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Investigating the solubility of wreath products group of degree 3p using numerical approach

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Abstract: Let p be a prime number (p > 3) and G a finite permutation group of degree 3p, generated via Wreath products of pairs of permutation groups. We, in this paper discuss the solubility of G using numerical approach. We applied the computational group theory (GAP) to enhance and validate our work.

Key words: Permutation Group, Solubility, Wreath Products, p-Groups and Sylow p-subgroup.

1. Introduction

The Wreath product of two permutation groups C and D denoted by $W = CwrD$ is the semi-direct product of P (a derived group of prime power order) by D, so that, $W = ((f,d)|f \in P, d \in D)$, with composition in $W := (f_1, d_1)(f_2, d_2) = \left(\left(f_1, f_2 d_1^{-1}\right)(d_1, d_2)\right) \quad \forall\ f_1, f_2 \in P$ and $d_1, d_2 \in D$ is a special form of permutation group. When the nature of the Wreath products groups is well understood it facilitates comprehension of certain types of subgroups of the symmetric groups.

According to Cameron(2013)[2], a group G is soluble or solvable if it has a series of subgroups,

$$G = H_{\mathrm{n}} \supset H_{\mathrm{n-1}} \supset \cdots \supset H_1 \supset H_0 = \{\mathrm{e}\}$$

With each subgroup H_{i} normal in $H_{\mathrm{i+1}}$ and the factor groups $H_{\mathrm{i+1}}/H_{\mathrm{i}}$ abelian. Solvable groups are significant as they allow us to differentiate between categories of groups.

In this work, we obtained more detailed description of the unique structure of Wreath product (permutation) groups of degree 3p that are not p-groups and investigate their solubility using numerical approach. This work is significant as it will form part of a growing database that will eventually be used in the needed review of the classification of finite simple group (CFSG).

There are some recent results on the solubility of permutations groups including the following:

Thanos (2006)[11] proved that If $|G| = p^k$ where p is a prime number then G is solvable. In other words every p-group where p is a prime number is solvable.

Bello et al. (2017)[1] used the concept of p-groups to construct locally solvable groups using two permutation groups by wreath product.

Gandi et al. (2019)[4] investigated solvable and Nilpotent concepts on dihedral groups of an even degree regular polygon.

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The results from the above papers and other findings on group concepts from the works of Kimura and Nakagawa, (1973)[8], Ito and Wada, (1972)[6] and Cai and Zhang, (2015)[3] will be used as valuable references towards achieving our desired objectives.

This work is organized in five sections. Section2 gives some preliminaries required for the work. In Section3 we state the main result of this paper with some illustrating examples. We also use the groups, algorithms and programming (GAP) to validate the solubility of permutation groups of degree $3p(p = 5, 7, 11, \cdots)$. Section4 contains conclusion and recommendation while Section4.2 is the list of references.

2. Materials and Methods

2.1. p-Group (Sylow, 1872)

If a group G has number of elements, $|G| = p^n$ where p is prime, it is called a p-group.

2.2. p-Subgroup (Sylow, 1872)

A subgroup H of a group $G(H \leq G)$ is called a p-subgroup G if H itself is a p-group, this is, $|H| = p^r$, for some $r \geq 0$ for all $H \in G$.

2.3. Sylow p-Subgroup (Sylow, 1872)

Let G be a group. If G is finite and $|G| = p^r m, r \geq 1$ where p and m are co-prime and $H \leq G$ such that $|H| = p^r$, we refer to H as a Sylow p-subgroup of G.

2.4. Sylow Theorems (Sylow, 1872)

Let G be a finite group of order n.

1. If p is a prime such that p^k is a divisor of $|G|$ for some $k \geq 0$, then G contains a subgroup of order p^k.

2. All Sylow p-subgroups of G are conjugate, and any p-subgroup of G is contained in a Sylow p-subgroup.

3. Let $n = mp^k$, with $(m, p) = 1$, and let n_p be the number of Sylow p-subgroups of G. Then $n_p \mid m$ and $n_p \equiv 1(\mathrm{mod} p)$.

2.5. Wreath product (Joseph and Audu, 1991)

The wreath product of two permutation groups C by D denoted by $W = CwrD$ is the semidirect product of P by D, so that,

$$W = \{(f, d) | f \in P, d \in D\}$$

with multiplication in W defined as

$$(f_1, d_1)(f_2, d_2) = \left(\left(f_1, f_2 d_1^{-1}\right)(d_1, d_2)\right) \quad \forall f_1, f_2 \in P \wedge d_1, d_2 \in D$$

Henceforth, we write fd instead of (f, d) for elements of W.

2.6. Theorem (Joseph and Audu, 1991)

Let D act on P as $f^d(\delta) = f(\delta d^{-1})$ where $f \in P, d \in D$ and $\delta \in \Delta$. Let W be group of all juxtaposed symbols f d, with $f \in P, d \in D$ and multiplication given by $(f_1, d_1)(f_2, d_2) = (f_1 f_2 d_1^{-1})(d_1, d_2)$ Then W is a group referred to as the semi-direct product of P by D with the action as defined

2.7. Theorem (Cameron, 2013)

If $|G| = pq$ where p and q are distinct prime numbers $(p < q)$ then, G is solvable.

2.8. Theorem (Thanos, 2006)

If $|G| = p^k$ where p is a prime number then G is solvable. In other words every p-group where p is a prime is solvable.

Proof. By induction on k.

1st step. For $k = 1$ our group is a cyclic group of prime order thus it is solvable by definition.

2nd step. Let the statement hold for all $n \leq k$.

3rd step. We will prove that it holds for $k = n + 1$. By corollary 3 since G is a p-group, the center of G denoted $Z(G) \neq \{e\}$. Also $Z(G)$ is a normal subgroup of G and $Z(G)$ is abelian. Thus Z(G) is solvable. Now $G/Z(G)$ is again a p-group or trivial. If it is trivial then $G = Z(G)$ thus G is abelian hence it is solvable. If it is not trivial then $|G/Z(G)| \leq p^n$. So by the inductive step it is solvable. Hence G is also solvable.

2.9. Corollary (Thonas, 2006)

If G has only one p-Sylow subgroup H then H is normal.

2.10. Corollary (Thonas, 2006)

If $H \trianglelefteq G$ and $\left|\frac{G}{H}\right| = p$ or p^2 then $\frac{G}{H}$ is abelian

2.11. Corollary (Thonas, 2006)

Let G be a finite group and H a Sylow p-subgroup of G. Then H is the only Sylow p-subgroup of G if and only if H is normal in G.

Proof

By Sylow theorem, the Sylow p-subgroups of G are the elements of the sets $\{g^{-1}Hg| \quad g \in G\}$ and this reduces to a singleton set if and only if $g^{-1}Hg = H$ for all $g \in G$; that is precisely when H is normal in G.

2.12. Proposition (Thonas, 2006)

Suppose G is a solvable group and H is a subgroup of G that is, $H \leq$ G. Then

1. H is solvable.
2. If $H \triangleleft G$, then G/H is solvable.

Proof

Start from a series with abelian slices. G: $G_0 \triangleright G_l \triangleright \ldots \triangleright \mathrm{G_n} = (l)$ Then $H = H \cap \mathrm{G_0} \triangleright \mathrm{H} \cap \; \mathrm{G_1} \triangleright \ldots \triangleright H \cap \mathrm{G_n} = \{1\}$. When H is normal, we use the canonical projection $\pi : \mathrm{G} \rightarrow \mathrm{G/H}$ to get $\mathrm{G/H} = \pi(\mathrm{G_0}) \triangleright \ldots \pi(\mathrm{G_n}) = \{1\}$; the quotients are abelian as well, so G/H is still solvable.

2.13. Theorem (Cameron, 2013)

If G is a group and H is a normal subgroup of G such that H is solvable and G/H is solvable then G is solvable.

3. Wreath product group of degree $3p$

3.1. Main Theorem

Let G be the Wreath product of two permutation groups C and D of degree 3p(p $>$ 3) and H the Sylow p-subgroup of G. Then (i) H is normal in G and is soluble (ii) G/H is soluble and (iii) G is soluble.

Proof

Now, the order of W that is, $|W| = 3\mathrm{p}^3$ or $3^p\mathrm{p}$

Case 1: $|W| = 3\mathrm{p}^3$

Let $N_{\mathrm{p}}(W)$ be the number of Sylow p-subgroups of the group W.

By Sylow theorem 2.8, we have

$$\mathrm{N_p} \equiv 1 \text{ modulo p and } \mathrm{N_p} \text{ divides } 3.$$

It follows from these constraints that $\mathrm{N_p} = 1$.

Let $H =$ p-Sylow subgroup of W. Then H is normal in W by corollary 2.11 proving (i).

Since $|H| = \mathrm{p}^3$, we have that H is a p-Group and by theorem 2.8 is Solvable proving (ii).

Also $|W : H| = 3$ implies that W/H is a p-Group hence Solvable by theorem 2.8.

Hence G is a solvable group by theorem 2.13 proving (iii).

Case 2: $|W| = 3^p\mathrm{p}$

By Sylow theorem 3.2.3, the number of Sylow p-subgroup of W, $\mathrm{N_p}$ of order p is congruent to 1 modulo p and it divides 3^p.

As $\mathrm{N_p}$ divides $|G|$ and $3^{\mathrm{p}} \equiv| 1 \bmod p$, it follows that $\mathrm{N_p} = 1$.

Let K be the unique Sylow p-subgroup of W. Then the subgroup K is a normal subgroup of W by Corollary 2.11. Since $|K| = p$, we have that K is a p-Group and by theorem 2.8 is Solvable. Also $|W : K| = 3^p$ implies that G/K is a p-Group hence Solvable by theorem 2.8.

By theorem 2.13, G is solvable as required.

3.2. Illustrating Example (1)

Let

$$C_1 = \{(1), (12345), (13524), (14253), (15432)\} \text{ and}$$
$$D_1 = \{(1), (6, 7)\}$$

acting on the sets $\Omega_1 = \{1, 2, 3, 4, 5\}$ and $\Delta_1 = \{6, 7\}$ respectively.

Let $P_1 = C_1^{\Delta_1} = \{f : \Delta_1 \to C_1\}$. Then $|P_1| = |C_1|^{|\Delta 1|} = 5^2 = 25$

The mappings in P_1 are as list below.

$f_1 : 6 \to (1), 7 \to (1)$

$f_2 : 6 \to (12345), 7 \to (12345)$

$f_3 : 6 \to (13524), 7 \to (13524)$,
$f_4 : 6 \to (14253), 7 \to (14253)$
$f_5 : 6 \to (15432), 7 \to (15432)$
$f_6 : 6 \to (1), 7 \to (12345)$
$f_7 : 6 \to (1), 7 \to (13524)$
$f_8 : 6 \to (1), 7 \to (14253)$
$f_9 : 6 \to (1), 7 \to (15432)$
$f_{10} : 6 \to (12345), 7 \to (1)$
$f_{11} : 6 \to (12345), 7 \to (13524)$
$f_{12} : 6 \to (12345), 7 \to (14253)$
$f_{13} : 6 \to (12345), 7 \to (15432)$
$f_{14} : 6 \to (13524), 7 \to (1)$
$f_{15} : 6 \to (13524), 7 \to (12345)$
$f_{16} : 6 \to (13524), 7 \to (14253)$
$f_{17} : 6 \to (13524), 7 \to (15432)$
$f_{18} : 6 \to (14253), 7 \to (1)$
$f_{19} : 6 \to (14253), 7 \to (12345)$
$f_{20} : 6 \to (14253), 7 \to (13524)$
$f_{21} : 6 \to (14253), 7 \to (15432)$
$f_{22} : 6 \to (15432), 7 \to (1)$
$f_{23} : 6 \to (15432), 7 \to (12345)$
$f_{24} : 6 \to (15432), 7 \to (13524)$
$f_{25} : 6 \to (15432), 7 \to (14253)$

We can easily verify that P is a group with respect to the operations $(f_1, f_2)(\delta) = f_1(\delta_1) f_1(\delta_1)$, where $\delta_1 \in \Delta_1$

We recall the definition of the action of D_1 on P as $f^{\mathrm{d}}(\delta_1) = f\left(\delta_1 d^{-1}\right)$ where $f \in P, d \in D_1$ and $\delta_1 \epsilon \Delta_1$, then D_1 acts on P as a groups.

We also recall the definition $W = C_1 wr D_1$, the semi-direct product of P by D_1 in that order; i.e. $W = \{(f, d) \mid f \epsilon P, \delta_1 \epsilon \Delta_1\}$

Now, W is a group with respect to the operation;

$(f_1, d_1)(f_2, d_2) = (f_1, f_2^{d_1^{-1}})(d_1, d_2)$, and accordingly, $d_1 = (1), d_2 = (6, 7)$.

Then the elements of W_1 are

$(f_1, d_1), (f_2, d_1), (f_3, d_1), (f_4, d_1), (f_5, d_1), (f_6, d_1), (f_7, d_1), (f_8 d_1), (f_9, d_1), (f_{10}, d_1), (f_{11}, d_1), (f_{12}, d_1)$,
$(f_{13}, d_1), (f_{14}, d_1), (f_{15}, d_1), (f_{16}, d_1), (f_{17}, d_1), (f_{18}, d_1), (f_{19}, d_1), (f_{20}, d_1), (f_{21}, d_1), (f_{22}, d_1), (f_{23}, d_1)(f_{24}, d_1)$,
$(f_{25}, d_1), (f_1, d_2), (f_2, d_2), (f_3, d_2), (f_4, d_2), (f_5, d_2), (f_6, d_2), (f_7, d_2), (f_8, d_2), (f_9, d_2), (f_{10}, d_2), (f_{11}, d_2)$,
$(f_{12}, d_2), (f_{13}, d_2), (f_{14}, d_2), (f_{15}, d_2), (f_{16}, d_2), (f_{17}, d_2), (f_{18}, d_2), (f_{19}, d_2), (f_{20}, d_2), (f_{21}, d_2), (f_{22}, d_2)$,
$(f_{23}, d_2), (f_{24}, d_2), (f_{25}, d_2)$.

Now, define action of W_1 on $\Omega_1 \times \Delta_1$ as

$(\beta, \delta_1) fd = (\beta f(\delta), d\delta)$ where $\beta \in \Omega_1$ and $\delta_1 \in \Delta_1$

Further, $\Omega_1 \times \Delta_1 = \{(1,6),(1,7),(2,6),(2,7),(3,6),(3,7),(4,6),(4,7),(5,6),(5,7)\}$
We obtain the following permutation by action of W_1 on Ω_1 x Δ_1

$$(1,6)f_1d_1 = (1f_1(6),d_1) = (1(1),6(1)) = (1,6)$$
$$(1,7)f_1d_1 = (1f_1(7),d_1) = (1(1),7(1)) = (1,7)$$
$$(2,6)f_1d_1 = (2f_1(6),d_1) = (2(1),6(1)) = (2,6)$$
$$(2,7)f_1d_1 = (2f_1(7),d_1) = (2(1),7(1)) = (2,7)$$
$$(3,6)f_1d_1 = (3f_1(6),d_1) = (3(1),6(1)) = (3,6)$$
$$(3,7)f_1d_1 = (3f_1(7),d_1) = (3(1),7(1)) = (3,7)$$
$$(4,6)f_1d_1 = (4f_1(6),d_1) = (4(1),6(1)) = (4,6)$$
$$(4,7)f_1d_1 = (4f_1(7),d_1) = (4(1),7(1)) = (4,7)$$
$$(5,6)f_1d_1 = (5f_1(6),d_1) = (5(1),6(1)) = (5,6)$$
$$(5,7)f_1d_1 = (5f_1(7),d_1) = (5(1),7(1)) = (5,7)$$

And in summary,

$$(\Omega_1 \times \Delta_1)\, f_1d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_2d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_3d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (3,6)(3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_4d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6)(3,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_5d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_7d_1 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (1,6)(3,7)(2,6)(4,7)(3,6)(5,7)(4,6)(1,7)(5,6)(2,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_4d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (4,7)(4,6)(5,7)(5,6)(1,7)(1,6)(2,7)(2,6)(3,7)(3,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_5d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (5,7)(5,6)(1,7)(1,6)(2,7)(2,6)(3,7)(3,6)(4,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_6d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_7 d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (1,7)(3,6)(2,7)(4,6)(3,7)(5,6)(4,7)(1,6)(5,7)(2,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_8 d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (1,7)(4,6)(2,7)(5,6)(3,7)(1,6)(4,7)(2,6)(5,7)(3,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_9 d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (1,7)(5,6)(2,7)(1,6)(3,7)(2,6)(4,7)(3,6)(5,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{10} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (2,7)(1,6)(3,7)(2,6)(4,7)(3,6)(5,7)(4,6)(1,7)(5,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{11} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{12} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (2,7)(4,6)(3,7)(5,6)(4,7)(1,6)(5,7)(2,6)(1,7)(3,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{13} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (2,7)(5,6)(3,7)(1,6)(4,7)(2,6)(5,7)(3,6)(1,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{14} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (3,7)(1,6)(4,7)(2,6)(5,7)(3,6)(1,7)(4,6)(2,7)(5,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{15} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (3,7)(2,6)(4,7)(3,6)(5,7)(4,6)(1,7)(5,6)(2,7)(1,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{16} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{17} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (3,7)(5,6)(4,7)(1,6)(5,7)(2,6)(1,7)(3,6)(2,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{18} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (4,7)(1,6)(5,7)(2,6)(1,7)(3,6)(2,7)(4,6)(3,7)(5,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{19} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (4,7)(2,6)(5,7)(3,6)(1,7)(4,6)(2,7)(5,6)(3,7)(1,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{20} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (4,7)(3,6)(5,7)(4,6)(1,7)(5,6)(2,7)(1,6)(3,7)(2,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{21} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{22} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (5,7)(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{23} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (5,7)(2,6)(1,7)(3,6)(2,7)(4,6)(3,7)(5,6)(4,7)(1,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{24} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (5,7)(3,6)(1,7)(4,6)(2,7)(5,6)(3,7)(1,6)(4,7)(2,6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1)\, f_{25} d_2 = \begin{pmatrix} (1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\ (5,7)(4,6)(1,7)(5,6)(2,7)(1,6)(3,7)(2,6)(4,7)(3,6) \end{pmatrix}$$

Renaming the symbols as

$$(1,6) \to 1, (1,7) \to 2, (2,6) \to 3, (2,7) \to 4, (3,6) \to 5, (3,7) \to 6, (4,6) \to 7, (4,7) \to 8, (5,6) \to 9, (5,7) \to 10,$$

The permutations in cyclic form are as follows.
$W_1 = \{(1), (6,7,8,9,10), (6,8,10,7,9), (6,9,7,10,8), (6,10,9,8,7), (1,2,3,4,5), (1,2,3,4,5)(6,7,8,9,10),$
$(1,2,3,4,5)(6,8,10,7,9), (1,2,3,4,5)(6,9,7,10,8), (1,2,3,4,5)(6,10,9,8,7), (1,3,5,2,4),$
$(1,3,5,2,4)(6,7,8,9,10), (1,3,5,2,4)(6,8,10,7,9), (1,3,5,2,4)(6,9,7,10,8), (1,3,5,2,4)(6,10,9,8,7),$
$(1,4,2,5,3), (1,4,2,5,3)(6,7,8,9,10), (1,4,2,5,3)(6,8,10,7,9), (1,4,2,5,3)(6,9,7,10,8), (1,4,2,5,3)(6,10,9,8,7),$
$(1,5,4,3,2), (1,5,4,3,2)(6,7,8,9,10), (1,5,4,3,2)(6,8,10,7,9), (1,5,4,3,2)(6,9,7,10,8),$
$(1,5,4,3,2)(6,10,9,8,7), (1,6)(2,7)(3,8)(4,9)(5,10), (1,6,2,7,3,8,4,9,5,10), (1,6,3,8,5,10,2,7,4,9),$
$(1,6,4,9,2,7,5,10,3,8), (1,6,5,10,4,9,3,8,2,7), (1,7,2,8,3,9,4,10,5,6), (1,7,3,9,5,6,2,8,4,10),$
$(1,7,4,10,2,8,5,6,3,9), (1,7,5,6,4,10,3,9,2,8), (1,7)(2,8)(3,9)(4,10)(5,6), (1,8,3,10,5,7,2,9,4,6),$
$(1,8,4,6,2,9,5,7,3,10), (1,8,5,7,4,6,3,10,2,9), (1,8)(2,9)(3,10)(4,6)(5,7), (1,8,2,9,3,10,4,6,5,7),$
$(1,9,4,7,2,10,5,8,3,6), (1,9,5,8,4,7,3,6,2,10), (1,9)(2,10)(3,6)(4,7)(5,8), (1,9,2,10,3,6,4,7,5,8),$
$(1,9,3,6,5,8,2,10,4,7), (1,10,5,9,4,8,3,7,2,6), (1,10)(2,6)(3,7)(4,8)(5,9),$
$(1,10,2,6,3,7,4,8,5,9), (1,10,3,7,5,9,2,6,4,8), (1,10,4,8,2,6,5,9,3,7)\}$.
Then the Wreath product $W_1 = C_{1\mathrm{wr}} D_1$ with degree $|C_1| \times |D_1| = 10$ and order given by

$|W_1| = |C_1|^{|\Delta_1|} \times |D_1| = 5^2 \times 2 = 50$ is soluble.

Proof
Let $H_2 = \mathrm{Syl}_2(W_1)$ and $H_5 = \mathrm{Syl}_5(W_1)$ be the Sylow 2-subgroups and Sylow 5-subgroups of W_1 respectively.
Routine calculation shows that W_1 has:

$H_2 = \{(1), (1,6)(2,7)(3,8)(4,9)(5,10)\} \leq W_1$ with $|\mathrm{Syl}_2(W_1)| = 2$,

and $H_5 = \{(1), (6,7,8,9,10), (6,8,10,7,9), (6,9,7,10,8), (6,10,9,8,7), (1,2,3,4,5), (1,2,3,4,5)(6,7,8,9,10),$
$(1,2,3,4,5)(6,8,10,7,9), (1,2,3,4,5)(6,9,7,10,8), (1,2,3,4,5)(6,10,9,8,7), (1,3,5,2,4), (1,3,5,2,4)(6,7,8,9,10),$
$(1,3,5,2,4)(6,8,10,7,9), (1,3,5,2,4)(6,9,7,10,8), (1,3,5,2,4)(6,10,9,8,7), (1,4,2,5,3), (1,4,2,5,3)(6,7,8,9,10),$
$(1,4,2,5,3)(6,8,10,7,9), (1,4,2,5,3)(6,9,7,10,8), (1,4,2,5,3)(6,10,9,8,7), (1,5,4,3,2), (1,5,4,3,2)(6,7,8,9,10),$
$(1,5,4,3,2)(6,8,10,7,9), (1,5,4,3,2)(6,9,7,10,8), (1,5,4,3,2)(6,10,9,8,7)\} \leq W_1$ with $|\mathrm{Syl}_5(W_1)| = 25$

Going by theorem 2.4, the number of Sylow 2-subgroups of W_1 denoted N_2 is given by $\mathrm{n}_2 = 1 \ +2\mathrm{k} \equiv 1(\bmod 2)$ and $\mathrm{N}_2 \mid 25$ (where $\mathrm{k} = \{0, 1, 2, \ldots.\}$). Therefore $\mathrm{N}_2 = 1$ or 5 or 25 implying that H_2 is not unique and hence not normal in W_1.

Also the number of Sylow 5-subgroups of W_1 denoted n_5 is given by $n_5 = 1 + 5\mathrm{k} \equiv 1 \pmod 5$ and $\mathrm{N}_5 \mid 2$ (where $\mathrm{k} = \{0, 1, 2, \ldots\}$).

It follows from the constraints that $\mathrm{N}_5 = 1$.

Let $K = \mathrm{Syl}_5(W_1)$ be the Sylow 5-subgroup of W_1. Then $K \leq W_1$ with $|K| = 5^2$. K is unique and it's normal in W_1 by corollary 2.11. Since $|K| = 5^2, K$ is a p-Group and by theorem 2.8 is Solvable. Also $|W_1 : K| = 3$ implies that W_1/K is also a p-Group hence Solvable by theorem 2.8. By theorem 2.13, we have that W_1 is solvable as required.

3.3. Illustrating Example (2)

Let C_2 be a group of degree 5 and D_2 a group of degree 3 acting on the sets $\Omega_2 = \{1, 2, 3, 4, 5\}$ and $\Delta_2 = \{6, 7, 8\}$ respectively. Then the Wreath product $W_2 = C_{2w}D_2$ with degree $|C_2| \times |D_2| = 15$ and order given by $|W_2| = |C_2|^{|\Delta 2|} \times |D_2| = 375 = 5^3 \times 3$ is soluble.

Proof:

Let $H_3 = \mathrm{Syl}_3(W_2)$ and $H_5 = \mathrm{Syl}_5(W_2)$ be the Sylow 3-subgroups and Sylow 5-subgroups of W_2 respectively. Routine calculation shows that W_2 has:

$H_3 = \{(1), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15), (1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10)\} \leq W_2$ with $|\mathrm{Syl}_3(W_2)| = 3$,

and $H_5 \leq W_2$ with $|\mathrm{Syl}_5(W_2)| = 125$

Going by theorem 2.4, the number of Sylow 3 -subgroups of W_2 denoted N_3 is given by $\mathrm{N}_3 = 1{+}3\mathrm{k} \equiv 1(\bmod 3)$ and $\mathrm{N}_3 \mid 125$ (where $\mathrm{k} = \{0, 1, 2, \ldots\}$). Therefore $\mathrm{N}_2 = 1$ or 25 implying that H_2 is not unique and hence not normal in W_2.

Also the number of Sylow 5-subgroups of W_2 denoted N_5 is given by $\mathrm{N}_5 = 1 + 5\mathrm{k} \equiv 1(\mathrm{mod} 5)$ and $\mathrm{N}_5 \mid 3($ where $\mathrm{k} = \{0, 1, 2, \ldots\}$).

It follows from the constraints that $\mathrm{N}_5 = 1$

Let $K = \mathrm{Syl}_5(W_2)$ be the Sylow 5-subgroup of W_2. Then $K \leq W_2$ with $|K| = 5^3$. K is unique and it's normal in W_2 by corollary 2.11. Since $|K| = 5^3, K$ is a p-Group and by theorem 2.8 is Solvable. Also $\mid W_2 : K \mid = 3$ implies that W_2/K is also a p-Group hence Solvable by theorem 2.8. By theorem 2.13, we have that W_2 is solvable as required.

3.4. Illustrating Example (3)

Let C_3 be a group of degree 7 and D_3 a group of degree 3 acting on the sets $\Omega_3 = \{1, 2, 3, 4, 5, 6, 7\}$ and $\Delta_4 = \{8, 9, 10\}$ respectively. Let $P_3 = C_3^{\Delta_3} = \{f : \Delta_3 \to C_3\}$. Then $|P_3| = |C_3|^{||3||} = 7^3 = 343$. Then the Wreath product $W_3 = C_{3w}D_3$ with degree $|C_3|\, x\, |D_3| = 21$ and order given by $|W_3| = |C_3|^{|\Delta 3|} \times |D_3| = 1029 = 7^3 \times 3$ is soluble.

Proof. Let $H_3 = \mathrm{Syl}_3(W_3)$ and $H_7 = \mathrm{Syl}_7(W_3)$ be the Sylow 3-subgroups and Sylow 7-subgroups of W_3 respectively. Routine calculation shows that W_3 has:

$H_3 = \{(1), (1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21),$
$(1,15,8)(2,16,9)\ (3,17,10)(4,18,11)(5,19,12)(6,20,13)(7,21,14)\} \leq W_3$ with $|\mathrm{Syl}_3\,(W_3)| = 3$, and $H_7 \leq W_3$ with $|\mathrm{Syl}_7\,(W_3)| = 343$

Going by theorem 2.4, the number of Sylow 3-subgroups of W_3 denoted N_3 is given by $\mathrm{N}_3 = 1+3\mathrm{k} \equiv 1\,(\mathrm{mod}3$) and $\mathrm{N}_3 \mid 343$ (where $\mathrm{k} = \{0,1,2,\ldots\}$). Therefore $\mathrm{N}_2 = 1$ or 343 implying that H_2 is not unique and hence not normal in W_3.

Also the number of Sylow 7 -subgroups of W_3 denoted N_7 is given by $\mathrm{N}_7 = 1+7\mathrm{k} \equiv 1\ (\mathrm{mod}7)$ and $\mathrm{N}_7 \mid 3($ where $\mathrm{k} = \{0,1,2,\ldots\}$).

It follows from the constraints that $\mathrm{N}_7 = 1$

Let $K = \mathrm{Syl}_7\,(W_3)$ be the Sylow 5-subgroup of W_3. Then $K \leq W_3$ with $|K| = 7^3$. K is unique and it's normal in W_3 by corollary 2.11. Since $|K| = 7^3, K$ is a p-Group and by theorem 2.8 is Solvable. Also $|W_3 : K| = 3$ implies that W_3/K is also a p-Group hence Solvable by theorem 2.8. By theorem 2.13, we have that W_3 is solvable as required. □

3.5. GAP Result-Validation

```
gap>
    gap> C1 := Group((1,2,3,4,5));
    Group([ (1,2,3,4,5) ])
    gap> D1 := Group((6,7));
    Group([ (6,7) ])
    gap> W1 := WreathProduct(C1,D1);
    Group([ (1,2,3,4,5), (6,7,8,9,10), (1,6)(2,7)(3,8)(4,9)(5,10) ])
    gap> Order(W1);
    50
    gap> Elements(W1);;
    gap>
    gap> IsSolvable(W1);
    true
    gap> S2 := SylowSubgroup(W1,2);
    Group([ (1,6)(2,7)(3,8)(4,9)(5,10) ])
    gap> Elements(S2);
    [ (), (1,6)(2,7)(3,8)(4,9)(5,10) ]
    gap> Order(S2);
    2
    gap> IsNormal(W1,S2);
    false
    gap> S5 := SylowSubgroup(W1,5);
    Group([ (1,4,2,5,3)(6,9,7,10,8), (1,2,3,4,5)(6,10,9,8,7) ])
    gap> Elements(S5);;
    gap> Order(S5);
```

```
25
gap> IsNormal(W1,S5);
true
gap>
gap> C2 := Group((1,2,3,4,5));
Group([ (1,2,3,4,5) ])
gap> D2 := Group((6,7,8));
Group([ (6,7,8) ])
gap> W2 := WreathProduct(C2,D2);
Group([ (1,2,3,4,5), (6,7,8,9,10), (11,12,13,14,15), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15)])
gap> Order(W2);
375
gap> Elements(W2);;
gap>
gap> IsSolvable(W2);
true
gap> S3 := SylowSubgroup(W2,3);
Group([ (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15) ])
gap> Elements(S3);
[ (), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15), (1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10) ]
gap> Order(S3);
3
gap> IsNormal(W2,S3);
false
gap> S5 := SylowSubgroup(W2,5);
Group([ (1,3,5,2,4)(6,8,10,7,9)(11,13,15,12,14), (1,2,3,4,5)(11,15,14,13,12), (1,5,4,3,2)(6,7,8,9,10) ])
gap> Elements(S5);;
gap> Order(S5);
125
gap> IsNormal(W2,S5);
true
gap>
gap> C3 := Group((1,2,3,4,5,6,7));
Group([ (1,2,3,4,5,6,7) ])
gap> D3 := Group((7,8,9));
Group([ (7,8,9) ])
gap> W3 := WreathProduct(C3,D3);
Group([(1,2,3,4,5,6,7),(8,9,10,11,12,13,14),(15,16,17,18,19,20,21), (1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)
(6,13,20)(7,14,21) ])
gap> Order(W3);
1029
```

```
gap> Elements(W3);;
gap>
gap> IsSolvable(W3);
true
gap> S3 := SylowSubgroup(W3,3);
Group([ (1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21) ])
gap> Elements(S3);
[ (), (1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21), (1,15,8)(2,16,9)(3,17,10)(4,18,11)
(5,19,12)(6,20,13)(7,21,14) ]
gap> Order(S3);
3
gap> IsNormal(W3,S3);
false
gap> S7 := SylowSubgroup(W3,7);
Group([ (1,6,4,2,7,5,3)(8,13,11,9,14,12,10)(15,20,18,16,21,19,17), (1,2,3,4,5,6,7) (15,21,20,19,18,17,16),
(1,7,6,5,4,3,2)(8,9,10,11,12,13,14) ])
gap> Elements(S7);:
gap> Order(S7);
343
gap> IsNormal(W3,S7);
true
gap>
```

4. Conclusion and Recommendation

4.1. Conclusion

We have shown that the Wreath products group of degree $3p$ is soluble as required.

4.2. Recommendation

This study can be extended by considering for further research, one or a combination of two or more of other theoretic properties such as simplicity, nilpotency, regularity, etc of same algebraic structures.

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