Investigating the solubility of wreath products group of degree 3p using numerical approach

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Received: 1 Dec 2021 • Accepted: 29 Mar 2022 • Published Online: 29 Apr 2022

Abstract: Let p be a prime number (p > 3) and G a finite permutation group of degree 3p, generated via Wreath products of pairs of permutation groups. We, in this paper discuss the solubility of G using numerical approach. We applied the computational group theory (GAP) to enhance and validate our work.

Key words: Permutation Group, Solubility, Wreath Products, p-Groups and Sylow p-subgroup.

1. Introduction

The Wreath product of two permutation groups C and D denoted by W = C wr D is the semi-direct product of P (a derived group of prime power order) by D, so that, W = ((f, d)|f ∈ P, d ∈ D), with composition in W := (f1, d1) (f2, d2) = ((f1 f2 d1 d1−1) (d1, d2)) ∀ f1, f2 ∈ P and d1, d2 ∈ D is a special form of permutation group. When the nature of the Wreath products groups is well understood it facilitates comprehension of certain types of subgroups of the symmetric groups.

According to Cameron(2013)[2], a group G is soluble or solvable if it has a series of subgroups,

\[ G = H_n \supset H_{n-1} \supset \cdots \supset H_1 \supset H_0 = \{e\} \]

With each subgroup \( H_i \) normal in \( H_{i+1} \) and the factor groups \( H_{i+1}/H_i \) abelian. Solvable groups are significant as they allow us to differentiate between categories of groups.

In this work, we obtained more detailed description of the unique structure of Wreath product (permutation) groups of degree 3p that are not p-groups and investigate their solubility using numerical approach. This work is significant as it will form part of a growing database that will eventually be used in the needed review of the classification of finite simple group (CFSG).

There are some recent results on the solubility of permutations groups including the following:

Thanos (2006)[11] proved that If \(|G| = p^k\) where \( p \) is a prime number then \( G \) is solvable. In other words every p-group where \( p \) is a prime number is solvable.

Bello et al. (2017)[1] used the concept of p-groups to construct locally solvable groups using two permutation groups by wreath product.

Gandi et al. (2019)[4] investigated solvable and Nilpotent concepts on dihedral groups of an even degree regular polygon.
The results from the above papers and other findings on group concepts from the works of Kimura and Nakagawa, (1973)[8], Ito and Wada, (1972)[6] and Cai and Zhang, (2015)[3] will be used as valuable references towards achieving our desired objectives.

This work is organized in five sections. Section 2 gives some preliminaries required for the work. In Section 3 we state the main result of this paper with some illustrating examples. We also use the groups, algorithms and programming (GAP) to validate the solubility of permutation groups of degree $3p(p = 5, 7, 11, \ldots)$. Section 4 contains conclusion and recommendation while Section 4.2 is the list of references.

2. Materials and Methods

2.1. p-Group (Sylow, 1872)

If a group $G$ has number of elements, $|G| = p^n$ where $p$ is prime, it is called a $p$-group.

2.2. p-Subgroup (Sylow, 1872)

A subgroup $H$ of a group $G(H \leq G)$ is called a $p$-subgroup $G$ if $H$ itself is a $p$-group, this is, $|H| = p^r$, for some $r \geq 0$ for all $H \in G$.

2.3. Sylow $p$-Subgroup (Sylow, 1872)

Let $G$ be a group. If $G$ is finite and $|G| = p^rm, r \geq 1$ where $p$ and $m$ are co-prime and $H \leq G$ such that $|H| = p^r$, we refer to $H$ as a Sylow $p$-subgroup of $G$.

2.4. Sylow Theorems (Sylow, 1872)

Let $G$ be a finite group of order $n$.

1. If $p$ is a prime such that $p^k$ is a divisor of $|G|$ for some $k \geq 0$, then $G$ contains a subgroup of order $p^k$.

2. All Sylow $p$-subgroups of $G$ are conjugate, and any $p$-subgroup of $G$ is contained in a Sylow $p$-subgroup.

3. Let $n = mp^k$, with $(m, p) = 1$, and let $n_p$ be the number of Sylow $p$-subgroups of $G$. Then $n_p | m$ and $n_p \equiv 1(\text{mod } p)$.

2.5. Wreath product (Joseph and Audu, 1991)

The wreath product of two permutation groups $C$ by $D$ denoted by $W = C wr D$ is the semidirect product of $P$ by $D$, so that,

$$W = \{(f, d)|f \in P, d \in D\}$$

with multiplication in $W$ defined as

$$(f_1, d_1)(f_2, d_2) = ((f_1, f_2d_1^{-1})(d_1, d_2)) \quad \forall f_1, f_2 \in P \land d_1, d_2 \in D$$

Henceforth, we write $fd$ instead of $(f, d)$ for elements of $W$. 
2.6. Theorem (Joseph and Audu, 1991)
Let $D$ act on $P$ as $f^d(\delta) = f(\delta d^{-1})$ where $f \in P, d \in D$ and $\delta \in \Delta$. Let $W$ be group of all juxtaposed symbols $f_d$, with $f \in P, d \in D$ and multiplication given by $(f_1, d_1)(f_2, d_2) = (f_1 f_2 d_1^{-1})(d_1, d_2)$ Then $W$ is a group referred to as the semi-direct product of $P$ by $D$ with the action as defined.

2.7. Theorem (Cameron, 2013)
If $|G| = pq$ where $p$ and $q$ are distinct prime numbers ($p < q$) then, $G$ is solvable.

2.8. Theorem (Thanos, 2006)
If $|G| = p^k$ where $p$ is a prime number then $G$ is solvable. In other words every $p$-group where $p$ is a prime is solvable.

Proof. By induction on $k$.

1st step. For $k = 1$ our group is a cyclic group of prime order thus it is solvable by definition.

2nd step. Let the statement hold for all $n \leq k$.

3rd step. We will prove that it holds for $k = n + 1$. By corollary 3 since $G$ is a $p$-group, the center of $G$ denoted $Z(G) \neq \{e\}$. Also $Z(G)$ is a normal subgroup of $G$ and $Z(G)$ is abelian. Thus $Z(G)$ is solvable. Now $G/Z(G)$ is again a $p$-group or trivial. If it is trivial then $G = Z(G)$ thus $G$ is abelian hence it is solvable. If it is not trivial then $|G/Z(G)| \leq p^n$. So by the inductive step it is solvable. Hence $G$ is also solvable.

2.9. Corollary (Thonas, 2006)
If $G$ has only one $p$-Sylow subgroup $H$ then $H$ is normal.

2.10. Corollary (Thonas, 2006)
If $H \trianglelefteq G$ and $|\frac{G}{H}| = p$ or $p^2$ then $\frac{G}{H}$ is abelian

2.11. Corollary (Thonas, 2006)
Let $G$ be a finite group and $H$ a Sylow $p$-subgroup of $G$. Then $H$ is the only Sylow $p$-subgroup of $G$ if and only if $H$ is normal in $G$.

Proof
By Sylow theorem, the Sylow $p$-subgroups of $G$ are the elements of the sets $\{g^{-1}Hg \mid g \in G\}$ and this reduces to a singleton set if and only if $g^{-1}Hg = H$ for all $g \in G$; that is precisely when $H$ is normal in $G$.

2.12. Proposition (Thonas, 2006)
Suppose $G$ is a solvable group and $H$ is a subgroup of $G$ that is, $H \leq G$. Then

1. $H$ is solvable.
2. If $H \triangleleft G$, then $G/H$ is solvable.

Proof
Start from a series with abelian slices. $G: G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = \{1\}$ Then $H = H \cap G_0$ $H \cap G_1$, $\ldots$, $H \cap G_n = \{1\}$. When $H$ is normal, we use the canonical projection $\pi: G \to G/H$ to get $G/H = \pi(G_0) \triangleright \ldots \triangleright \pi(G_n) = \{1\}$; the quotients are abelian as well, so $G/H$ is still solvable.
2.13. Theorem (Cameron, 2013)
If \( G \) is a group and \( H \) is a normal subgroup of \( G \) such that \( H \) is solvable and \( G/H \) is solvable then \( G \) is solvable.

3. Wreath product group of degree \( 3p \)

3.1. Main Theorem
Let \( G \) be the Wreath product of two permutation groups \( C \) and \( D \) of degree \( 3p(p > 3) \) and \( H \) the Sylow \( p \)-subgroup of \( G \). Then (i) \( H \) is normal in \( G \) and is soluble (ii) \( G/H \) is soluble and (iii) \( G \) is soluble.

Proof
Now, the order of \( W \) that is, \(|W| = 3p^3 \) or \( 3^p \)

Case 1: \(|W| = 3p^3 \)
Let \( N_p(W) \) be the number of Sylow \( p \)-subgroups of the group \( W \).
By Sylow theorem 2.8, we have

\[ N_p \equiv 1 \text{ modulo } p \text{ and } N_p \text{ divides } 3. \]

It follows from these constraints that \( N_p = 1 \).
Let \( H = p \)-Sylow subgroup of \( W \). Then \( H \) is normal in \( W \) by corollary 2.11 proving (i).
Since \(|H| = p^3 \), we have that \( H \) is a \( p \)-Group and by theorem 2.8 is Solvable proving (ii).
Also \(|W : H| = 3 \) implies that \( W/H \) is a \( p \)-Group hence Solvable by theorem 2.8.
Hence \( G \) is a soluble group by theorem 2.13 proving (iii).

Case 2: \(|W| = 3^p \)
By Sylow theorem 3.2.3, the number of Sylow \( p \)-subgroup of \( W \), \( N_p \) of order \( p \) is congruent to 1 modulo \( p \) and it divides \( 3^p \).
As \( N_p \) divides \(|G| \) and \( 3^p \equiv 1 \text{ mod } p \), it follows that \( N_p = 1 \).
Let \( K \) be the unique Sylow \( p \)-subgroup of \( W \). Then the subgroup \( K \) is a normal subgroup of \( W \) by Corollary 2.11. Since \(|K| = p \), we have that \( K \) is a \( p \)-Group and by theorem 2.8 is Solvable. Also \(|W : K| = 3^p \) implies that \( G/K \) is a \( p \)-Group hence Solvable by theorem 2.8.
By theorem 2.13, \( G \) is solvable as required.

3.2. Illustrating Example (1)
Let

\[ C_1 = \{(1),(12345),(13524),(14253),(15432)\} \text{ and } \]
\[ D_1 = \{(1),(6,7)\} \]
acting on the sets \( \Omega_1 = \{1,2,3,4,5\} \) and \( \Delta_1 = \{6,7\} \) respectively.
Let \( P_1 = C_1^\Delta_1 = \{ f : \Delta_1 \to C_1 \} \). Then \(|P_1| = |C_1|^{|\Delta_1|} = 5^2 = 25 \)
The mappings in \( P_1 \) are as list below.
\[ f_1 : 6 \to (1), 7 \to (1) \]
\[ f_2 : 6 \to (12345), 7 \to (12345) \]
\( f_3 : 6 \to (13524), 7 \to (13524), \)
\( f_4 : 6 \to (14253), 7 \to (14253) \)
\( f_5 : 6 \to (15432), 7 \to (15432) \)
\( f_6 : 6 \to (1), 7 \to (12345) \)
\( f_7 : 6 \to (1), 7 \to (13524) \)
\( f_8 : 6 \to (1), 7 \to (14253) \)
\( f_9 : 6 \to (1), 7 \to (15432) \)
\( f_{10} : 6 \to (12345), 7 \to (1) \)
\( f_{11} : 6 \to (12345), 7 \to (13524) \)
\( f_{12} : 6 \to (12345), 7 \to (14253) \)
\( f_{13} : 6 \to (12345), 7 \to (15432) \)
\( f_{14} : 6 \to (13524), 7 \to (1) \)
\( f_{15} : 6 \to (13524), 7 \to (12345) \)
\( f_{16} : 6 \to (13524), 7 \to (14253) \)
\( f_{17} : 6 \to (13524), 7 \to (15432) \)
\( f_{18} : 6 \to (14253), 7 \to (1) \)
\( f_{19} : 6 \to (14253), 7 \to (12345) \)
\( f_{20} : 6 \to (14253), 7 \to (13524) \)
\( f_{21} : 6 \to (14253), 7 \to (15432) \)
\( f_{22} : 6 \to (15432), 7 \to (1) \)
\( f_{23} : 6 \to (15432), 7 \to (12345) \)
\( f_{24} : 6 \to (15432), 7 \to (15432) \)
\( f_{25} : 6 \to (15432), 7 \to (14253) \)

We can easily verify that \( P \) is a group with respect to the operations \((f_1, f_2) (\delta) = f_1 (\delta_1) f_2 (\delta_2)\), where \( \delta_1 \in \Delta_1 \)

We recall the definition of the action of \( D_1 \) on \( P \) as \( f^d (\delta_1) = f (\delta_1 d^{-1}) \) where \( f \in P, d \in D_1 \) and \( \delta_1 \in \Delta_1 \), then \( D_1 \) acts on \( P \) as a groups.

We also recall the definition \( W = C_1 \wr D_1 \), the semi-direct product of \( P \) by \( D_1 \) in that order; i.e. \( W = \{(f, d) \mid f \in P, \delta_1 \in \Delta_1 \} \)

Now, \( W \) is a group with respect to the operation;
\[
(f_1, d_1) (f_2, d_2) = (f_1 f_2 d_2^{-1})(d_1, d_2), \text{ and accordingly, } d_1 = (1), d_2 = (6, 7).
\]

Then the elements of \( W_1 \) are
\[
(1, 1), (f_2, d_1), (f_3, d_1), (f_4, d_1), (f_5, d_1), (f_6, d_1), (f_7, d_1), (f_8 d_1), (f_9, d_1), (f_{10}, d_1), (f_{11}, d_1), (f_{12}, d_1),
\]
\[
(f_{13}, d_1), (f_{14}, d_1), (f_{15}, d_1), (f_{16}, d_1), (f_{17}, d_1), (f_{18}, d_1), (f_{19}, d_1), (f_{20}, d_1), (f_{21}, d_1), (f_{22}, d_1), (f_{23}, d_1), (f_{24}, d_1),
\]
\[
(f_{25}, d_1), (f_1, d_2), (f_2, d_2), (f_3, d_2), (f_4, d_2), (f_5, d_2), (f_6, d_2), (f_7, d_2), (f_8, d_2), (f_9, d_2), (f_{10}, d_2), (f_{11}, d_2),
\]
\[
(f_{12}, d_2), (f_{13}, d_2), (f_{14}, d_2), (f_{15}, d_2), (f_{16}, d_2), (f_{17}, d_2), (f_{18}, d_2), (f_{19}, d_2), (f_{20}, d_2), (f_{21}, d_2), (f_{22}, d_2),
\]
\[
(f_{23}, d_2), (f_{24}, d_2), (f_{25}, d_2).
\]

Now, define action of \( W_1 \) on \( \Omega_1 \times \Delta_1 \) as
\[
(\beta, \delta_1) f d = (\beta f (\delta), d \delta) \text{ where } \beta \in \Omega_1 \text{ and } \delta_1 \in \Delta_1.
\]
Further, $\Omega_1 \times \Delta_1 = \{(1, 6), (1, 7), (2, 6), (2, 7), (3, 6), (3, 7), (4, 6), (4, 7), (5, 6), (5, 7)\}$

We obtain the following permutation by action of $W_1$ on $\Omega_1 \times \Delta_1$

$$(1, 6)f_1d_1 = (1f_1(6), d_1) = (1(1), 6(1)) = (1, 6)$$

$$(1, 7)f_1d_1 = (1f_1(7), d_1) = (1(1), 7(1)) = (1, 7)$$

$$(2, 6)f_1d_1 = (2f_1(6), d_1) = (2(1), 6(1)) = (2, 6)$$

$$(2, 7)f_1d_1 = (2f_1(7), d_1) = (2(1), 7(1)) = (2, 7)$$

$$(3, 6)f_1d_1 = (3f_1(6), d_1) = (3(1), 6(1)) = (3, 6)$$

$$(3, 7)f_1d_1 = (3f_1(7), d_1) = (3(1), 7(1)) = (3, 7)$$

$$(4, 6)f_1d_1 = (4f_1(6), d_1) = (4(1), 6(1)) = (4, 6)$$

$$(4, 7)f_1d_1 = (4f_1(7), d_1) = (4(1), 7(1)) = (4, 7)$$

$$(5, 6)f_1d_1 = (5f_1(6), d_1) = (5(1), 6(1)) = (5, 6)$$

$$(5, 7)f_1d_1 = (5f_1(7), d_1) = (5(1), 7(1)) = (5, 7)$$

And in summary,

$$(\Omega_1 \times \Delta_1) f_1d_1 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1) f_2d_1 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7)(1, 6)(1, 7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1) f_3d_1 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7)(1, 6)(1, 7)(2, 6)(2, 7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1) f_4d_1 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (4, 6)(4, 7)(5, 6)(5, 7)(1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1) f_5d_1 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (5, 6)(5, 7)(1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1) f_7d_1 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (1, 6)(3, 7)(2, 6)(4, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1) f_4d_2 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (4, 7)(4, 6)(5, 7)(1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1) f_5d_2 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (5, 7)(5, 6)(1, 7)(1, 6)(2, 7)(2, 6)(3, 7)(3, 6)(4, 7)(4, 6) \end{pmatrix}$$

$$(\Omega_1 \times \Delta_1) f_6d_2 = \begin{pmatrix} (1, 1)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \\ (1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6)(5, 7) \end{pmatrix}$$
\[
(\Omega_1 \times \Delta_1) f_{7d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(1,7)(3,6)(2,7)(4,6)(3,7)(5,6)(4,7)(1,6)(5,7)(2,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{8d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(1,7)(4,6)(2,7)(5,6)(3,7)(1,6)(4,7)(2,6)(5,7)(3,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{9d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(1,7)(5,6)(2,7)(1,6)(3,7)(2,6)(4,7)(3,6)(5,7)(4,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{10d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(2,7)(1,6)(3,7)(2,6)(4,7)(3,6)(5,7)(4,6)(1,7)(5,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{11d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{12d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(2,7)(4,6)(3,7)(5,6)(4,7)(1,6)(5,7)(2,6)(1,7)(3,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{13d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(2,7)(5,6)(3,7)(1,6)(4,7)(2,6)(5,7)(3,6)(1,7)(4,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{14d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(3,7)(1,6)(4,7)(2,6)(5,7)(3,6)(1,7)(4,6)(2,7)(5,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{15d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(3,7)(2,6)(4,7)(3,6)(5,7)(4,6)(1,7)(5,6)(2,7)(1,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{16d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(3,7)(4,6)(4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{17d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(3,7)(5,6)(4,7)(1,6)(5,7)(2,6)(1,7)(3,6)(2,7)(4,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{18d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(4,7)(1,6)(5,7)(2,6)(1,7)(3,6)(2,7)(4,6)(3,7)(5,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{19d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(4,7)(2,6)(5,7)(3,6)(1,7)(4,6)(2,7)(5,6)(3,7)(1,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{20d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(4,7)(3,6)(5,7)(4,6)(1,7)(5,6)(2,7)(1,6)(3,7)(2,6)
\end{pmatrix}
\]

\[
(\Omega_1 \times \Delta_1) f_{21d_2} = \begin{pmatrix}
(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)(4,7)(5,6)(5,7) \\
(4,7)(5,6)(5,7)(1,6)(1,7)(2,6)(2,7)(3,6)(3,7)(4,6)
\end{pmatrix}
\]
The permutations in cyclic form are as follows.

Renaming the symbols as

\[(\Omega_1 \times \Delta_1) f_{22} d_2 = \begin{pmatrix} (1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6) \\
(5, 7)(1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6) \end{pmatrix} \]

\[(\Omega_1 \times \Delta_1) f_{23} d_2 = \begin{pmatrix} (1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6) \\
(5, 7)(2, 6)(1, 7)(3, 6)(2, 7)(4, 6)(3, 7)(5, 6)(4, 7)(1, 6) \end{pmatrix} \]

\[(\Omega_1 \times \Delta_1) f_{24} d_2 = \begin{pmatrix} (1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6) \\
(5, 7)(3, 6)(1, 7)(4, 6)(2, 7)(5, 6)(3, 7)(1, 6)(4, 7)(2, 6) \end{pmatrix} \]

\[(\Omega_1 \times \Delta_1) f_{25} d_2 = \begin{pmatrix} (1, 6)(1, 7)(2, 6)(2, 7)(3, 6)(3, 7)(4, 6)(4, 7)(5, 6) \\
(5, 7)(4, 6)(1, 7)(5, 6)(2, 7)(1, 6)(3, 7)(2, 6)(4, 7)(3, 6) \end{pmatrix} \]

Renaming the symbols as

\[(1, 6) \rightarrow 1, (1, 7) \rightarrow 2, (2, 6) \rightarrow 3, (2, 7) \rightarrow 4, (3, 6) \rightarrow 5, (3, 7) \rightarrow 6, (4, 6) \rightarrow 7, (4, 7) \rightarrow 8, (5, 6) \rightarrow 9, (5, 7) \rightarrow 10, \]

The permutations in cyclic form are as follows.

\[W_1 = \{(1, 6, 7, 8, 9, 10), (6, 8, 10, 7, 9), (6, 9, 7, 10, 8), (6, 10, 9, 8, 7), (1, 2, 3, 4, 5), (1, 2, 3, 4, 5)(6, 7, 8, 9, 10), (1, 2, 3, 4, 5)(6, 8, 10, 7, 9), (1, 2, 3, 4, 5)(6, 9, 7, 10, 8), (1, 2, 3, 4, 5)(6, 10, 9, 8, 7), (1, 3, 5, 2, 4), (1, 3, 5, 2, 4)(6, 8, 10, 7, 9), (1, 3, 5, 2, 4)(6, 9, 7, 10, 8), (1, 3, 5, 2, 4)(6, 10, 9, 8, 7), (1, 4, 2, 5, 3), (1, 4, 2, 5, 3)(6, 8, 10, 7, 9), (1, 4, 2, 5, 3)(6, 9, 7, 10, 8), (1, 4, 2, 5, 3)(6, 10, 9, 8, 7), (1, 5, 4, 3, 2)(1, 5, 4, 3, 2)(6, 7, 8, 9, 10), (1, 5, 4, 3, 2)(6, 8, 10, 7, 9), (1, 5, 4, 3, 2)(6, 9, 7, 10, 8), (1, 5, 4, 3, 2)(6, 10, 9, 8, 7), (1, 6, 2, 7, 3, 8, 4, 9, 5, 10), (1, 6, 3, 8, 5, 10, 2, 7, 4, 9), (1, 6, 4, 9, 2, 7, 5, 10, 3, 8), (1, 6, 5, 10, 4, 9, 3, 8, 2, 7), (1, 7, 2, 8, 3, 9, 4, 10, 5, 6), (1, 7, 3, 9, 5, 6, 2, 8, 4, 10), (1, 7, 4, 10, 2, 8, 5, 6, 3, 9), (1, 7, 5, 6, 4, 10, 3, 9, 2, 8), (1, 7)(2, 8)(3, 9)(4, 10)(5, 6), (1, 8, 3, 10, 5, 7, 2, 9, 4, 6), (1, 8, 4, 6, 2, 9, 5, 7, 3, 10), (1, 8, 5, 7, 4, 6, 3, 10, 2, 9), (1, 8)(2, 9)(3, 10)(4, 6)(5, 7), (1, 8, 9, 2, 3, 10, 4, 6, 5, 7), (1, 9, 4, 7, 2, 10, 5, 8, 3, 6), (1, 9, 5, 8, 4, 7, 3, 6, 2, 10), (1, 9)(2, 10)(3, 6)(4, 7)(5, 8), (1, 9, 2, 10, 3, 6, 4, 7, 5, 8), (1, 9, 3, 6, 5, 8, 2, 10, 4, 7), (1, 10, 5, 9, 4, 8, 3, 7, 2, 6), (1, 10)(2, 6)(3, 7)(4, 8)(5, 9), (1, 10, 2, 6, 3, 7, 4, 8, 5, 9), (1, 10, 3, 7, 5, 9, 2, 6, 4, 8), (1, 10, 4, 8, 2, 6, 5, 9, 3, 7)\}.

Then the Wreath product \(W_1 = C_{1 wr} D_1\) with degree \(|C_1| \times |D_1| = 10\) and order given by

\[|W_1| = |C_1|^{|\Delta_1|} \times |D_1| = 5^2 \times 2 = 50\] is soluble.

**Proof**

Let \(H_2 = \text{Syl}_2(W_1)\) and \(H_5 = \text{Syl}_5(W_1)\) be the Sylow 2-subgroups and Sylow 5-subgroups of \(W_1\) respectively.

Routine calculation shows that \(W_1\) has:

\[H_2 = \{(1, 6)(2, 7)(3, 8)(4, 9)(5, 10)\} \leq W_1\] with \(|\text{Syl}_2(W_1)| = 2\),

and \(H_5 = \{(1, 6, 7, 8, 9, 10), (6, 8, 10, 7, 9), (6, 9, 7, 10, 8), (6, 10, 9, 8, 7), (1, 2, 3, 4, 5), (1, 2, 3, 4, 5)(6, 7, 8, 9, 10), (1, 2, 3, 4, 5)(6, 8, 10, 7, 9), (1, 2, 3, 4, 5)(6, 9, 7, 10, 8), (1, 2, 3, 4, 5)(6, 10, 9, 8, 7), (1, 3, 5, 2, 4), (1, 3, 5, 2, 4)(6, 7, 8, 9, 10), (1, 3, 5, 2, 4)(6, 8, 10, 7, 9), (1, 3, 5, 2, 4)(6, 9, 7, 10, 8), (1, 3, 5, 2, 4)(6, 10, 9, 8, 7), (1, 4, 2, 5, 3), (1, 4, 2, 5, 3)(6, 7, 8, 9, 10), (1, 4, 2, 5, 3)(6, 8, 10, 7, 9), (1, 4, 2, 5, 3)(6, 9, 7, 10, 8), (1, 4, 2, 5, 3)(6, 10, 9, 8, 7), (1, 5, 4, 3, 2), (1, 5, 4, 3, 2)(6, 7, 8, 9, 10), (1, 5, 4, 3, 2)(6, 8, 10, 7, 9), (1, 5, 4, 3, 2)(6, 9, 7, 10, 8), (1, 5, 4, 3, 2)(6, 10, 9, 8, 7)\} \leq W_1\] with \(|\text{Syl}_5(W_1)| = 25\).
Going by theorem 2.4, the number of Sylow 2-subgroups of $W_1$ denoted $N_2$ is given by $n_2 = 1 + 2k \equiv 1 \pmod{2}$ and $N_2 \mid 25$ (where $k = \{0,1,2,\ldots\}$). Therefore $N_2 = 1$ or 5 or 25 implying that $H_2$ is not unique and hence not normal in $W_1$.

Also the number of Sylow 5-subgroups of $W_1$ denoted $n_5$ is given by $n_5 = 1 + 5k \equiv 1 \pmod{5}$ and $N_5 \mid 2$ (where $k = \{0,1,2,\ldots\}$).

It follows from the constraints that $N_5 = 1$.

Let $K = \text{Syl}_5(W_1)$ be the Sylow 5-subgroup of $W_1$. Then $K \leq W_1$ with $|K| = 5^2$. $K$ is unique and it’s normal in $W_1$ by corollary 2.11. Since $|K| = 5^2$, $K$ is a p-Group and by theorem 2.8 is Solvable. Also $|W_1 : K| = 3$ implies that $W_1/K$ is also a p-Group hence Solvable by theorem 2.8. By theorem 2.13, we have that $W_1$ is solvable as required.

### 3.3. Illustrating Example (2)

Let $C_2$ be a group of degree 5 and $D_2$ a group of degree 3 acting on the sets $\Omega_2 = \{1,2,3,4,5\}$ and $\Delta_2 = \{6,7,8\}$ respectively. Then the Wreath product $W_2 = C_{2^w}D_2$ with degree $|C_2| \times |D_2| = 15$ and order given by $|W_2| = |C_2|^{\Delta_2} \times |D_2| = 375 = 5^3 \times 3$ is soluble.

**Proof:**

Let $H_3 = \text{Syl}_3(W_2)$ and $H_5 = \text{Syl}_5(W_2)$ be the Sylow 3-subgroups and Sylow 5-subgroups of $W_2$ respectively. Routine calculation shows that $W_2$ has:

- $H_3 = \{(1),(1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15),(1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10)\} \leq W_2$ with $|\text{Syl}_3(W_2)| = 3$,
- $H_5 \leq W_2$ with $|\text{Syl}_5(W_2)| = 125$

Going by theorem 2.4, the number of Sylow 3-subgroups of $W_2$ denoted $N_3$ is given by $N_3 = 1 + 3k \equiv 1 \pmod{3}$ and $N_3 \mid 125$ (where $k = \{0,1,2,\ldots\}$). Therefore $N_2 = 1$ or 25 implying that $H_2$ is not unique and hence not normal in $W_2$.

Also the number of Sylow 5-subgroups of $W_2$ denoted $N_5$ is given by $N_5 = 1 + 5k \equiv 1 \pmod{5}$ and $N_5 \mid 3$ (where $k = \{0,1,2,\ldots\}$).

It follows from the constraints that $N_5 = 1$

Let $K = \text{Syl}_5(W_2)$ be the Sylow 5-subgroup of $W_2$. Then $K \leq W_2$ with $|K| = 5^3$. $K$ is unique and it’s normal in $W_2$ by corollary 2.11. Since $|K| = 5^3$, $K$ is a p-Group and by theorem 2.8 is Solvable. Also $|W_2 : K| = 3$ implies that $W_2/K$ is also a p-Group hence Solvable by theorem 2.8. By theorem 2.13, we have that $W_2$ is solvable as required.

### 3.4. Illustrating Example (3)

Let $C_3$ be a group of degree 7 and $D_3$ a group of degree 3 acting on the sets $\Omega_3 = \{1,2,3,4,5,6,7\}$ and $\Delta_3 = \{8,9,10\}$ respectively. Let $P_3 = C_3^{\Delta_3} = \{f : \Delta_3 \rightarrow C_3\}$. Then $|P_3| = |C_3|^{\mid \Delta_3 \mid} = 7^3 = 343$. Then the Wreath product $W_3 = C_{3^w}D_3$ with degree $|C_3| \times |D_3| = 21$ and order given by $|W_3| = |C_3|^{\mid \Delta_3 \mid} \times |D_3| = 1029 = 7^3 \times 3$ is soluble.

**Proof.** Let $H_3 = \text{Syl}_3(W_3)$ and $H_7 = \text{Syl}_7(W_3)$ be the Sylow 3-subgroups and Sylow 7-subgroups of $W_3$ respectively. Routine calculation shows that $W_3$ has:
\[ H_3 = \{(1), (1, 8, 15)(2, 9, 16)(3, 10, 17)(4, 11, 18)(5, 12, 19)(6, 13, 20)(7, 14, 21), \\
(1, 15, 8)(2, 16, 9)(3, 17, 10)(4, 18, 11)(5, 19, 12)(6, 20, 13)(7, 21, 14)\} \leq W_3 \text{ with } |Syl_3(W_3)| = 3, \text{ and } H_7 \leq W_3 \text{ with } |Syl_7(W_3)| = 343 \]

Going by theorem 2.4, the number of Sylow 3-subgroups of \( W_3 \) denoted \( N_3 \) is given by \( N_3 = 1 + 3k \equiv 1 \pmod{3} \) and \( N_3 \mid 343 \) (where \( k = \{0, 1, 2, \ldots\} \) ). Therefore \( N_2 = 1 \) or 343 implying that \( H_2 \) is not unique and hence not normal in \( W_3 \).

Also the number of Sylow 7 -subgroups of \( W_3 \) denoted \( N_7 \) is given by \( N_7 = 1 + 7k \equiv 1 \pmod{7} \) and \( N_7 \mid 3 \) (where \( k = \{0, 1, 2, \ldots\} \) ).

It follows from the constraints that \( N_7 = 1 \)

Let \( K = Syl_7(W_3) \) be the Sylow 5-subgroup of \( W_3 \). Then \( K \leq W_3 \) with \( |K| = 7^3 \). \( K \) is unique and it’s normal in \( W_3 \) by corollary 2.11. Since \( |K| = 7^3 \), \( K \) is a \( p \)-Group and by theorem 2.8 is Solvable. Also \( |W_3 : K| = 3 \) implies that \( W_3/K \) is also a \( p \)-Group hence Solvable by theorem 2.8. By theorem 2.13, we have that \( W_3 \) is solvable as required.

### 3.5. GAP Result-Validation

```gap
gap> C1 := Group((1,2,3,4,5));
group((1,2,3,4,5))
gap> D1 := Group((6,7));
group((6,7))
gap> W1 := WreathProduct(C1,D1);
grup((1,2,3,4,5),(6,7,8,9,10),(1,6)(2,7)(3,8)(4,9)(5,10))
gap> Order(W1);
50

gap> Elements(W1);;

gap> IsSolvable(W1);
true

gap> S2 := SylowSubgroup(W1,2);

gap> IsNormal(W1,S2);
ture

gap> S5 := SylowSubgroup(W1,5);

gap> Order(S5);
2

false

gap> Order(S5);
```
gap> IsNormal(W1,S5);
true

gap> C2 := Group([(1,2,3,4,5)]);
Group([(1,2,3,4,5)])

gap> D2 := Group([(6,7,8)]);
Group([(6,7,8)])

gap> W2 := WreathProduct(C2,D2);
Group([(1,2,3,4,5), (6,7,8,9,10), (11,12,13,14,15), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15)])

gap> Order(W2);
375

gap> Elements(W2);;

gap> IsSolvable(W2);
true

gap> S3 := SylowSubgroup(W2,3);
Group([(1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15)])

gap> Elements(S3);
[ (), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15), (1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10) ]

gap> Order(S3);
3

gap> IsNormal(W2,S3);
false

gap> S5 := SylowSubgroup(W2,5);
Group([(1,3,5,2,4)(6,8,10,7,9)(11,13,15,12,14), (1,2,3,4,5)(11,15,14,13,12), (1,5,4,3,2)(6,7,8,9,10)])

gap> Elements(S5);;

gap> Order(S5);
125

gap> IsNormal(W2,S5);
true

gap> C3 := Group([(1,2,3,4,5,6,7)]);
Group([(1,2,3,4,5,6,7)])

gap> D3 := Group([(7,8,9)]);
Group([(7,8,9)])

gap> W3 := WreathProduct(C3,D3);
Group([(1,2,3,4,5,6,7),(8,9,10,11,12,13,14),(15,16,17,18,19,20,21), (1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21)])

gap> Order(W3);
1029
gap > Elements(W3);

gap >
gap > IsSolvable(W3);
    true

gap > S3 := SylowSubgroup(W3,3);
Group([ [ 1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21) ])

gap > Elements(S3);
[ (), (1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21), (1,15,8)(2,16,9)(3,17,10)(4,18,11)
(5,19,12)(6,20,13)(7,21,14) ]

gap > Order(S3);
3

gap > IsNormal(W3,S3);
    false

gap > S7 := SylowSubgroup(W3,7);
Group([ [ 1,6,4,2,7,5,3)(8,13,11,9,14,12,10)(15,20,18,16,21,19,17), (1,2,3,4,5,6,7)(15,21,20,19,18,17,16),
(1,7,6,5,4,3,2)(8,9,10,11,12,13,14) ])

gap > Elements(S7);

gap > Order(S7);
343

gap > IsNormal(W3,S7);
    true

gap >

4. Conclusion and Recommendation

4.1. Conclusion
We have shown that the Wreath products group of degree $3p$ is soluble as required.

4.2. Recommendation
This study can be extended by considering for further research, one or a combination of two or more of other
theoretic properties such as simplicity, nilpotency, regularity, etc of same algebraic structures.

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