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Generalized approximation Hyers-Ulam-Rassias type stability of generalized homomorphisms in quasi-Banach algebras

LY VAN AN *
Faculty of Mathematics Teacher Education, Tay Ninh University, Ninh Trung,
Ninh Son, Tay Ninh Province, Vietnam.

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Abstract: In this paper, we study to solve the Hyers-Ulam-Rassias stability of generalized homomorphisms in quasi-Banach algebras, associated to Jensen type additive functional equation with 2k-variables. furthermore we investigated the generalized Hyers-Ulam-Rassias stability and superstability of generalized homomorphisms in quasi-Banach algebras.

Key words: Additive functional equation, Jensen functional equation, Homomorphisms in quasi-Banach algebras, Hyers-Ulam-Rassias stability; p-Banach-Algebras.

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ are two linear spaces on the same field $\mathbb{K}$, and $f:\mathbf{X}\to\mathbf{Y}$ be a linear mapping. We use the notation $\|\cdot\|_{\mathbf{X}}$ ($\|\cdot\|_{\mathbf{Y}}$) for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investigate the stability of generalized homomorphisms when $\mathbf{X}$ is a quasi-algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a p- Banach algebras with p-norm $\|\cdot\|_{\mathbf{Y}}$.

In fact, when $\mathbf{X}$ is a quasi-Banach algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a p- Banach algebras with p-norm $\|\cdot\|_{\mathbf{Y}}$ we solve and prove the Hyers-Ulam-Rassias type stability of generalized Homomorphisms in quasi-Banach algebras, associated to the Jensen type additive functional equation.

$$mf\Big(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}x_{k+j}}{m}\Big)=\sum_{j=1}^{k}f\big(x_j\big)+\sum_{j=1}^{k}f\big(x_{k+j}\big) \tag{1}$$

The study the stability of generalized homomorphisms in quasi-Banach algebras originated from a question of S.M. Ulam [22], concerning the stability of group homomorphisms.

Let $(\mathbf{G},*)$ be a group and let $(\mathbf{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Geven $\epsilon>0$, does there exist a $\delta>0$ such that if $f:\mathbf{G}\to\mathbf{G}'$ satisfies

$$d\big(f(x*y),f(x)\circ f(y)\big)<\delta,\forall x\in\mathbf{G}$$

then there is a homomorphism $h:\mathbf{G}\to\mathbf{G}'$ with

$$d\big(f(x),h(x)\big)<\epsilon,\forall x\in\mathbf{G}$$

*Correspondence: lyvanan145@gmail.com

Hyers gave a first affirmative answes the question of Ulam as follows:

D. H. Hyers [8] Let $\epsilon \geq 0$ and let $f : \mathbf{E_1} \to \mathbf{E_2}$ be a mapping between Banach space and f satisfty Hyers inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon,$$

for all $x, y \in \mathbf{E_1}$ and some $\epsilon \geq 0$. It was shown that the limit

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in \mathbf{E_1}$ and that $T : \mathbf{E_1} \to \mathbf{E_2}$ is that unique additive mapping satisfying

$$\|f(x) - T(x)\| \leq \epsilon, \forall x \in \mathbf{E_1}.$$

If $f(tx)$ is continuous in the real variable t for each fixed $x \in \mathbf{E}_1$, then T is linear and if f is continuous at a single point of $\mathbf{E}_1$ then $T : \mathbf{E_1} \to \mathbf{E_2}$ is also continuous.

Next
Result was proved by J.M. Rassias [15]. J.M. Rassias assumed the following weaker inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \|x\|^p \|y\|^p, \forall x, y \in \mathbf{E}_1$$

where $\theta > 0$ and real p, q such that $r = p + q \neq 1$, and retained the condition of continuity $f(tx)$ in t for fixed x.
And J.M. Rassias [16,17] investigated that it is possible to replace in the above Hyers inequality by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. The stability phenomenon that was introduced and proved by J.M. Rassias is called the Hyers-Ulam-Rassias stability.

The stability problems for several functional equations have been extensively investigated by a number of authors and and there are many interesting results concerning this probem. Such as in in 2008 Choonkil Park [10] have established the and investigateed the $Hyers-Ulam-Rassias$ stability of homomorphisms in quasi-Banach algebras the following Jensen functional equation

$$2f(\frac{x+y}{2}) = f(x) + f(y)$$

and
next in 2009 M. Éhaghi Gordji and M. Bavand Savadkouhi [9] have established the and investigateed the approximation of generalized stability of homomorphisms in quasi-Banach algebras the following Jensen functional equation

$$rf(\frac{x+y}{r}) = f(x) + f(y)$$

. Recently, in [3,9,10] the authors studied the on Hyers-Ulam-Rassias type stability of generalized homomorphisms in quasi-Banach algebras, associated to following Jensen type additive functional equation.

$$mf\Big(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}x_{k+j}}{m}\Big)=\sum_{j=1}^{k}f\big(x_j\big)+\sum_{j=1}^{k}f\big(x_{k+j}\big)$$

ie the functional equation with $2k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the functional (1). Thus, the results in this paper are generalization of those in [3,9,10] for functional equation with $2k$-variables.

The paper is organized as followns:

In section preliminarie we remind some basic notations in [6,10,11,12,13,14,15,17] such as quasi-Banach algebras, p-Banach algebras, the important lemma of linear space and solutions of the Jensen function equation.

Section 3 is devoted to prove the Hyers-Ulam-Rassias type stabilityof generalized homomorphims in quasi-Banach algebras of the Jensen type additive functional equation (1) when $\mathbf{X}$ is a quasi-Banach algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a $p-Banach$ algebras with p-norm $\|\cdot\|_{\mathbf{Y}}$.

2. preliminaries

2.1. Quasi-normed space - Quasi-Banach algebras.

Definition 2.1. Let $\mathbf{X}$ be a real linear space. A quasi-norm is a real-valued function on $\mathbf{X}$ satisfying the following :

1. $\|x\|\geq 0$ for all $x\in\mathbf{X}$ and $\|x\|=0$ if and only if $x=0$.
2. $\|\lambda x\|=|\lambda|\|x\|$ for all $\lambda\in\mathbf{R}$ and all $x\in\mathbf{X}$.
3. There is a constant $K\geq 1$ such that

$$\|x+y\|\leq K\big(\|x\|+\|y\|\big),\forall x,y\in\mathbf{X}.$$

The pair $\big(\mathbf{X},\|\cdot\|\big)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on $\mathbf{X}$.

The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a $p-norm$ $\big(0<p\leq 1\big)$ if

$$\|x+y\|^p\leq\|x\|^p+\|y\|^p\forall x,y\in\mathbf{X}.$$

In this case, a quasi-Banach space is called a $p-Banach$ space

$Note_1$: Given a p-norm, the formula $d\big(x,y\big):=\|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [18] (see also [6]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms.

$Note_2$: every homomorphism is a generalized homomorphism, but the converse is false, in general. For instance, let $\mathbf{X}$ be an algebra over $\mathbb{C}$ and let $f:\mathbf{X}\to\mathbf{X}$ be a non-zero homomorphism on $\mathbf{X}$. Then, we have $if\big(xy\big)=if\big(x\big)f\big(y\big)$. This means that i is a generalized homomorphism.

Definition 2.2. Let $\big(\mathbf{X}, \|\cdot\|\big)$ be a quasi-normed space. The quasi-normed space $\big(\mathbf{X}, \|\cdot\|\big)$ is called a quasi-normed algebras if $\mathbf{X}$ is an algebras and there is a constan $K > 0$ such that

$$\|x.y\| \leq K\|x\|\|y\|$$

Definition 2.3. A quasi-Banach algebras is a complete quasi-normed algebras.
If the quasi-norm $\|\cdot\|$ is a p-norm then quasi-Banach is called a p-Banach algebras.

2.2. Some concepts of generalized homomorphism

Definition 2.4. A $\mathbb{C}$-linear mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is called a homomorphism in quasi-Banach algebras if $\phi : \mathbf{X} \to \mathbf{Y}$ such that $\phi\big(xy\big) = \phi\big(x\big)\phi\big(y\big)$ for all $x, y \in \mathbf{X}$.

Definition 2.5. A $\mathbb{C}$-linear mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is called a generalized homomorphism in quasi-Banach algebras if there exists a homomorphism $\phi' : \mathbf{X} \to \mathbf{Y}$ such that $\phi\big(xy\big) = \phi\big(x\big)\phi'\big(y\big)$ for all $x, y \in \mathbf{X}$.

Lemma 2.1. *Let $\mathbf{X}$ and $\mathbf{Y}$ be linear spaces and let $f : \mathbf{X} \to \mathbf{Y}$ be an additive mapping such that $f\big(\alpha x\big) = \alpha f\big(x\big)$ for all $x, y \in \mathbf{X}$ and all $\alpha \in \mathbb{L}^1 := \big\{\alpha \in \mathbb{C}; \big|\alpha\big| = 1\big\}$ Then the mapping f is $\mathbb{C}$-linear.*

2.3. Solutions of the equation.

The functional equation

$$2f\big(\frac{x+y}{2}\big) = f\big(x\big) + f\big(y\big)$$

is called the Jensen equation. In particular, every solution of the Jensen equation is said to be an $Jensen - additive\ \ mapping$.

3. Stability of generalized homomorphisms quasi-Banach algebras

Now, we study the generalized homomorphisms related to equation of (1). Note that for (1), when $\mathbf{X}$ is a quasi-Banach algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a $p-Banach$ algebras with p-norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the generalized homomorphisms mapping relate to (1). These results are give in the following.
Here we assume that m is a positive integer and $\beta \in \mathbb{L}^1$

Theorem 3.1. *Let $f : \mathbf{X} \to \mathbf{Y}$ is a mapping with $f\big(0\big) = 0$ for which there exist a mapping $g : \mathbf{X} \to \mathbf{Y}$ with $g\big(0\big) = 0$, $g\big(1\big) = 1$ and a function*

$$\varphi : \mathbf{X^{4k}} \to \mathbb{R}^+$$

such that

$$\begin{aligned}
&\bigg\|mf\Big(\frac{\beta}{m}\sum_{j=1}^{k}x_j + \frac{\beta}{m}\sum_{j=1}^{k}x_{k+j} + \frac{1}{m}\prod_{j=1}^{k}x_{2k+j}x_{3k+j}\Big)\\
&-\beta\sum_{j=1}^{k}f\big(x_j\big) - \beta\sum_{j=1}^{k}f\big(x_{k+j}\big) - \prod_{j=1}^{k}f\big(x_{2k+j}\big)g\big(x_{3k+j}\big)\bigg\|_{\mathbf{Y}}\\
&\leq \varphi\big(x_1, ..., x_k, ..., x_{2k}, ..., x_{3k}, ..., x_{4k}\big)
\end{aligned} \tag{2}$$

$$\Big\|g\Big(\beta\prod_{j=1}^{k}x_jx_{k+j}+\beta\prod_{j=1}^{k}x_{2k+j}x_{3k+j}\Big)-\beta\prod_{j=1}^{k}g(x_j)g(x_{k+j})-\beta\prod_{j=1}^{k}g(x_{2k+j})g(x_{3k+j})\Big\|_{\mathbf{Y}}$$
$$\leq\varphi(x_1,...,x_k,...,x_{2k},...,x_{3k},...,x_{4k}) \tag{3}$$

and

$$\widetilde{\varphi}(x_1,...,x_k,...,x_{2k},...,x_{3k},...,x_{4k})$$
$$=\sum_{i=1}^{\infty}\frac{\varphi\big((2k)^ix_1,...,(2k)^ix_k,...,(2k)^ix_{2k},...,(2k)^ix_{3k},...,(2k)^ix_{4k}\big)}{(2k)^i}<\infty \tag{4}$$

for all x_j, $x_{k+j},x_{2k+j},x_{3k+j}\in X$ for all $j=1\to k$. Then there exists a unique generalized homomorphism $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\Big\|f(x)-H(x)\Big\|_{\mathbf{Y}}\leq\frac{1}{2k}\widetilde{\varphi}(x,...,x,...,x,...,0,...,0),\forall x\in\mathbf{X}. \tag{5}$$

Proof. Case I: Putting $m=\beta=1$.
Letting $x_j=x_{k+j}=x$ and $x_{2k+1}=x_{3k+j}=0$ for all $j=1\to k$ by the hypothesis (2), we have

$$\Bigg\|f(2kx)-2kf(x)\Bigg\|_{\mathbf{Y}}\leq\varphi(x,...,x,...,x,0,...,0,...,0). \tag{6}$$

for all $x\in\mathbf{X}$. So

$$\Bigg\|\frac{f(2kx)}{2k}-f(x)\Bigg\|_{\mathbf{Y}}\leq\frac{1}{2k}\varphi(x,...,x,...,x,0,...,0,...,0). \tag{7}$$

for all $x\in\mathbf{X}$. Sence $\mathbf{Y}$ is a $p-Banach$ algebra,

$$\Bigg\|\frac{1}{(2k)^l}f\Big((2k)^lx\Big)-\frac{1}{(2k)^m}f\Big((2k)^mx\Big)\Bigg\|_{\mathbf{Y}}^p$$
$$\leq\sum_{j=l}^{m-1}\Bigg\|\frac{1}{(2k)^j}f\Big((2k)^jx\Big)-\frac{1}{(2k)^{j+1}}f\Big((2k)^{j+1}x\Big)\Bigg\|_{\mathbf{Y}}^p$$
$$\leq\frac{1}{(2k)^p}\sum_{m=1}^{k}\frac{\varphi^p\big((2k)^jx,...,(2k)^jx,...,(2k)^jx,0,...,0,...,0\big)}{(2k)^{pj}}. \tag{8}$$

for all $x\in\mathbf{X}$. Sence $\mathbf{Y}$ is a $p-Banach$ algebras

for all nonnegative integers m and l with $m>l$ and $\forall x\in\mathbf{X}$. It follows from (8) that the sequence $\Big\{\frac{1}{(2k)^n}f\Big((2k)^nx\Big)\Big\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\Big\{\frac{1}{(2k)^n}f\Big((2k)^nx\Big)\Big\}$ coverges.
So one can define the mapping $H:\mathbf{X}\to\mathbf{Y}$ by

$$H(x):=\lim_{n\to\infty}\frac{1}{(2k)^n}f\big((2k)^nx\big)$$

for all $x \in \mathbf{X}$.

Case II: Putting $x_{2k+j} = x_{3k+j} = 0$, $m = 1$. and replacing x_j, x_{k+j} by $(2k)^n x_j, (2k)^n x_{k+j}$ respectively, in (2) and multiply both sides of (2) by $\frac{1}{\left(2k\right)^n}$

$$\left\| \frac{f\Big((2k)^n\Big(\beta\sum_{j=1}^k x_j + \beta\sum_{j=1}^k x_{k+j}\Big)\Big)}{(2k)^n} - \beta\sum_{j=1}^k \frac{f\big((2k)^n x_j\big)}{(2k)^n} - \beta\sum_{j=1}^k \frac{f\big((2k)^n x_{k+j}\big)}{(2k)^n} \right\|_{\mathbf{Y}}$$
$$\leq \frac{\varphi\big((2k)^n x_1, ..., (2k)^n x_k, ..., (2k)^n x_{2k}, 0, ..., 0, ..., 0\big)}{(2k)^n} \tag{9}$$

for all $\beta \in L^1$, x_j, $x_{k+j}, x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$. Pass the limit as $n \to \infty$ in (9) we have

$$H\Big(\beta\sum_{j=1}^k x_j + \beta\sum_{j=1}^k x_{k+j}\Big) = \beta\sum_{j=1}^k H\big(x_j\big) + \beta\sum_{j=1}^k H\big(x_{k+j}\big) \tag{10}$$

for all $\beta \in L^1$, x_j, $x_{k+j}, x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$. By lemma 2.6 the mapping H is $\mathbb{C}$-linear

Now we prove the uniqueness of H. Assume that $H_1 : \mathbf{X} \to \mathbf{Y}$ is an additive mapping satisfing (5). Then we have

$$\begin{aligned}
&\Big\|H\big(x\big) - H_1\big(x\big)\Big\|_{\mathbf{Y}}\\
&= \frac{1}{\left(2k\right)^n}\Big\|H\big(\left(2k\right)^n x\big) + H_1\big(\left(2k\right)^n x\big)\Big\|_{\mathbf{Y}}\\
&\leq \frac{1}{\left(2k\right)^n}\Big(\Big\|H\big(\left(2k\right)^n x\big) - f\big(\left(2k\right)^n x\big)\Big\|_{\mathbf{Y}} + \Big\|f\big(\left(2k\right)^n x\big) - H_1\big(\left(2k\right)^n x\big)\Big\|_{\mathbf{Y}}\Big)\\
&\leq \frac{2}{\left(2k\right)^{n+1}}\widetilde{\varphi}\big(\left(2k\right)^n x, ..., \left(2k\right)^n x, ..., \left(2k\right)^n x, 0, ..., 0, ..., 0\big)
\end{aligned} \tag{11}$$

.

which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $H\big(x\big) = H_1\big(x\big)$ for all $x \in \mathbf{X}$. This proves the uniqueness of H. Thus the mapping $H_1 : \mathbf{X} \to \mathbf{Y}$ is a unique homomorphism satisfying (5).

Case III:
Putting $x_j = x_{k+j} = 0$, $\beta = m = 1$. and replacing x_{2k+j}, x_{3k+j} by $(2k)^{kn} x_{2k+j}$, $(2k)^{kn} x_{3k+j}$ respectively, in (2) and multiply both sides of (2) by $\frac{1}{\left(2k\right)^{2kn}}$ we get

$$\left\| \frac{f\Big((2k)^{2kn}\big(\prod_{j=1}^k x_{2k+j}x_{3k+j}\big)\Big)}{(2k)^{2kn}} - \prod_{j=1}^k \frac{f\big((2k)^n x_{2k+j}\big)}{(2k)^{kn}} \frac{g\big((2k)^n x_{3k+j}\big)}{(2k)^{kn}} \right\|_{\mathbf{Y}}$$
$$\leq \frac{\varphi\big(0, ..., 0, ..., 0, (2k)^{kn} x_{2k+1}, ..., (2k)^{kn} x_{3k+1}, ..., (2k)^{kn} x_{4k}\big)}{(2k)^{2kn}} \tag{12}$$

for all $x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$. Pass the limit as $n \to \infty$ in (12) we have

$$H\Big(\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big) = \prod_{j=1}^{k} H\big(x_{2k+j}\big)\prod_{j=1}^{k} H_1\big(x_{3k+j}\big) \tag{13}$$

for all $\beta \in L^1$, x_j, $x_{k+j}, x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$.

Next we claim that H_1 is homomorphism.
Case IV:
Putting $x_{k+j} = x_{3k+j} = 1$ for all $j = 1 \to k$, and replacing x_j, x_{2k+j} by $(2k)^{kn}x_j, (2k)^{kn}x_{2k+j}$ respectively, in (3) and multiply both sides of (3) by $\frac{1}{\big(2k\big)^{kn}}$

$$\begin{aligned}&\Bigg\|\frac{g\Big((2k)^{kn}\Big(\beta\prod_{j=1}^{k} x_jx_{k+j} + \beta\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big)\Big)}{(2k)^{kn}} - \beta\frac{\prod_{j=1}^{k} g\big((2k)^{kn}x_j\big)}{(2k)^{kn}}\\ &- \beta\frac{\prod_{j=1}^{k} g\big((2k)^{kn}x_{j+3k}\big)}{(2k)^{kn}}\Bigg\|_{\mathbf{Y}} \le \varphi\big(x_1, ..., x_k, ..., x_{2k}, ..., x_{3k}, ..., x_{4k}\big)\end{aligned} \tag{14}$$

for all $x_j, x_{2k+j} \in \mathbf{X}$ for all $j = 1 \to k$ and $\beta \in L^1$. Pass the limit as $n \to \infty$ in (17) we have

$$H_1\Big(\beta\prod_{j=1}^{k} x_j + \beta\prod_{j=1}^{k} x_{2k+j}\Big) = \beta\prod_{j=1}^{k} H_1\big(x_j\big) + \beta\prod_{j=1}^{k} H_1\big(x_{j+2k}\big) \tag{15}$$

By lemma 2.6 the mapping H is $\mathbb{C}$-linear

Case V:
Putting $x_{2k+j} = x_{3k+j} = 0$ for all $j = 1 \to k, \beta = 1$ in (3)

$$\Bigg\|g\Big(\prod_{j=1}^{k} x_jx_{k+j}\Big) - \prod_{j=1}^{k} g\big(x_j\big)g\big(x_{k+j}\big)\Bigg\|_{\mathbf{Y}} \le \varphi\big(x_1, ..., x_k, x_{k+1}, ..., x_{2k}, 0..., 0, ..., 0, ..., 0\big) \tag{16}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$

and replacing x_j, x_{k+j} by $(2k)^{kn}x_j, (2k)^{kn}x_{k+j}$ respectively, in (3) and multiply both sides of (3) by $\frac{1}{\big(2k\big)^{2kn}}$

$$\begin{aligned}&\Bigg\|\frac{g\Big((2k)^{2kn}\Big(\prod_{j=1}^{k} x_jx_{k+j}\Big)\Big)}{(2k)^{2kn}} - \frac{\prod_{j=1}^{k} g\big((2k)^{kn}x_j\big)}{(2k)^{kn}}\frac{\prod_{j=1}^{k} g\big((2k)^{kn}x_{j+k}\big)}{(2k)^{kn}}\Bigg\|_{\mathbf{Y}}\\ &\le \frac{\varphi\big((2k)^n x_1, ..., (2k)^n x_k, ..., (2k)^n x_{2k}, 0, ..., 0, ..., 0\big)}{(2k)^{2kn}}\end{aligned} \tag{17}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. Pass the limit as $n \to \infty$ in (17) we have

$$H_1\Big(\prod_{j=1}^{k} x_j \prod_{j=1}^{k} x_{k+j}\Big) = \prod_{j=1}^{k} H_1\big(x_j\big) \prod_{j=1}^{k} H_1\big(x_{j+k}\big) \tag{18}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. It then follows from (13) that H is a generalized homomorphism. $\square$

Corollary 3.1. *Suppose $f : \mathbf{X} \to \mathbf{Y}$ is a mapping with $f\big(0\big) = 0$ for which there exist constant $\epsilon > 0, p \neq 0$ and a mappig $g : \mathbf{X} \to \mathbf{Y}$ with $g\big(0\big) = 0$, $g\big(1\big) = 1$ and a fuction*

$$\varphi : \mathbf{X^{4k}} \to \mathbb{R}^+$$

such that

$$\begin{aligned}&\bigg\|mf\Big(\frac{\beta}{m}\sum_{j=1}^{k} x_j + \frac{\beta}{m}\sum_{j=1}^{k} x_{k+j} + \frac{1}{m}\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big)\\
&-\beta\sum_{j=1}^{k} f\Big(x_j\Big) - \beta\sum_{j=1}^{k} f\Big(x_{k+j}\Big) - \prod_{j=1}^{k} f\big(x_{2k+j}\big)g\big(x_{3k+j}\big)\bigg\|_{\mathbf{Y}}\\
&\leq \epsilon\Big(\sum_{j=1}^{k}\Big\|x_j\Big\|_{\mathbf{X}}^p + \sum_{j=1}^{k}\Big\|x_{k+j}\Big\|_{\mathbf{X}}^p \sum_{j=1}^{k}\Big\|x_{2k+j}\Big\|_{\mathbf{X}}^p + \sum_{j=1}^{k}\Big\|x_{3k+j}\Big\|_{\mathbf{X}}^p\Big)\end{aligned} \tag{19}$$

$$\begin{aligned}&\bigg\|g\Big(\beta\prod_{j=1}^{k} x_j x_{k+j} + \beta\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big) - \beta\prod_{j=1}^{k} g\big(x_j\big)g\big(x_{k+j}\big) - \beta\prod_{j=1}^{k} g\big(x_{2k+j}\big)g\big(x_{3k+j}\big)\bigg\|_{\mathbf{Y}}\\
&\leq \epsilon\Big(\sum_{j=1}^{k}\Big\|x_j\Big\|_{\mathbf{X}}^p + \sum_{j=1}^{k}\Big\|x_{k+j}\Big\|_{\mathbf{X}}^p \sum_{j=1}^{k}\Big\|x_{2k+j}\Big\|_{\mathbf{X}}^p + \sum_{j=1}^{k}\Big\|x_{3k+j}\Big\|_{\mathbf{X}}^p\Big)\end{aligned} \tag{20}$$

for all x_j , $x_{k+j}, x_{2k+j}, x_{3k+j} \in X$ for all $j = 1 \to k$ and $\beta \in T^1$. Then there exists a unique generalized homomorphism $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\Big\|f\big(x\big) - H\big(x\big)\Big\|_{\mathbf{Y}} \leq \frac{2k\epsilon}{\big|(2k)^p - 2k\big|}\Big\|x\Big\|_{\mathbf{X}}^p \tag{21}$$

for all $x \in \mathbf{X}$

Corollary 3.2. *Suppose $f : \mathbf{X} \to \mathbf{Y}$ is a mapping with $f\big(0\big) = 0$ for which there exist constant $\lambda > 0, p \neq 0$ and a mappig $g : \mathbf{X} \to \mathbf{Y}$ with $g\big(0\big) = 0$, $g\big(1\big) = 1$ and a function*

$$\varphi : \mathbf{X^{4k}} \to \mathbb{R}^+$$

such that

$$\begin{aligned}&\bigg\|mf\Big(\frac{\beta}{m}\sum_{j=1}^{k} x_j + \frac{\beta}{m}\sum_{j=1}^{k} x_{k+j} + \frac{1}{m}\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big)\\
&-\beta\sum_{j=1}^{k} f\big(x_j\big) - \beta\sum_{j=1}^{k} f\big(x_{k+j}\big) - \prod_{j=1}^{k} f\big(x_{2k+j}\big)g\big(x_{3k+j}\big)\bigg\|_{\mathbf{Y}} \leq \lambda\end{aligned} \tag{22}$$

$$\Big\| g\Big(\beta\prod_{j=1}^{k} x_j x_{k+j} + \beta\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big) - \beta\prod_{j=1}^{k} g(x_j)g(x_{k+j}) \\ - \beta\prod_{j=1}^{k} g(x_{2k+j})g(x_{3k+j})\Big\|_{\mathbf{Y}} \leq \lambda \tag{23}$$

for all $x_j,\ x_{k+j}, x_{2k+j}, x_{3k+j} \in X$ *for all* $j = 1 \to k$ *and* $\beta \in T^1$. *Then there exists a unique generalized homomorphism* $H : \mathbf{X} \to \mathbf{Y}$ *such that*

$$\Big\| f(x) - H(x)\Big\|_{\mathbf{Y}} \leq \frac{\lambda}{2} \tag{24}$$

for all $x \in \mathbf{X}$

Theorem 3.2. *Suppose* $f : \mathbf{X} \to \mathbf{Y}$ *is a mapping with* $f(0) = 0$ *for which there exist a mapping* $g : \mathbf{X} \to \mathbf{Y}$ *with* $g(0) = 0$, $g(1) = 1$ *and a function*

$$\varphi : \mathbf{X^{4k}} \to \mathbb{R}^+$$

such that

$$\Big\| mf\Big(\frac{\beta}{m}\sum_{j=1}^{k} x_j + \frac{\beta}{m}\sum_{j=1}^{k} x_{k+j} + \frac{1}{m}\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big) \\ - \beta\sum_{j=1}^{k} f(x_j) - \beta\sum_{j=1}^{k} f(x_{k+j}) - \prod_{j=1}^{k} f(x_{2k+j})g(x_{3k+j})\Big\|_{\mathbf{Y}} \\ \leq \varphi(x_1, ..., x_k, ..., x_{2k}, ..., x_{3k}, ..., x_{4k}) \tag{25}$$

$$\Big\| g\Big(\beta\prod_{j=1}^{k} x_j x_{k+j} + \beta\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big) - \beta\prod_{j=1}^{k} g(x_j)g(x_{k+j}) - \beta\prod_{j=1}^{k} g(x_{2k+j})g(x_{3k+j})\Big\|_{\mathbf{Y}} \\ \leq \varphi(x_1, ..., x_k, ..., x_{2k}, ..., x_{3k}, ..., x_{4k}) \tag{26}$$

and

$$\widetilde{\varphi}(x_1, ..., x_k, ..., x_{2k}, ..., x_{3k}, ..., x_{4k}) \\ = \sum_{i=1}^{\infty}(2k)^i\varphi\Big(\frac{x_1}{(2k)^i}, ..., \frac{x_k}{(2k)^i}, ..., \frac{x_{2k}}{(2k)^i}, ..., \frac{x_{3k}}{(2k)^i}, ..., \frac{x_{4k}}{(2k)^i}\Big) < \infty \tag{27}$$

for all $x_j,\ x_{k+j}, x_{2k+j}, x_{3k+j} \in X$ *for all* $j = 1 \to k$ *and* $\beta \in T^1$. *Then there exists a unique generalized homomorphism* $H : \mathbf{X} \to \mathbf{Y}$

Proof. The proof is similar to the proof of theorem 3.1.

□

Theorem 3.3. *Suppose* $p \neq 1, \epsilon > 0$ *and* $f : \mathbf{X} \to \mathbf{Y}$ *is a mapping with* $f(0) = 0$ *for which there exist a mapping* $g : \mathbf{X} \to \mathbf{Y}$ *with* $g(0) = 0$, $g(1) = 1$ *such that*

$$\Big\| mf\Big(\frac{\beta}{m}\sum_{j=1}^{k} x_j + \frac{\beta}{m}\sum_{j=1}^{k} x_{k+j} + \frac{1}{m}\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big)$$
$$- \beta\sum_{j=1}^{k} f(x_j) - \beta\sum_{j=1}^{k} f(x_{k+j}) - \prod_{j=1}^{k} f(x_{2k+j})g(x_{3k+j})\Big\|_{\mathbf{Y}}$$
$$\leq \epsilon \sum_{j=1}^{k} \big\|f(x_{2k+j})\big\|_{\mathbf{Y}} \tag{28}$$

$$\Big\| g\Big(\beta\prod_{j=1}^{k} x_j x_{k+j} + \beta\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big) - \beta\prod_{j=1}^{k} g(x_j)g(x_{k+j}) - \beta\prod_{j=1}^{k} g(x_{2k+j})g(x_{3k+j})\Big\|_{\mathbf{Y}}$$
$$\leq \epsilon\Big(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^p + \sum_{j=1}^{k}\|x_{k+j}\|_{\mathbf{X}}^p + \sum_{j=1}^{k}\|x_{2k+j}\|_{\mathbf{X}}^p + \sum_{j=1}^{k}\|x_{3k+j}\|_{\mathbf{X}}^p\Big) \tag{29}$$

for all x_j , $x_{k+j}, x_{2k+j}, x_{3k+j} \in X$ *for all* $j = 1 \to k$ *and all* $\beta \in \mathbb{L}^1$. *Then there exists a unique generalized homomorphism* $H : \mathbf{X} \to \mathbf{Y}$

Proof. In this theorem I only prove the case $p < 1$ and the case $p > 1$ the proof is similar.

Case I: Putting $x_{2k+j} = x_{3k+j} = 0$, $m = 1$. in (28)

$$\Big\|f\Big(\beta\sum_{j=1}^{k} x_j + \beta\sum_{j=1}^{k} x_{k+j}\Big) - \beta\sum_{j=1}^{k} f\Big(x_j\Big) - \beta\sum_{j=1}^{k} f\Big(x_{k+j}\Big)\Big\|_{\mathbf{Y}} \leq \epsilon\sum_{j=1}^{k}\Big\|f(0)\Big\|_{\mathbf{Y}} = 0 \tag{30}$$

for all $\beta \in L^1$, x_j, $x_{k+j}, x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$. Thus we have

$$f\Big(\beta\sum_{j=1}^{k} x_j + \beta\sum_{j=1}^{k} x_{k+j}\Big) = \beta\sum_{j=1}^{k} f(x_j) + \beta\sum_{j=1}^{k} f(x_{k+j}) \tag{31}$$

for all $\beta \in L^1$, x_j, $x_{k+j}, x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$. By lemma 2.6 the mapping f is $\mathbb{C}$-linear .

Case II:
Putting $x_j = x_{k+j} = 0$, $\beta = m = 1$. in (28)

$$\Big\|f\Big(\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big) - \beta\prod_{j=1}^{k} f(x_{2k+j})g(x_{3k+j})\Big\|_{\mathbf{Y}} \leq \epsilon\sum_{j=1}^{k}\Big\|f(x_{2k+i})\Big\|_{\mathbf{Y}} \tag{32}$$

and replacing x_{2k+j}, x_{3k+j} by $(2k)^{kn}x_{2k+j}$,
$(2k)^{kn}x_{3k+j}$ respectively, in (28) and multiply both sides of (28) by $\frac{1}{\left(2k\right)^{2kn}}$ we get

$$\begin{aligned}\Bigg\|\frac{f\Big((2k)^{2kn}\Big(\prod_{j=1}^k x_{2k+j}x_{3k+j}\Big)\Big)}{(2k)^{2kn}} - \beta\prod_{j=1}^k \frac{f\Big((2k)^n x_{2k+j}\Big)}{(2k)^{kn}}\frac{g\Big((2k)^n x_{3k+j}\Big)}{(2k)^{kn}}\Bigg\|_{\mathbf{Y}} \\ \leq \frac{\epsilon}{(2k)^{2kn}}\sum_{j=1}^k \Big\|f\big((2k)^{nk}x_{2k+i}\big)\Big\|_{\mathbf{Y}} \qquad (33)\end{aligned}$$

for all $x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$. Pass the limit as $n \to \infty$ in (34) we have

$$\Bigg\|f\Big(\prod_{j=1}^k x_{2k+j}x_{3k+j}\Big) - \prod_{j=1}^k f\Big(x_{2k+j}\Big)\frac{g\Big((2k)^n x_{3k+j}\Big)}{(2k)^{kn}}\Bigg\|_{\mathbf{Y}} \leq \frac{\epsilon}{(2k)^{kn}}\sum_{j=1}^k \Big\|f\big(x_{2k+i}\big)\Big\|_{\mathbf{Y}} \qquad (34)$$

for all $x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$. Pass the limit as $n \to \infty$ in (34) we have

$$f\Big(\prod_{j=1}^k x_{2k+j}x_{3k+j}\Big) = \prod_{j=1}^k f\big(x_{2k+j}\big)\prod_{j=1}^k f_1\big(x_{3k+j}\big) \qquad (35)$$

for all $\beta \in L^1$, x_j, $x_{k+j}, x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$.

Next we claim that H_1 is homomorphism.
Case III.
Putting $x_{k+j} = x_{3k+j} = 1$ for all $j = 1 \to k$, and replacing x_j, x_{2k+j} by $(2k)^{kn}x_j, (2k)^{kn}x_{2k+j}$ respectively, in (29) and multiply both sides of (29) by $\frac{1}{\left(2k\right)^{kn}}$

$$\begin{aligned}&\Bigg\|\frac{g\Big((2k)^{kn}\Big(\beta\prod_{j=1}^k x_jx_{k+j} + \beta\prod_{j=1}^k x_{2k+j}x_{3k+j}\Big)\Big)}{(2k)^{kn}} - \beta\frac{\prod_{j=1}^k g\big((2k)^{kn}x_j\big)}{(2k)^{kn}} \\ &- \beta\frac{\prod_{j=1}^k g\big((2k)^{kn}x_{j+3k}\big)}{(2k)^{kn}}\Bigg\|_{\mathbf{Y}} \leq \frac{1}{\left(2k\right)^{kn}}\Big(\sum_{j=1}^k \|(2k)^{kn}x_j\|_{\mathbf{X}}^p + \sum_{j=1}^k \|(2k)^{kn}\|_{\mathbf{X}}^p \\ &+ \sum_{j=1}^k \|(2k)^{kn}x_{2k+j}\|_{\mathbf{X}}^p + \sum_{j=1}^k \|(2k)^{kn}\|_{\mathbf{X}}^p\Big) \qquad (36)\end{aligned}$$

for all $x_j, x_{2k+j} \in \mathbf{X}$ for all $j = 1 \to k$ and $\beta \in L^1$. Pass the limit as $n \to \infty$ in (36) we have

$$H_1\Big(\beta\prod_{j=1}^k x_j + \beta\prod_{j=1}^k x_{2k+j}\Big) = \beta\prod_{j=1}^k H_1\big(x_j\big) + \beta\prod_{j=1}^k H_1\big(x_{j+2k}\big) \qquad (37)$$

By lemma 2.6 the mapping H is $\mathbb{C}$-linear

Case IV.
Putting $x_j = x_{k+j} = 0$ for all $j = 1 \to k, \beta = 1$ in (29)

$$\left\| g\Big(\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big) - \prod_{j=1}^{k} g\big(x_{2k+j}\big)g\big(x_{3k+j}\big)\right\|_{\mathbf{Y}} \leq \epsilon\Big(\sum_{j=1}^{k}\|x_{2k+j}\|_{\mathbf{X}}^{p} + \sum_{j=1}^{k}\|x_{3k+j}\|_{\mathbf{X}}^{p}\Big) \tag{38}$$

for all $x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$

and replacing x_{2k+j}, x_{3k+j} by $(2k)^{kn}x_{2k+j}, (2k)^{kn}x_{3k+j}$ respectively, in (29) and multiply both sides of (29) by $\frac{1}{\big(2k\big)^{2kn}}$

$$\left\| \frac{g\Big((2k)^{2kn}\Big(\prod_{j=1}^{k} x_{2k+j}x_{3k+j}\Big)\Big)}{(2k)^{2kn}} - \frac{\prod_{j=1}^{k} g\big((2k)^{kn}x_{2k+j}\big)}{(2k)^{kn}}\frac{\prod_{j=1}^{k} g\big((2k)^{kn}x_{3k+j}\big)}{(2k)^{kn}}\right\|_{\mathbf{Y}}$$
$$\leq \frac{\epsilon}{(2k)^{2kn}}\Big(\sum_{j=1}^{k}\|(2k)^{kn}x_{2k+j}\|_{\mathbf{X}}^{p} + \sum_{j=1}^{k}\|(2k)^{kn}x_{3k+j}\|_{\mathbf{X}}^{p}\Big) \tag{39}$$

for all $x_{2k+j}, x_{3k+j} \in \mathbf{X}$ for all $j = 1 \to k$. Pass the limit as $n \to \infty$ in (36) we have

$$H_1\Big(\prod_{j=1}^{k} x_j \prod_{j=1}^{k} x_{k+j}\Big) = \prod_{j=1}^{k} H_1\big(x_j\big) \prod_{j=1}^{k} H_1\big(x_{j+k}\big) \tag{40}$$

from (3.4) that f is a generalized homomorphism. Similarly, one can show the result for the case $p > 1$.

□

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