in\mathcal{H}$, we prove that there exists $U\in D(\mathcal{A})$ satisfying

$$\left(\mathcal{I}+\mathcal{A}\right)U=G, \tag{12}$$

which gives

$$\begin{aligned}
&-v+u=g_1,\\
&-(p(x)u_x+2\delta(x)v_x-\eta\theta)_x+\textstyle\int_{\tau_1}^{\tau_2}\mu(s)z(.,1,s)ds\\
&\qquad\qquad+\left(\textstyle\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds+m\left(x\right)\right)v=m\left(x\right)g_2,\\
&kq_x+\eta v_x+\theta=g_3,\\
&k\theta_x+\left(\beta+\tau\right)q=\tau g_4,\\
&z_\rho+sz=sg_5.
\end{aligned}\tag{13}$$

Then, from $(13)_1$ and $(13)_4$, we obtain

$$v=u-g_1\in H_0^1(0,L), \tag{14}$$

$$\theta_x=\frac{\tau}{k}g_4-\frac{\beta+\tau}{k}q\in L_*^2(0,L), \tag{15}$$

by using (15), we get

$$\theta=\frac{\tau}{k}\int_0^x g_4(y)dy-\frac{\beta+\tau}{k}\int_0^x q(y)dy,$$

and then

$$\theta\left(0,t\right)=\theta\left(L,t\right)=0.$$

The last equation in Equation (13), (14) and the fact $z(x,0)=v(x)=u-g_1(x)$ give

$$z(x,\rho,s)=ue^{-s\rho}-e^{-s\rho}g_1+se^{-s\rho}\int_0^\rho e^{s\upsilon}g_5(x,\upsilon,s)d\upsilon. \tag{16}$$

From equation (14)-(16), we can verify that $u$ and $q$ satisfying

$$\begin{aligned}
&-(p(x)u_x+2\delta(x)v_x)_x-\tfrac{\eta(\beta+\tau)}{k}q+u\textstyle\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds+\gamma u(x)=f_1,\\
&-k^2q_x+\left(\beta+\tau\right)\textstyle\int_0^x q(y)dy-k\eta u_x=f_2,\\
&-v_x+u_x=f_3,
\end{aligned}\tag{17}$$

where

$$\begin{array}{l}\gamma = m\left(x\right) + \int_{\tau_1}^{\tau_2}\left|\mu(s)\right| ds,\\ z_0(x,s) = e^{-s}g_1 - se^{-s}\int_0^1 e^{s\upsilon}g_5(x,\upsilon,s)d\upsilon,\\ f_1 = \gamma g_1(x) + m\left(x\right)g_2(x) - \frac{\eta\tau}{k}g_4(x) + \int_{\tau_1}^{\tau_2}\mu(s)z_0(x,s)ds \in L^2(0,L),\\ f_2 = -k\eta g_1(x) + \tau\int_0^x g_4(y)dy - kg_3 \in L^2(0,L),\\ f_3 = g_{1x} \in L^2(0,L).\end{array}$$

The variational formulation corresponding to Equation (17) is then

$$B((u,q),(\widetilde{u},\widetilde{q})) = F(\widetilde{u},\widetilde{q}), \tag{18}$$

where $B:\left[H_0^1(0,L)\times L_*^2(0,L)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}B((u,q),(\widetilde{u},\widetilde{q})) &= \int_0^L \left(p(x)+2\delta(x)\right)u_x\widetilde{u}_x dx - \frac{\eta\left(\beta+\tau\right)}{k}\int_0^L q\widetilde{u}dx\\ &\quad +\left(\beta+\tau\right)\int_0^L q\widetilde{q}dx + \gamma\int_0^L u\widetilde{u}dx\\ &\quad +\frac{\left(\beta+\tau\right)^2}{k^2}\int_0^L\left(\int_0^x q(y)dy\int_0^x\widetilde{q}(y)dy\right)dx\\ &\quad +\frac{\eta\left(\beta+\tau\right)}{k}\int_0^L u\widetilde{q}dx + \int_0^L u\widetilde{u}\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}dsdx,\end{aligned}$$

and $F: H_0^1(0,L)\times L_*^2(0,L)\to\mathbb{R}$ is the linear functional defined by

$$F(\widetilde{u},\widetilde{q}) = \int_0^L f_1\widetilde{u}dx + \frac{(\beta+\tau)}{k^2}\int_0^L f_2\int_0^x\widetilde{q}(y)dydx + \int_0^L 2f_3\delta(x)\widetilde{u}_x dx.$$

For $V = H_0^1(0,L)\times L_*^2(0,L)$ equipped with the norm

$$\left\|(u,q)\right\|_V^2 = \left\|u\right\|_2^2 + \left\|u_x\right\|_2^2 + \left\|q\right\|_2^2,$$

where $\left\|.\right\|_2$ is the usual norm.

One can easily see that $B$ and $F$ are bounded. Also, we get

$$\begin{aligned}B((u,q),(u,q)) &= \int_0^L\left(p(x)+2\delta(x)\right)u_x^2dx + (\beta+\tau)\int_0^L q^2dx\\ &\quad +\gamma\int_0^L u^2dx + \frac{\beta+\tau}{k^2}\int_0^L\left(\int_0^x q(y)dy\right)^2dx\\ &\quad +\int_0^L u^2dx\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds\\ &\geq c\left\|(u,q)\right\|_V^2.\end{aligned}$$

Then, $B$ is coercive. Consequently, by the Lax–Milgram lemma (see [8] Corollary 5.8 p. 138-140), system (17) has a unique solution

$$u\in H_0^1(0,L),\quad q\in L_*^2(0,L).$$

If $\widetilde{q} \equiv 0 \in L^2_*(0,L)$, then Equation (18) reduces to

$$\begin{aligned} & -\int_0^L (p(x)u_x)_x \widetilde{u}dx + \gamma \int_0^L u\widetilde{u}dx - \frac{\eta(\beta+\tau)}{k}\int_0^L q\widetilde{u}dx \\ + & \int_0^L u\widetilde{u}\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}dsdx - \int_0^L (2\delta(x)u_x)_x \widetilde{u}dx \\ = & \int_0^L f_1\widetilde{u}dx + \int_0^L (2f_3\delta(x))_x \widetilde{u}dx, \quad \forall \widetilde{u} \in H_0^1(0,L) . \end{aligned}$$

That is

$$-(p(x)u_x)_x + \gamma u - \frac{\eta(\beta+\tau)}{k}q - (2\delta(x)u_x)_x + u\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds = f_1 + (2f_3\delta(x))_x ,$$

then, we have

$$\begin{aligned} (p(x)u_x + 2\delta(x)u_x)_x = & \ \gamma u + u\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds - \tfrac{\eta(\beta+\tau)}{k}q - f_1 \\ & -(2f_3\delta(x))_x \in L^2(0,L). \end{aligned}$$

Hence,

$$u \in H_0^1(0,L) \cap H^2(0,L).$$

Similarly, if $\widetilde{u} \equiv 0 \in H_0^1(0,L)$, we obtain

$$q \in H_*^1(0,L).$$

Moreover, from (14) and (15) we deduce that

$$v, \theta \in H_0^1(0,L).$$

Consequently, $\mathcal{A}$ is a maximal monotone operator. Then, $D(\mathcal{A})$ is dense in $\mathcal{H}$ (see Proposition 7.1 in [8] ).

On the other hand, we show that operator $\mathcal{B}$ is Lipschitz continuous. In fact, if $U = (u,v,\theta,q,z)^T$ and $\widetilde{U} = (\widetilde{u},\widetilde{v},\widetilde{\theta},\widetilde{q},\widetilde{z})^T$ belong to $\mathcal{H}$, we have

$$\left\|\mathcal{B}U - \mathcal{B}\widetilde{U}\right\|_{\mathcal{H}}^2 = c\left|h\right|^2, \tag{19}$$

where $h = \widetilde{v} - v$. Using the embedding of $H^1(0,L)$ into $L^\infty(0,L)$ (see [8] Theorem 8.8, p. 212) and (7), one sees that

$$|h| \le c\left\|v - \widetilde{v}\right\|_{L^\infty(0,L)} \le c\left\|U - \widetilde{U}\right\|_{\mathcal{H}}. \tag{20}$$

Combining (19) and (20), then $\mathcal{B}$ is Lipschitz continuous in $\mathcal{H}$ ( see [6]). Consequently, $\mathcal{A}+\mathcal{B}$ is the infinitesimal generator of a linear contraction $C_0$-semigroup on $\mathcal{H}$. Hence, the result of Proposition 2.1 follows (see [19], [26]) and the references therein. □

To state our decay result, we introduce the energy functional associated to (5)-(6), namely,

$$\begin{aligned} \mathcal{E}(t,\varphi,\psi,\theta,q,z) = & \ \frac{1}{2}\int_0^L \left\{p(x)u_x^2 + m(x)u_t^2 + \theta^2 + \tau q^2\right\}dx \\ & + \frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z^2(x,\rho,s,t)dsd\rho dx, \end{aligned} \tag{21}$$

we denote $\mathcal{E}(t) = \mathcal{E}(t, \varphi, \psi, \theta, q, z)$ and $\mathcal{E}(0) = \mathcal{E}(0, \varphi_0, \psi_0, \theta_0, q_0, f_0)$ for simplicity of notations.

### 3. Exponential stability

In this section, we introduce some lemmas allow us to achieve our goal, which is the proof of the stability result.

**Lemma 3.1.** *[23]: Let $h \in H_0^1(0, L)$. Then it holds*

$$\int_0^L |h|^2 dx \leq l \int_0^L |h_x|^2 dx, \quad l = \frac{L^2}{\pi^2}. \tag{22}$$

**Lemma 3.2.** *[2] : Let $(u, u_t, \theta, q)$ be the solution to system (1)-(2), with an initial datum in $D(\mathcal{A})$. Then, for any $t > 0$, there exists a sequence of real numbers (depending on t), denoted by $\xi_i \in [0, L](i = 1, ..., 6)$, such that:*

$$\int_0^L p(x) u_x^2 dx = p(\xi_1) \int_0^L u_x^2 dx, \quad \int_0^L m(x) u_t^2 dx = m(\xi_2) \int_0^L u_t^2 dx,$$
$$\int_0^L m(x) u^2 dx = m(\xi_3) \int_0^L u^2 dx, \quad \int_0^L \delta(x) u^2 dx = \delta(\xi_4) \int_0^L u^2 dx,$$
$$\int_0^L \delta(x) u_x^2 dx = \delta(\xi_5) \int_0^L u_x^2 dx, \quad \int_0^L \delta(x) u_{xt}^2 dx = \delta(\xi_6) \int_0^L u_{xt}^2 dx.$$

**Lemma 3.3.** *Let $(u, v, \theta, q, z)$ be the solution of (5)-(6), then the energy $\mathcal{E}$ is non-increasing function and satisfies, for all $t \geq 0$,*

$$\begin{aligned} \mathcal{E}'(t) &= -2\int_0^L \delta(x) u_{xt}^2 dx - \beta \int_0^L q^2 dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x, 1, s, t) ds dx \\ &\quad -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x, 1, s, t) ds dx + \frac{1}{2}\int_0^L u_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)| ds dx \\ &\leq -\beta \int_0^L q^2 dx - c\int_0^L u_{xt}^2 dx \leq 0. \end{aligned} \tag{23}$$

*where $c > 0$ is constant.*

*Proof.* Multiplying the equations in $(5)_1$,$(5)_2$, and $(5)_3$ by $u_t$,$\theta$ and $q$, respectively, and integrate over $(0, L)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L \left\{p(x)u_x^2 + m(x)u_t^2 + \theta^2 + \tau q^2\right\} dx \\ &= -\beta \int_0^L q^2 dx - 2\int_0^L \delta(x) u_{xt}^2 dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x, 1, s, t) ds dx. \end{aligned} \tag{24}$$

Multiplying the last equation in (5) by $|\mu(s)| z$, integrating the product over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, and recall that $z(x, 0, s, t) = u_t$, yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu(s)| z^2(x, \rho, s, t) ds d\rho dx \\ &= -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x, 1, s, t) ds dx + \frac{1}{2}\int_0^L u_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)| ds dx. \end{aligned} \tag{25}$$

Now, a combination of (24) and (25) gives

$$\begin{aligned}\mathcal{E}'(t) &= -2\int_0^L \delta(x)\, u_{xt}^2 dx - \beta \int_0^L q^2 dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds dx \\ &\quad -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t) ds dx + \frac{1}{2}\int_0^L u_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds dx. \end{aligned} \tag{26}$$

Meanwhile, using Young's and Cauchy–Schwarz inequalities (see [8] p.92), we have

$$\begin{aligned} & -\int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds dx \\ \leq\ & \frac{1}{2}\underbrace{\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds}_{<\eta} \int_0^L u_t^2 dx + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t) ds dx. \end{aligned} \tag{27}$$

Substitution of (27) into (26), using (7), Lemma 3.2 and (22) gives

$$\begin{aligned}\mathcal{E}'(t) &= -2\int_0^L \delta(x)\, u_{xt}^2 dx - \beta \int_0^L q^2 dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds dx \\ &\quad -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t) ds dx + \frac{1}{2}\int_0^L u_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds dx \\ &\leq -\beta \int_0^L q^2 dx - 2\int_0^L \delta(x)\, u_{xt}^2 dx + \eta \int_0^L u_t^2 dx \\ &\leq -\beta \int_0^L q^2 dx - 2\delta(\xi_6) \int_0^L u_{xt}^2 dx + l\eta \int_0^L u_{xt}^2 dx \\ &\leq -\beta \int_0^L q^2 dx - (2\delta(\xi_6) - l\eta) \int_0^L u_{xt}^2 dx \\ &\leq -\beta \int_0^L q^2 dx - c\int_0^L u_{xt}^2 dx \leq 0, \end{aligned}$$

which concludes the proof. □

**Lemma 3.4.** *Let $(u, v, \theta, q, z)$ be the solution of (5)-(6). Then, for $\epsilon_1, \epsilon_2 > 0$, the functional*

$$I_1(t) := \tau \int_0^L \theta \left( \int_0^x q(t,y) dy \right) dx, \tag{28}$$

*satisfies*

$$I_1'(t) \leq -(\kappa - \beta\varepsilon_1) \int_0^L \theta^2 dx + \varepsilon_2 \tau\eta \int_0^L u_t^2 dx + \left( \tau + \frac{\tau\eta}{\varepsilon_2} + \frac{l\beta}{4\varepsilon_1} \right) \int_0^L q^2 dx. \tag{29}$$

*Proof.* Taking the derivative of (28) and using $(5)_2$, $(5)_3$, (22), integration by parts and Young's inequality, we obtain (29). □

**Lemma 3.5.** *Let $(u, v, \theta, q, z)$ be the solution of (5)-(6). Then the functional*

$$I_2(t) := \int_0^L \left(\delta(x)u_x^2 + m(x)u_t u\right) dx, \tag{30}$$

*satisfies*

$$\begin{aligned} I_2'(t) \quad &\leq \quad -\left(p(\xi_1) - \epsilon_3\eta\,(l+1)\right)\int_0^L u_x^2 dx + m(\xi_2)\int_0^L u_t^2 dx + \frac{\eta}{4\epsilon_3}\int_0^L \theta^2 dx \\ &\quad + \frac{1}{4\epsilon_3}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx, \end{aligned} \tag{31}$$

*for any $\varepsilon_3 > 0$.*

*Proof.* Differentiating Equation (30) gives

$$I_2'(t) = -\int_0^L p(x)u_x^2 dx + \int_0^L m(x)u_t^2 dx - \eta\int_0^L \theta_x u dx - \int_0^L u\int_{\tau_1}^{\tau_2} \mu(s)z(x,1,s,t)dsdx.$$

Using Young's inequality, we have for $\epsilon_3 > 0$

$$-\eta\int_0^L \theta_x u dx = \eta\int_0^L u_x\theta dx \leq \eta\epsilon_3\int_0^L u_x^2 dx + \frac{\eta}{4\epsilon_3}\int_0^L \theta^2 dx,$$

from Young's inequality, (7) and (22), we find

$$\begin{aligned} & -\int_0^L u\int_{\tau_1}^{\tau_2} \mu(s)z(x,1,s,t)dsdx \\ \leq \quad & \epsilon_3\underbrace{\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds}_{<\eta}\int_0^L u^2 dx + \frac{1}{4\epsilon_3}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx \\ \leq \quad & l\eta\epsilon_3\int_0^L u_x^2 dx + \frac{1}{4\epsilon_3}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx, \end{aligned}$$

application of Lemma 3.2 and the last inequality completes the proof. □

**Lemma 3.6.** *Let $(u, v, \theta, q, z)$ be the solution of (5)-(6). Then, for some positive constant $\eta_1$, the functional*

$$I_3(t) = \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)dsd\rho dx, \tag{32}$$

*satisfies*

$$\begin{aligned} I_3'(t) \quad &\leq \quad -\eta_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu(s)|\, z^2(x,\rho,s,t)dsd\rho dx \\ &\quad -\eta_1\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx + \eta\int_0^L u_t^2 dx. \end{aligned} \tag{33}$$

*Proof.* Differentiating (32) and using the last equation in (5), we obtain

$$\begin{aligned} I_3'(t) &= -2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\mu(s)|\,z(x,\rho,s,t)z_\rho(x,\rho,s,t)dsd\rho dx \\ &= -\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} |\mu(s)|\,\frac{\partial}{\partial\rho}[e^{-s\rho}z^2(x,\rho,s,t)]dsd\rho dx \\ &\quad -\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\,z^2(x,\rho,s,t)dsd\rho dx \\ &= -\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\,[e^{-s}z^2(x,1,s,t)-z^2(x,0,s,t)]dsdx \\ &\quad -\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\,z^2(x,\rho,s,t)dsd\rho dx, \end{aligned}$$

using the fact that $z(x,0,s,t)=u_t$ and $e^{-s}\le e^{-s\rho}\le 1$ we get for all $\rho\in[0,1]$

$$\begin{aligned} I_3'(t) &\le \int_0^L\int_{\tau_1}^{\tau_2} e^{-s}\,|\mu(s)|\,z^2(x,1,s,t)dsdx + \underbrace{\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds}_{<\eta}\int_0^L u_t^2dx \\ &\quad -\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s}\,|\mu(s)|\,z^2(x,\rho,s,t)dsd\rho dx. \end{aligned}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s}\le -e^{-\tau_2}$ for all $s\in[\tau_1,\tau_2]$. Finally, setting $\eta_1=-e^{-\tau_2}$ and recalling (7), we obtain (33). □

Next, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional $\mathcal{E}$.

**Lemma 3.7.** *For $N$ sufficiently large, the functional defined by*

$$\mathcal{L}(t) := NE\,(t) + N_1I_1(t) + I_2(t) + N_2I_3(t). \tag{34}$$

*where $N_1$ and $N_2$ are positive real numbers to be chosen appropriately later, satisfies*

$$c_1'\mathcal{E}(t)\le\mathcal{L}(t)\le c_2'\mathcal{E}(t),\quad \forall t\ge 0. \tag{35}$$

*where $c_1'$ and $c_2'$ are positive constants.*

*Proof.* Let

$$\mathfrak{L}(t) := N_1I_1(t) + I_2(t) + N_2I_3(t).$$

then, exploiting Young's, Poincaré's (see [8] p.218) and Cauchy-Schwarz inequalities, (21), and the fact that

$e^{-s\rho} \leq 1,$ we obtain

$$\begin{aligned}
|\mathfrak{L}(t)| &\leq N_1\tau \int_0^L \left|\theta\left(\int_0^x q(t,y)dy\right)\right| dx + \int_0^L \delta(x)u_x^2 dx + \int_0^L m(x)\left|u_t u\right| dx \\
&\quad + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|e^{-s\rho}\mu(s)\right| z^2(x,\rho,s)dsd\rho dx \\
&\leq \int_0^L \delta(x)u_x^2 dx + \frac{1}{2}\int_0^L m(x)u^2 dx + \frac{1}{2}\int_0^L m(x)u_t^2 dx + N_1\tau l \int_0^L |\theta q|\, dx \\
&\quad + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z^2(x,\rho,s)dsd\rho dx \\
&\leq \frac{1}{2}\int_0^L m(x)u_t^2 dx + \frac{\|\delta(x)\|_\infty}{\lambda}\int_0^L p(x)u_x^2 dx + \frac{l\,\|m(x)\|_\infty}{2\lambda}\int_0^L p(x)u_x^2 dx \\
&\quad + \frac{N_1\tau l}{2}\int_0^L \theta^2 dx + \frac{N_1\tau l}{2}\int_0^L q^2 dx \\
&\quad + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z^2(x,\rho,s)dsd\rho dx \\
&\leq cE\left(t\right),
\end{aligned}$$

where $\lambda = \inf_{x\in[0,L]}\{p(x)\}$, and $c > 0$. Consequently,

$$|\mathcal{L}(t) - N\mathcal{E}(t)| \leq c\mathcal{E}(t),$$

which yields

$$\left(N - c\right)\mathcal{E}(t) \leq \mathcal{L}(t) \leq \left(N + c\right)\mathcal{E}(t).$$

Choosing $N$ large enough, we obtain estimate (35). □

Now, we are ready to state and prove the main result of this section.

**Theorem 3.1.** *Let $U = (u, v, \theta, q, z)$ be the solution of (5)-(6), assume that **(H1)** and **(H2)** are satisfied, then the energy $\mathcal{E}$ satisfies, for all $t \geq 0$,*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 t},$$

*where $c_1$ and $c_2$ are positive constants.*

*Proof.* We differentiate (34), and recall (22), (23), (31), (29), and (33), to obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & N\left(-\beta\int_0^L q^2dx - 2\int_0^L \delta\left(x\right)u_{xt}^2dx + \eta\int_0^L u_t^2dx\right) + m(\xi_2)\int_0^L u_t^2dx \\
& -\left(p(\xi_1) - \eta\epsilon_3\left(l+1\right)\right)\int_0^L u_x^2dx + \frac{\eta}{4\epsilon_3}\int_0^L \theta^2dx + N_2\eta\int_0^L u_t^2dx \\
& +\frac{1}{4\epsilon_3}\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|z^2(x,1,s,t)dsdx + N_1\left(\tau + \frac{\tau\eta}{\epsilon_2} + \frac{l\beta}{4\varepsilon_1}\right)\int_0^L q^2dx \\
& +N_1\left(-\left(\kappa - \beta\varepsilon_1\right)\int_0^L \theta^2dx + \epsilon_2\tau\eta\int_0^L u_t^2dx\right) \\
& -N_2\eta_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right|z^2(x,\rho,s,t)dsd\rho dx \\
& -N_2\eta_1\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|z^2(x,1,s,t)dsdx \\
\leq\ & -\left\{N\left(2\delta\left(\xi_6\right) - \eta l\right) - N_2l\eta - m(\xi_2)l - N_1\epsilon_2\tau\eta l\right\}\int_0^L u_{xt}^2dx \\
& -\left\{p(\xi_1) - \eta\epsilon_3\left(l+1\right)\right\}\int_0^L u_x^2dx - \left\{N_1\left(\kappa - \beta\varepsilon_1\right) - \frac{\eta}{4\epsilon_3}\right\}\int_0^L \theta^2dx \\
& -\left\{N\beta - N_1\left(\tau + \frac{\tau\eta}{\epsilon_2} + \frac{\beta l}{4\varepsilon_1}\right)\right\}\int_0^L q^2dx \\
& -\left\{\eta_1N_2 - \frac{1}{4\epsilon_4}\right\}\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|z^2(x,1,s,t)dsdx \\
& -\eta_1N_2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|z^2(x,\rho,s,t)dsd\rho dx
\end{aligned}$$

At this point, we take $\epsilon_2 = 1$ and choose $\epsilon_1$ and $\epsilon_3$ small enough such that

$$\varepsilon_1 < \frac{k}{\beta}, \quad \epsilon_3 < \frac{p(\xi_1)}{\eta(l+1)},$$

then we choose $N_1$ and $N_2$ large enough so that

$$N_1\left(\kappa - \beta\varepsilon_1\right) - \frac{\eta}{4\epsilon_3} > 0, \quad \eta_1N_2 - \frac{1}{4\varepsilon_3} > 0.$$

Once $N_1$ and $N_2$ are fixed, we then choose $N$ large enough so that

$$N\left(2\delta\left(\xi_6\right) - l\eta\right) - N_2l\eta - m(\xi_2)l - N_1\tau\eta l > 0,$$

$$N\beta - N_1\left(\tau + \tau\eta + \frac{\beta l}{4\varepsilon_1}\right) > 0.$$

Thus, using (21), we arrive at

$$\mathcal{L}'(t) \leq -c\mathcal{E}(t), \quad \forall t > 0. \tag{36}$$

A combination of (35) and (36) gives

$$\mathcal{L}'(t) \leq -c_2\mathcal{L}(t), \quad \forall t > 0, \tag{37}$$

where $c_2 = c/c_2'$, a simple integration of (37) over $(0, t)$ yields

$$c_1'\mathcal{E}(t) \leq \mathcal{L}(t) \leq \mathcal{L}(0)e^{-c_2 t} \ , \ \forall t > 0.$$

Taking $c_1 = \mathcal{L}(0)/c_1'$ which completes the proof. □

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