New properties of $gb^*$–closed map and $gb^*$-open map in topological spaces

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Abstract: In this paper, we some of their properties of semi-open, $\alpha$-open, pre-open, $\alpha$-open sets to induced topology, we compare with the semi-open or ($b$-open, pre-open, $\alpha$-open) set if for the topological space and the topological space induced, finally we introduce a new properties for $gb^*$-closed maps and $gb^*$-open map in topological space induced.

Key words: semi-open set, $\alpha$-open set, $gb^*$-closed map, $gb^*$-open map.

1. Introduction and preliminaries

Different types of closed and open were studied by various researchers. The notion of $b$-open sets was introduced by D. Andrijevic [4] in the year 1968. A. A. Omari et al [1] made an analytical study and result the concepts of generalized $b$-closed sets in topological spaces. The idea of introduced regular generalized $b$-closed map in topological space by S. Sekar and K. Mariappa [13] in 2013. A new class of generalised $b$ star-closed map in Topological Spaces was introduced in 2017 by S. Sekar et al [14].

The aim of this paper is to continue the study of new properties of $b$ star-closed maps in topological spaces one of its properties are based on the induced topology and compares the state of a set with respect to the topological subspace and the total topological space in the following positions. Throughout the present paper, $(\mathcal{E}, \mathcal{T})$ and $(\mathcal{F}, \Sigma)$ will denote topological spaces with no separation properties assumed. For a subset $\Omega$ of a topological space $(\mathcal{E}, \mathcal{T})$, $cl(\Omega)$ and $int(\Omega)$ will denote the closure and interior of $\Omega$ in $(\mathcal{E}, \mathcal{T})$, respectively.

We recall the following definitions which are useful in the sequel.

\textbf{Definition 1.1.} Let a subset $\Omega$ of a topological space $(\mathcal{E}, \mathcal{T})$ is called

(1) a pre-open set [2], if $\Omega \subseteq int(cl(\Omega))$.

(2) a semi-open set [10], if $\Omega \subseteq cl(int(\Omega))$.

(3) a $\alpha$-open set [2], if $\Omega \subseteq int(cl(int(\Omega)))$.

(4) a $b$-open set [4], if $\Omega \subseteq cl(int(\Omega)) \cup int(cl(\Omega))$. 
Remark 1.1. The $b$-closed (resp. semi-closed, pre-closed, $\alpha$-closed) of a subset $\Omega$ of a space $(\mathcal{E}, \mathcal{T})$ is the intersection of all $b$-closure (resp. semi-closure, pre-closure, $\alpha$-closure) sets that contain $\Omega$ and is denoted by $\text{bcl}(\Omega)$ (resp. $\text{scl}(\Omega)$, $\text{pcl}(\Omega)$, $\text{acl}(\Omega)$).

Remark 1.2. The $b$-open (resp. semi-open, pre-open, $\alpha$-open) of a contained in $\Omega$ of a space $(\mathcal{E}, \mathcal{T})$ is the union of all $b$-interior (resp. semi-interior, pre-interior, $\alpha$-interior) sets that are contained in $\Omega$ and is denoted by $\text{b-int}(\Omega)$ (resp. $\text{s-int}(\Omega)$, $\text{p-int}(\Omega)$, $\text{\alpha-int}(\Omega)$).

Definition 1.2. Let a subset $\Omega$ of a topological space $(\mathcal{E}, \mathcal{T})$ is called

1. a generalized closed set (briefly $g$-closed) [9], if $\text{cl}(\Omega) \subseteq \theta$ whenever $\Omega \subseteq \theta$ and $\theta$ is open in $\mathcal{E}$.
2. a generalized $b$-closed set (briefly $gb$-closed) [1], if $\text{bcl}(\Omega) \subseteq \theta$ whenever $\Omega \subseteq \theta$ and $\theta$ is open in $\mathcal{E}$.
3. a $\alpha$-generalized $\ast$-closed set (briefly $\alpha g^\ast$-closed) [8], if $\text{cl}(\Omega) \subseteq \text{int}(\theta)$ whenever $\Omega \subseteq \theta$ and $\theta$ is $\alpha$-open in $\mathcal{E}$.
4. a $g^\ast s$-closed set (briefly $g^\ast s$-closed) [3], if $\text{scl}(\Omega) \subseteq \theta$ whenever $\Omega \subseteq \theta$ and $\theta$ is $gs$-open in $\mathcal{E}$.
5. a regular generalized $b$-closed set (briefly $rgb$-closed) [5] if $\text{scl}(\Omega) \subseteq \theta$ whenever $\Omega \subseteq \theta$ and $\theta$ is regular open in $\mathcal{E}$.

Definition 1.3. [14] Let $\mathcal{E}$ and $\mathcal{F}$ be topological spaces. A map $\phi : (\mathcal{E}, \mathcal{T}) \to (\mathcal{F}, \mathcal{T})$ is called generalized $b$-star-closed (briefly, $gb^\ast$-closed map) if the image of every closed set in $\mathcal{E}$ is $gb^\ast$-closed in $\mathcal{F}$.

Definition 1.4. [14] Let $\mathcal{E}$ and $\mathcal{F}$ be topological spaces. A map $\phi : (\mathcal{E}, \mathcal{T}) \to (\mathcal{F}, \mathcal{T})$ is called generalized $b$-star open (briefly, $gb^\ast$-open) if the image of every open set in $\mathcal{E}$ is $gb^\ast$-open in $\mathcal{F}$.

2. Main Results

In this section, we study the behavior of pre-open sets, $b$-open sets, $\alpha$-open sets and $b$-closed sets and $g$-closed sets in a sub-space. For simplification, we introduce the following Remark and Notation.

Notation 2.1. Let $\mathcal{E}_0$ be a nonempty set endowed with the induced topology of $(\mathcal{E}, \mathcal{T})$, we denote $(\mathcal{E}_0, \mathcal{T}_0)$.

Notation 2.2. Let a subset $\Omega$ of a topological space $(\mathcal{E}_0, \mathcal{T}_0)$, we give the following set $\text{cl}_{\mathcal{E}_0}(\Omega)$, $\text{int}_{\mathcal{E}_0}(\Omega)$, $\text{bcl}_{\mathcal{E}_0}(\Omega)$, $\text{b-int}_{\mathcal{E}_0}(\Omega)$ and $\text{scl}_{\mathcal{E}_0}(\Omega)$ according to their definitions but in relation to the induced topology.

Remark 2.1. Let a subset $\Omega$ of a topological space $(\mathcal{E}_0, \mathcal{T}_0)$, we have

$$\text{cl}_{\mathcal{E}_0}(\Omega) = \text{cl}(\Omega) \cap \mathcal{E}_0, \quad \text{int}(\Omega) \subseteq \text{int}_{\mathcal{E}_0}(\Omega). \quad (1)$$

The principle of the following theorem, we study the behavior of pre-open sets a sub-space.

Theorem 2.1. Let a subset $\Omega$ of a topological space $\mathcal{E}_0$, such as $\text{cl}(\Omega) \subset \mathcal{E}_0 \subset \mathcal{E}$ and $\Omega$ is pre-open in $(\mathcal{E}, \mathcal{T})$. Then $\Omega$ is pre-open in $(\mathcal{E}_0, \mathcal{T}_0)$, but not conversely.

Proof. Let $\Omega$ is pre-open in $(\mathcal{E}, \mathcal{T})$, by (1), we get that $\Omega \subseteq \text{int}(\text{cl}(\Omega)) \subseteq \text{int}_{\mathcal{E}_0}(\text{cl}(\Omega))$, since $\text{cl}_{\mathcal{E}_0}(\Omega) = \text{cl}(\Omega) \cap \mathcal{E}_0 = \text{cl}(\Omega)$, then $\Omega \subseteq \text{int}_{\mathcal{E}_0}(\text{cl}(\Omega) \cap \mathcal{E}_0) = \text{int}_{\mathcal{E}_0}(\text{cl}_{\mathcal{E}_0}(\Omega))$. Hence $\Omega$ is pre-open in $(\mathcal{E}_0, \mathcal{T}_0)$.

Remark 2.2. If $(\mathbb{R}, \mathcal{T}_u)$ endowed with the usual topology, then $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ is the induced topology.
Example 2.1. Let \( E = \mathbb{R} \), \( E_0 = \mathbb{N} \) and \( \Omega = \{1, 2\} \), we have \( \text{cl} (\Omega) = \Omega \), \( \text{int} (\text{cl} (\Omega)) = \emptyset \) and \( \text{int}_{E_0} (\text{cl}_{E_0} (\Omega)) = \Omega \). Hence \( \Omega \) is pre-open in the topology for \( \mathbb{N} \) and \( \Omega \) is not pre-open in \( \mathbb{R} \).

The following result is a direct consequence of Theorem 2.1.

Corollary 2.1. Let a subset \( \Omega \) of a topological space \( E_0 \), such as \( \text{cl} (\Omega) \subset E_0 \subset E \), then \( p \text{-int} (\Omega) \subset \text{pint}_{E_0} (\Omega) \).

Proof. Let a subset \( \Omega \) of a topological space \( E_0 \), we note \( T_0 = \{ \theta \subset \Omega : \theta \text{ is pre-open in } (E_0, T_0) \} \) and \( \Upsilon = \{ \theta \subset \Omega : \theta \text{ is pre-open in } (E, T) \} \). By Theorem 2.1, we get \( \Upsilon \subset T_0 \), which implies that
\[
\text{p-int} (\Omega) = \bigcup_{\theta \in \Upsilon} \theta \subset \bigcup_{\theta \in T_0} \theta = \text{p-int}_{E_0} (\Omega).
\]

The principle of the following theorem, we study the behavior of \( b \)-open sets a sub-space.

Theorem 2.2. Let a subset \( \Omega \) of a topological space \( E_0 \), such as \( \text{cl} (\Omega) \subset E_0 \subset E \) and \( \Omega \) is \( b \)-open in \( (E_0, T_0) \). Then \( \Omega \) is \( b \)-open in \( (E, T) \), but not conversely.

Proof. Let a subset \( \Omega \) of a topological space \( E_0 \), by (1), we have
\[
\Omega \subset \text{cl}_{E_0} (\text{int}_{E_0} (\Omega)) \cup \text{int}_{E_0} (\text{cl}_{E_0} (\Omega)) \subset (\text{cl} (\text{int}_{E_0} (\Omega)) \cap E_0) \cup \text{int}_{E_0} (\text{cl} (\Omega) \cap E_0)
\]
\[
\subset (\text{cl} (\text{int} (\Omega)) \cap E_0) \cup \text{int} (\text{cl} (\Omega) \cap E_0) \subset (\text{cl} (\text{int} (\Omega)) \cap E_0) \cup \text{int} (\text{cl} (\Omega) \cap E_0).
\]
Hence, \( \text{cl} (\text{int} (\Omega)) \cap E_0 \subset \text{cl} (\text{int} (\Omega)) \) and \( \text{int} (\text{cl} (\Omega) \cap E_0) \subset \text{int} (\text{cl} (\Omega)) \), which implies that \( \Omega \subset \text{cl} (\text{int} (\Omega)) \cup \text{int} (\text{cl} (\Omega)) \), then \( \Omega \) is semi-open in \( (E, T) \).

Example 2.2. Let \( E = \mathbb{R} \), \( E_0 = \mathbb{N} \) and \( \Omega = \{1, 2\} \), then \( \text{int} (\text{cl} (\Omega)) = \text{cl} (\text{int} (\Omega)) = \emptyset \) and \( \text{cl}_{E_0} (\text{int}_{E_0} (\Omega)) = \text{int}_{E_0} (\text{cl}_{E_0} (\Omega)) = \Omega \). Hence \( \Omega \) is \( b \)-open in \( (\mathbb{N}, \mathcal{P} (\mathbb{N})) \) and \( \Omega \) is not \( b \)-open in \( (\mathbb{R}, T_0) \).

Corollary 2.2. Let a subset \( \Omega \) of a topological space \( E_0 \), such as \( \text{cl} (\Omega) \subset E_0 \subset E \), then \( b \text{-int}_{E_0} (\Omega) \subset b \text{-int} (\Omega) \).

Proof. Let \( \Omega \) be set in \( E_0 \), we note \( \Theta_0 = \{ \theta \subset \Omega : \theta \text{ is } b \text{-open in } (E_0, T_0) \} \) and \( \Theta = \{ \theta \subset \Omega : \theta \text{ is } b \text{-open in } (E, T) \} \). By Theorem 2.2, we get \( \Theta_0 \subset \Theta \), which implies that
\[
b \text{-int}_{E_0} (\Omega) = \bigcup_{\theta \in \Theta_0} \theta \subset \bigcup_{\theta \in \Theta} \theta = b \text{-int} (\Omega).
\]

The principle of the following theorem, we study the behavior of \( a \)-open sets a sub-space.

Theorem 2.3. Let a subset \( \Omega \) of a topological space \( E_0 \), such as \( \text{cl} (\Omega) \subset E_0 \subset E \) and \( \Omega \) is \( a \)-open in \( (E, T) \). Then \( \Omega \) is \( a \)-open in \( (E_0, T_0) \), but not conversely.

Proof. By remark 2.1, we have
\[
\text{cl} (\text{int} (\Omega)) \subset \text{cl} (\text{int}_{E_0} (\Omega)) \subset \text{cl} (\text{int}_{E_0} (\Omega)) \cap E_0 = \text{cl}_{E_0} (\text{int}_{E_0} (\Omega)).
\]
Then,
\[
\text{int} (\text{cl} (\text{int} (\Omega))) \subset \text{int} (\text{cl}_{E_0} (\text{int}_{E_0} (\Omega))) \subset \text{int}_{E_0} (\text{cl}_{E_0} (\text{int}_{E_0} (\Omega))).
\]
Since \( \Omega \) is \( a \)-open in \( (E, T) \), then \( \Omega \subset \text{int}_{E_0} (\text{cl}_{E_0} (\text{int}_{E_0} (\Omega))) \). Hence \( \Omega \) is \( a \)-open in \( (E_0, T_0) \).
Example 2.3. Let $\mathcal{E} = \mathbb{R}$, $\mathcal{E}_0 = \mathbb{N}$ and $\Omega = \{1, 2\}$, then $\text{int} (\text{cl} (\text{int} (\Omega))) = \emptyset$ and $\text{int}_{\mathcal{E}_0} (\text{cl}_{\mathcal{E}_0} (\text{int}_{\mathcal{E}_0} (\Omega))) = \Omega$. Hence $\Omega$ is $\alpha$-open in the topology for $\mathbb{N}$ and $\Omega$ is not $\alpha$-open in the topology for $\mathbb{R}$.

Corollary 2.3. Let a subset $\Omega$ of a topological space $\mathcal{E}_0$, such as $cl (\Omega) \subset \mathcal{E}_0 \subset \mathcal{E}$, then $\alpha\text{-int} (\Omega) \subseteq \alpha\text{-int}_{\mathcal{E}_0} (\Omega)$.

**Proof.** Let $\Omega$ be in $\mathcal{E}_0$, we note $\Xi_0 = \{\theta \subset \Omega : \theta$ is $\alpha$-open in $(\mathcal{E}_0, \mathcal{T}_0)$ and $\Xi = \{\theta \subset \Omega : \theta$ is $\alpha$-open in $(\mathcal{E}, \mathcal{T})\}$. By Theorem 2.3, we get $\Xi \subset \Xi_0$, which implies that

$$\alpha\text{-int} (\Omega) = \bigcup_{\theta \in \Xi} \theta \subseteq \bigcup_{\theta \in \Xi_0} \theta = \alpha\text{-int}_{\mathcal{E}_0} (\Omega).$$

The principle of the following theorem, we study the behavior of $g$-closed sets a sub-space.

**Theorem 2.4.** Let a subset $\Omega$ of a topological space $\mathcal{E}_0$, such as $\Omega \subset \mathcal{E}_0 \subset \mathcal{E}$. If $\Omega$ is $g$-closed in $(\mathcal{E}_0, \mathcal{T}_0)$, then $\Omega$ is $g$-closed in $(\mathcal{E}, \mathcal{T})$.

**Proof.** Assume $\Omega$ is $g$-closed in $(\mathcal{E}_0, \mathcal{T}_0)$. Let $\theta$ be open in $(\mathcal{E}, \mathcal{T})$ and $\Omega \subseteq \theta$, with $\Omega \subseteq \theta \cap \mathcal{E}_0$, since $\theta \cap \mathcal{E}_0$ is open in $(\mathcal{E}_0, \mathcal{T}_0)$, then $cl (\Omega) \subseteq \theta \cap \mathcal{E}_0 \subseteq \theta$. Hence $\Omega$ is $g$-closed in $\mathcal{E}$.

Theorem 2.5. Let a subset $\Omega$ of a topological space $\mathcal{E}_0$, such as $\mathcal{E}_0$ is open in $(\mathcal{E}, \mathcal{T})$, then $\Omega$ is $g$-closed in $(\mathcal{E}, \mathcal{T})$, if and only if $\Omega$ is $g$-closed in $(\mathcal{E}_0, \mathcal{T}_0)$.

**Proof.** By Theorem 2.4, just show that, $\Omega$ is $g$-closed in $(\mathcal{E}, \mathcal{T})$ implies that $\Omega$ is $g$-closed in $(\mathcal{E}_0, \mathcal{T}_0)$. Indeed, let $\theta_0$ be open in $\mathcal{E}_0$, there is an open $\theta$ in $\mathcal{E}$, such that $\theta_0 = \theta \cap \mathcal{E}_0$, if $\Omega \subseteq \theta_0 = \theta \cap \mathcal{E}_0$, since $\theta \cap \mathcal{E}_0$ is open in $(\mathcal{E}, \mathcal{T})$, give us $cl (\Omega) \subseteq \theta_0$ and $cl_{\mathcal{E}_0} (\Omega) \subseteq cl_{\mathcal{E}_0} (\Omega)$, then $\Omega$ is $g$-closed in $(\mathcal{E}_0, \mathcal{T}_0)$.

The following result is some properties of $gbr^*$-closed maps.

**Theorem 2.6.** Let $\mathcal{E}_0$ be a closed set and let $\phi : (\mathcal{E}, \mathcal{T}) \to (\mathcal{F}, \Sigma)$ is $gbr^*$-closed map, then $\phi : (\mathcal{E}_0, \mathcal{T}_0) \to (\mathcal{F}, \Sigma)$ is $gbr^*$-closed map, but not conversely.

**Proof.** Let $\phi : (\mathcal{E}, \mathcal{T}) \to (\mathcal{F}, \Sigma)$ is $gbr^*$-closed map and $\vartheta_0$ be an closed set in $(\mathcal{E}_0, \mathcal{T}_0)$. So, it exists is a closed set $\vartheta$ in $(\mathcal{E}, \mathcal{T})$, such that $\vartheta_0 = \vartheta \cap \mathcal{E}_0$. As $\mathcal{E}_0$ is closed in $(\mathcal{E}, \mathcal{T})$. Hence $\phi (\vartheta_0)$ is $gbr^*$-closed in $(\mathcal{F}, \Sigma)$. Then $\phi : (\mathcal{E}_0, \mathcal{T}_0) \to (\mathcal{F}, \Sigma)$ is $gbr^*$-closed.

**Example 2.4.** Let $\phi : (\mathcal{E}, \mathcal{T}) \to (\mathcal{E}, \Sigma)$ be a map, such that $\phi (x) = x$, $\phi (y) = x$, $\phi (z) = y$, $\phi (t) = z$, where $\mathcal{E} = \{x, y, z, t\}$, $\mathcal{T} = \{\mathcal{E}, \emptyset, \{x\}, \{y, z, t\}\}$, $\mathcal{E} = \{x, y, \emptyset, \{y, z, t\}\}$. Then $\phi : (\mathcal{E}_0, \mathcal{T}_0) \to (\mathcal{E}, \Sigma)$ is $gbr^*$-closed map, but $\phi : (\mathcal{E}, \mathcal{T}) \to (\mathcal{E}, \Sigma)$ is not $gbr^*$-closed map.

**Lemma 2.1.** Let $\Omega_1$ and $\Omega_2$ are $gbr^*$-closed in $(\mathcal{E}, \mathcal{T})$, then $\Omega_1 \cup \Omega_2$ is $gbr^*$-closed in $(\mathcal{E}, \mathcal{T})$.

**Proof.** Let $\Omega_1$ and $\Omega_2$ are $gbr^*$-closed in $(\mathcal{E}, \mathcal{T})$, if $bcl (\Omega_1 \cup \Omega_2) \subseteq \theta$, with $\theta$ is $g^*$ open in $(\mathcal{E}, \mathcal{T})$, we have $bcl (\Omega_1) \subseteq bcl (\Omega_1 \cup \Omega_2) \subseteq \theta$, then $\Omega_1 \subseteq \theta$, the same way we find $\Omega_2 \subseteq \theta$. Hence $\Omega_1 \cup \Omega_2 \subseteq \theta$.

**Theorem 2.7.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ sets in $\mathcal{E}$ and let $\phi : (\mathcal{E}, \mathcal{T}) \to (\mathcal{F}, \Sigma)$ is map, such as $\mathcal{E}_1 \cup \mathcal{E}_2 = \mathcal{E}$. If $\phi_{\mathcal{E}_1} : (\mathcal{E}_1, \mathcal{T}_1) \to (\mathcal{F}, \Sigma)$ and $\phi_{\mathcal{E}_2} : (\mathcal{E}_2, \mathcal{T}_2) \to (\mathcal{F}, \Sigma)$ are $gbr^*$-closed map, then $\phi : (\mathcal{E}, \mathcal{T}) \to (\mathcal{F}, \Sigma)$ is $gbr^*$-closed map.
Proof. Let \( \phi_{\mathcal{E}_1} : (\mathcal{E}_1, T_1) \rightarrow (\mathcal{F}, \Sigma) \), \( \phi_{\mathcal{E}_2} : (\mathcal{E}_2, T_2) \rightarrow (\mathcal{F}, \Sigma) \) are \( gb^* \)-closed map and \( \vartheta \) be an closed set in \( \mathcal{E} \), then

\[
\phi(\vartheta) = \phi(\vartheta \cap \mathcal{E}) = \phi(\vartheta \cap (\mathcal{E}_1 \cup \mathcal{E}_2)) = \phi((\vartheta \cap \mathcal{E}_1) \cup (\vartheta \cap \mathcal{E}_2))
\]

We have \( \vartheta \cap \mathcal{E}_1 \) and \( \vartheta \cap \mathcal{E}_2 \) are a closed in \( (\mathcal{E}_1, T_1) \) and \( (\mathcal{E}_2, T_2) \), respectively, then \( \phi(\vartheta \cap \mathcal{E}_1) \) and \( \phi(\vartheta \cap \mathcal{E}_2) \) are \( gb^* \)-closed in \( (\mathcal{F}, \Sigma) \). By lemma 2.1, we get that \( \phi(\vartheta \cap \mathcal{E}_1) \) and \( \phi(\vartheta \cap \mathcal{E}_2) \) are \( gb^* \)-closed in \( (\mathcal{F}, \Sigma) \). Hence \( \phi(\vartheta) \) is \( gb^* \)-closed in \( (\mathcal{F}, \Sigma) \). Then \( \phi : (\mathcal{E}, T) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-closed.

The following corollary is a new generalization of the Theorem 2.7.

Corollary 2.4. Let \( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n \) be sets, such as \( \mathcal{E} = \bigcup_{i=1}^{n} \mathcal{E}_i \), if \( \phi_{\mathcal{E}_i} : (\mathcal{E}_i, T_i) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-closed map, for \( i \in [1, n] \cap \mathbb{N} \), then \( \phi : (\mathcal{E}, T) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-closed map.

The following result is some properties of \( gb^* \)-open map.

Theorem 2.8. Let \( \mathcal{E}_0 \) be a open set in \( (\mathcal{E}, T) \) and let \( \phi : (\mathcal{E}, T) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-open map, then \( \phi : (\mathcal{E}_0, T_0) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-open map, but not conversely.

Proof. Let \( \phi : (\mathcal{E}, T) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-closed map and \( \theta_0 \) be an open set in \( (\mathcal{E}_0, T_0) \). So, it exists is a open set \( \theta \) in \( (\mathcal{E}, T) \), such that \( \theta_0 = \theta \cap \mathcal{E}_0 \). As \( \mathcal{E}_0 \) is open in \( (\mathcal{E}, T) \). Hence \( \phi(\theta_0) \) is \( gb^* \)-open in \( (\mathcal{F}, \Sigma) \). Then \( \phi : (\mathcal{E}_0, T_0) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-open.

Example 2.5. Let \( \phi : (\mathcal{E}, T) \rightarrow (\mathcal{F}, \Sigma) \) be a map such that \( \phi(x) = x \), \( \phi(y) = x \), \( \phi(c) = y \), \( \phi(d) = c \), where \( \mathcal{E} = \{x, y, c, d\}, T = \{\mathcal{E}, \emptyset, \{x\}, \{y, c, d\}\}, \Sigma = \{\mathcal{E}, \emptyset, \{y\}, \{x, y, c\}\} \) and \( \mathcal{E}_0 = \{x\} \). Then \( \phi : (\mathcal{E}_0, T) \rightarrow (\mathcal{E}, \Sigma) \) is \( gb^* \)-open map, but \( \phi : (\mathcal{E}_0, T) \rightarrow (\mathcal{E}, \Sigma) \) is not \( gb^* \)-open map.

Lemma 2.2. Let \( \Omega_1 \) and \( \Omega_2 \) are \( gb^* \)-open in \( (\mathcal{E}, T) \), such as \( \Omega_1 \cap \Omega_2 \neq \emptyset \), then \( \Omega_1 \cup \Omega_2 \) is \( gb^* \)-open in \( (\mathcal{E}, T) \).

Proof. Let \( \Omega_1 \) and \( \Omega_2 \) are \( gb^* \)-open in \( (\mathcal{E}, T) \), such as \( \Omega_1 \cap \Omega_2 \neq \emptyset \), we have \( \emptyset \) is \( gb^* \)-closed, then \( (\Omega_1 \cup \Omega_2)^c \) is \( gb^* \)-closed in \( (\mathcal{E}, T) \), thus \( \Omega_1 \cup \Omega_2 \) is \( gb^* \)-open in \( (\mathcal{E}, T) \).

Theorem 2.9. Let \( \mathcal{E}_1, \mathcal{E}_2 \) be sets disjoint, such as \( \mathcal{E}_1 \cup \mathcal{E}_2 = \mathcal{E} \). If \( \phi_{\mathcal{E}_1} : (\mathcal{E}_1, T_1) \rightarrow (\mathcal{F}, \Sigma) \) and \( \phi_{\mathcal{E}_2} : (\mathcal{E}_2, T_2) \rightarrow (\mathcal{F}, \Sigma) \) are \( gb^* \)-open map, \( \phi : \mathcal{E} \rightarrow \mathcal{F} \) is an injective map, then \( \phi : (\mathcal{E}, T) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-open map.

Proof. Let \( \phi_{\mathcal{E}_1} : (\mathcal{E}_1, T_1) \rightarrow (\mathcal{F}, \Sigma) \) and \( \phi_{\mathcal{E}_2} : (\mathcal{E}_2, T_2) \rightarrow (\mathcal{F}, \Sigma) \) are \( gb^* \)-open map, \( \theta \) be an open set in \( (\mathcal{E}, T) \), then

\[
\phi(\theta) = \phi(\theta \cap \mathcal{E}) = \phi(\theta \cap \mathcal{E}_1) \cup \phi(\theta \cap \mathcal{E}_2).
\]

We have \( \theta \cap \mathcal{E}_1 \) and \( \theta \cap \mathcal{E}_2 \) are a open in \( (\mathcal{E}_1, T_1) \) and \( (\mathcal{E}_2, T_2) \), respectively, then \( \phi(\theta \cap \mathcal{E}_1) \) and \( \phi(\theta \cap \mathcal{E}_2) \) are \( gb^* \)-open in \( (\mathcal{F}, \Sigma) \), now we show that \( \phi(\theta \cap \mathcal{E}_1) \cap \phi(\theta \cap \mathcal{E}_2) = \emptyset \). Suppose that \( \phi(\theta \cap \mathcal{E}_1) \cap \phi(\theta \cap \mathcal{E}_2) \neq \emptyset \), let \( y \in \phi(\theta \cap \mathcal{E}_1) \cap \phi(\theta \cap \mathcal{E}_2) \), there exists \( x_1 \in \mathcal{E}_1 \) and \( x_2 \in \mathcal{E}_2 \), such that \( x_1 \neq x_2 \) and \( y = \phi(x_1) = \phi(x_2) \), which contradicts, by \( \phi \) is injective map. By lemma 2.2, we get \( \phi(\theta) \) is \( gb^* \)-open in \( (\mathcal{F}, \Sigma) \). Then \( \phi : (\mathcal{E}, T) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-open.

The following results are consequences of Theorem 2.9.

Corollary 2.5. Let \( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n \) be sets, such as \( \mathcal{E} = \bigcup_{i=1}^{n} \mathcal{E}_i \) and for \( i \neq j \), \( \mathcal{E}_i \cap \mathcal{E}_j = \emptyset \). If, for \( i \in [1, n] \cap \mathbb{N} \), \( \phi_{\mathcal{E}_i} \) is \( gb^* \)-open map, then \( \phi : (\mathcal{E}, T) \rightarrow (\mathcal{F}, \Sigma) \) is \( gb^* \)-open map.
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References