An Overview on $\mu_N$ Strongly Nowhere Dense Sets

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Abstract: In this article we have introduced some new types of sets such as $\mu_N$ strongly dense, $\mu_N$ strongly nowhere dense, $\mu_N$ strongly first category sets, $\mu_N$ strongly nowhere residual sets and their attributes are explained briefly. Also by making use of these we have retrieved $\mu_N$ strongly Baire space and its properties are to be described.

Key words: $\mu_N$ strongly dense, $\mu_N$ strongly nowhere dense, $\mu_N$ strongly first category sets, $\mu_N$ strongly nowhere residual sets

1. Introduction

Zadeh’s concept of fuzziness has a huge impact on all fields of mathematics. C.L.Chang[3] later combined the ideas of fuzziness with topological spaces, laying the groundwork for the theory of fuzzy topological spaces. K.T.Attanasov[1] discovered intuitionistic fuzzy sets, and with his friend Stoeva[2], he expanded his research to reveal a generalisation to intuitionistic L-fuzzy sets. F.Smarandache[7] directed his attention to the degree of indeterminacy and proposed the neutrosophic sets. Following that, A.A.Salama and Albowi[13] discovered the neutrosophic topological spaces using neutrosophic sets. We[12] created Generalized topological spaces via neutrosophic sets using all of the works as inspiration and named it as TS. In $\mu_N$ TS the concept of Baire space was putforth by us and here we extended our research ideas into the strong natures of $\mu_N$ Baire space.

2. Necessities

Definition 2.1. [14] Let $X$ be a non-empty fixed set. A Neutrosophic set [ NS for short ] $A$ is an object having the form $A = \{ < x, \mu_A(x), \sigma_A(x), \gamma_A(x) > : x \in X \}$ where $\mu_A(x)$, $\sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function, the degree of indeterminacy and the degree of non-membership function respectively of each element $x \in X$ to the set $A$.

Remark 2.1. [14] Every intuitionistic fuzzy set $A$ is a non empty set in $X$ is obviously on Neutrosophic sets having the form $A = \{ < \mu_A(x), 1-(\mu_A(x)+\sigma_A(x)), \gamma_A(x) > : x \in X \}$. Since our main purpose is to construct the tools for developing Neutrosophic Set and Neutrosophic topology , we must introduce the neutrosophic sets $0_N$ and $1_N$ in $X$ as follows: $0_N$ may be defined as follows 

($0_1$) $0_N = \{ < x, 0, 0, 1 > : x \in X \}$

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1. \( N \) may be defined as follows
(1) \( N \) may be defined as follows
\[
1_N = \{< x, 1, 0, 0 > : x \in X \}
\]

**Definition 2.2.** [14] Let \( A = \{< \mu_A, \sigma_A, \gamma_A >\} \) be a NS on \( X \), then the complement of the set \( A \) [for short] may be defined as three kinds of complements:

\[
(C_1) C(A) = A = \{< x, 1 - \mu_A(x), 1 - \sigma_A(x), 1 - \gamma_A(x) > : x \in X \}
\]

**Definition 2.3.** [14] Let \( X \) be a non-empty set and neutrosophic sets \( A \) and \( B \) in the form \( A = \{< x, \mu_A(x), \sigma_A(x), \gamma_A(x) > : x \in X \} \) and \( B = \{< x, \mu_B(x), \sigma_B(x), \gamma_B(x) > : x \in X \} \). Then we may consider two possibilities for definitions for subsets \( A \subseteq B \).

\[
(A \subseteq B) \iff \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x), \gamma_A(x) \geq \gamma_B(x) \forall x \in X
\]

**Proposition 2.1.** [14] For any neutrosophic set \( A \), the following conditions holds:

0\( N \subseteq A \), 0\( N \subseteq 0_N \)

\( A \subseteq 1_N \), 1\( N \subseteq 1_N \)

**Definition 2.4.** [14] Let \( X \) be a non-empty set and \( A = \{< x, \mu_A(x), \sigma_A(x), \gamma_A(x) > : x \in X \} \) \( B = \{< x, \mu_B(x), \sigma_B(x), \gamma_B(x) > : x \in X \} \) are NSs. Then \( A \cap B \) may be defined as:

\[
(I_1) A \cap B = \{< x, \mu_A(x) \land \mu_B(x), \sigma_A(x) \land \sigma_B(x), \gamma_A(x) \lor \gamma_B(x) > : x \in X \}
\]

\( A \cup B \) may be defined as:

\[
(I_1) A \cup B = \{< x, \mu_A(x) \lor \mu_B(x), \sigma_A(x) \lor \sigma_B(x), \gamma_A(x) \land \gamma_B(x) > : x \in X \}
\]

**Definition 2.5.** [12] A \( \mu_N \) topology is a non-empty set \( X \) is a family of neutrosophic subsets in \( X \) satisfying the following axioms:

\[
(\mu_N, \emptyset) \subseteq \mu_N
\]

(\( \mu_N \)) union of any number of \( \mu_N \) open sets is \( \mu_N \) open.

**Remark 2.2.** [12] The elements of \( \mu_N \) are \( \mu_N \)-open sets and their complement is called \( \mu_N \) closed sets.

**Definition 2.6.** [12] The \( \mu_N \)-Closure of \( A \) is the intersection of all \( \mu_N \) closed sets containing \( A \).

**Definition 2.7.** [12] The \( \mu_N \)-Interior of \( A \) is the union of all \( \mu_N \) open sets contained in \( A \).

**Definition 2.8.** [13] A neutrosophic set \( A \) in NTS is called neutrosophic dense if there exists no neutrosophic closed sets \( B \) in \( (X, T) \) such that \( A \subseteq B \subseteq 1_N \).

**Definition 2.9.** [13] The \( \mu_N \) Topological spaces is said to be \( \mu_N \) Baire’s Space if \( \mu_N \text{Int}(\bigcup_{i=1}^{\infty} G_i) = 0_N \) where \( G_i \)’s are \( \mu_N \) nowhere dense set in \( (X, \mu_N) \).

**Proposition 2.2.** [13] Let \( (X, \mu_N) \) be a \( \mu_N \) TS. Then the following are equivalent.

1. \( (X, \mu_N) \) is \( \mu_N \) Baire’s Space.
2. \( \mu_N \text{Int}(A) = 0_N \), for all \( \mu_N \) first category set in \( (X, \mu_N) \).
3. \( \mu_N \text{Cl}(A) = 1_N \), for every \( \mu_N \) Residual set in \( (X, \mu_N) \).
3. \( \mu_N \) Strongly Nowhere Dense sets:

**Definition 3.1.** Let \((X, \mu_N)\) be a \( \mu_N \) Topological Space. A neutrosophic sets \( \zeta \) defined on \((X, \mu_N)\) is called \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\) if \( \zeta \cap \bar{\zeta} \) is a \( \mu_N \) nowhere dense set in \((X, \mu_N)\). That is, \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\) if \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) = 0_N \) in \((X, \mu_N)\).

**Proposition 3.1.** If \( \zeta \) is a \( \mu_N \) nowhere dense set in a \( \mu_N \) Topological Space \((X, \mu_N)\), then \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proof.** Let \( \zeta \) be a \( \mu_N \) nowhere dense set in a \( \mu_N \) Topological Space in \((X, \mu_N)\), then \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta)) = 0_N \) in \((X, \mu_N)\). Since \( \zeta \cap \bar{\zeta} \subseteq \zeta \) in \((X, \mu_N)\). We obtain that \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(\zeta)) \) and hence \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) \subseteq 0_N \). That is, \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) = 0_N \). Hence \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proposition 3.2.** If \( \bar{\zeta} \) is a \( \mu_N \) nowhere dense set in a \( \mu_N \) Topological Space in \((X, \mu_N)\), then \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proof.** Suppose that \( \bar{\zeta} \) is a \( \mu_N \) nowhere dense set in a \( \mu_N \) Topological Space in \((X, \mu_N)\), then \( \mu_N \text{Int}(\mu_N \text{Cl}(\bar{\zeta})) = 0_N \) in \((X, \mu_N)\). Since \( \zeta \cap \bar{\zeta} \subseteq \bar{\zeta} \), \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(\bar{\zeta})) \) and hence \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) \subseteq 0_N \) that implies us \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) = 0_N \). Hence \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proposition 3.3.** If \( \mu_N \text{Cl}(\mu_N \text{Int}\bar{\zeta}) = 1_N \), for a neutrosophic set \( \zeta \) defined on \((X, \mu_N)\), then \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proof.** Suppose that \( \mu_N \text{Cl}(\mu_N \text{Int}\bar{\zeta}) = 1_N \) in \((X, \mu_N)\). Then we deduce that \( \mu_N \text{Cl}(\mu_N \text{Int}\bar{\zeta}) = 0_N \) which implies us \( \mu_N \text{Int}(\mu_N \text{Cl} \bar{\zeta}) = 0_N \). We obtain \( \mu_N \text{Int}(\mu_N \text{Cl} \bar{\zeta}) = 0_N \). Thus, \( \zeta \) is a \( \mu_N \) nowhere dense set in \((X, \mu_N)\). By using proposition 3.2, \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proposition 3.4.** If \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\), then \( \bar{\zeta} \) is also a \( \mu_N \) strongly nowhere dense set.

**Proof.** Let \( \zeta \) be a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\) which entails us that \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) = 0_N \) in \((X, \mu_N)\). Now \( \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \bar{\zeta})) = \mu_N \text{Int}(\mu_N \text{Cl}(\bar{\zeta} \cap \zeta)) = 0_N \). This implies us \( \bar{\zeta} \) is also a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proposition 3.5.** If \( \zeta \) is a \( \mu_N \) nowhere dense set in \( \mu_N \) Topological space then \( \bar{\zeta} \) is \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proof.** Let \( \zeta \) be a \( \mu_N \) nowhere dense set in \((X, \mu_N)\). Now by using Proposition 3.2 we get \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\) and by proposition 3.4 we obtain that \( \bar{\zeta} \) is \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proposition 3.6.** If \( \zeta \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\), then \( \mu_N \text{Cl}(\zeta \cup \bar{\zeta}) = 1_N \).
Proof. Let $\zeta$ be a $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$ then we obtain that $\mu_N Cl(\zeta \cup \zeta) = 0_N \Rightarrow \mu_N Cl(\mu_N Int(\zeta \cap \zeta)) = 1_N \Rightarrow \mu_N Cl(\mu_N Int(\zeta \cap \zeta)) = 1_N$. But $\mu_N Cl(\mu_N Int(\zeta \cup \zeta)) \subseteq \mu_N Cl(\zeta \cup \zeta)$ implies us that $\mu_N Cl(\zeta \cup \zeta) = 1_N$. Hence we get $\mu_N Cl(\zeta \cup \zeta) = 1_N$.

Proposition 3.7. If $\mu_N Int P$ is a $\mu_N$ dense set, for a neutrosophic set $P$ defined on a $\mu_N$ TS $(X, \mu_N)$ then $P$ is $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$.

Proof. Suppose that $\mu_N Int P$ is a $\mu_N$ dense set in $(X, \mu_N)$, then $\mu_N Cl(\mu_N Int P) = 1_N$ in $(X, \mu_N)$. Then $\mu_N Cl(\mu_N Int P) = 0_N$. This implies us $\mu_N Int(\mu_N Cl P) = 0_N$ in $(X, \mu_N)$. Since $P \cap \bar{P} \subseteq \bar{P}$ we deduce that $\mu_N Int(\mu_N Cl(P \cap \bar{P})) \subseteq \mu_N Int(\mu_N Cl P)$ and hence $\mu_N Int(\mu_N Cl(P \cap \bar{P})) \subseteq 0_N$. That is $\mu_N Int(\mu_N Cl(P \cap \bar{P})) = 0_N$ which implies us that $P$ is $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$.

Proposition 3.8. If $P$ is a neutrosophic set defined on $(X, \mu_N)$ such that $\mu_N Int(\mu_N Fr(P)) = 0_N$ in a $\mu_N$ Topological space then $P$ is a $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$.

Proof. Let $P$ be a neutrosophic set defined on $(X, \mu_N)$ such that $\mu_N Int(\mu_N Fr(P)) = 0_N$. Since $\mu_N Fr(P) = \mu_N Cl(\mu_N Cl(P)) = 0_N$. Now $\mu_N Cl(\mu_N Cl(P)) = 0_N$. Hence we get $\mu_N Int(\mu_N Fr(P)) = 0_N$ that implies us $P$ is a $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$.

Definition 3.2. A neutrosophic set in a $\mu_N$ Topological space is called $\mu_N$ simply open set in $(X, \mu_N)$ if $\mu_N Fr(P)$ is $\mu_N$ nowhere dense set in $(X, \mu_N)$. In otherwords, $P$ is $\mu_N$ simply open set iff $\mu_N Int(\mu_N Cl(\mu_N Fr(P))) = 0_N$ in $(X, \mu_N)$.

Proposition 3.9. If $P$ is a $\mu_N$ simply open set in a $\mu_N$ Topological space $(X, \mu_N)$, then $P$ is a $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$.

Proof. Let $P$ be a simply open set in $(X, \mu_N)$. Then $\mu_N Int(\mu_N Cl(\mu_N Fr(P))) = 0_N$ in $(X, \mu_N)$. But $\mu_N Int(\mu_N Fr(P)) \subseteq \mu_N Int(\mu_N Cl(\mu_N Fr(P)))$. From this we obtain that $\mu_N Int(\mu_N Fr(P)) = 0_N$ in $(X, \mu_N)$. Then by using proposition 3.8 we obtain that $P$ is a $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$.

Remark 3.1. The converse of the above proposition need not be true. This can be illustrated in the example given below.

Example 3.1. Let $(X, \mu_N)$ be a $\mu_N$ TS where $X = \{a, b\}$ and we define neutrosophic sets $\delta_1 = \{< 0.6, 0.4, 0.8 >, < 0.8, 0.6, 0.9 >\}, \delta_2 = \{< 0.6, 0.3, 0.8 >, < 0.9, 0.2, 0.7 >\}, \delta_3 = \{< 0.5, 0.4, 0.9 >, < 0.7, 0.8, 0.9 >\}, \delta_4 = \{< 0.4, 0.6, 0.9 >, < 0.6, 0.8, 0.9 >\}, \delta_5 = \{< 0.3, 0.7, 0.9 >, < 0.5, 0.9, 0.9 >\}$ and $\mu_N = \{0_N, \delta_1, \delta_2, \delta_3, \delta_4\}$ be a $\mu_N$ TS here the $\mu_N$ simply open sets are $\{0_N, \delta_2, \delta_5, 1_N\}$ and the $\mu_N$ strongly nowhere dense sets are $\{0_N, \delta_2, \delta_3, \delta_5, 1_N\}$. Here $\delta_5$ is $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$ but not $\mu_N$ simply open set in $(X, \mu_N)$.

Proposition 3.10. If $P$ is a $\mu_N$ closed set with $\mu_N Int(P) = 0_N$ in a $\mu_N$ TS $(X, \mu_N)$, then $P$ is a $\mu_N$ strongly nowhere dense set in $(X, \mu_N)$.
Proof. Let \( P \) be a \( \mu_N \) closed set with \( \mu_N \text{Int}(P) = 0_N \) in \((X, \mu_N)\). Then \( \mu_N \text{Int} \mu_N \text{Cl}(\mu_N \text{Cl}(P \cap \mu_N \text{Cl}(P))) = \mu_N \text{Int} \mu_N \text{Cl}(P \cap \mu_N \text{Cl}(P)) = \mu_N \text{Int} \mu_N \text{Cl}(P \cap 0_N) = \mu_N \text{Int}(\mu_N \text{Cl}(P)) = \mu_N \text{Int} \mu_N \text{Cl}(P) = 0_N \). Hence, we get \( P \) is a \( \mu_N \) simply open set in \((X, \mu_N)\). By proposition 3.9 deduce that \( P \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\). \( \square \)

**Proposition 3.11.** If \( P \) is a \( \mu_N \) open set and \( \mu_N \) dense set in a \( \mu_N \) Topological space \((X, \mu_N)\), then \( P \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\).

**Proof.** Let \( P \) be a \( \mu_N \) open and \( \mu_N \) dense set in \((X, \mu_N)\). Then \( \overline{P} \) is a \( \mu_N \) closed set with \( \mu_N \text{Int} \overline{P} = \mu_N \text{Int} \mu_N \text{Cl}(\eta \cap \mu_N \text{Cl}(\eta)) = 0_N \) in \((X, \mu_N)\). Then by using proposition 3.10 we retrieve that \( \overline{P} \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\) and by using the proposition 3.4 we obtain that \( \overline{P} \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\) which implies us that \( P \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\). \( \square \)

**Proposition 3.12.** Every subset of \( \mu_N \) strongly nowhere dense set is \( \mu_N \) strongly nowhere dense set.

**Proof.** Let \( P \) be a \( \mu_N \) strongly nowhere dense set, then \( \mu_N \text{Int}(\mu_N \text{Cl}(P \cap P)) = 0_N \). If \( \zeta \subseteq P \) we have \( \zeta \cap P \subseteq \zeta \cap \overline{P} \Rightarrow \mu_N \text{Int}(\mu_N \text{Cl}(\zeta \cap \overline{P})) = 0_N \). Therefore \( \zeta \) is a \( \mu_N \) strongly nowhere dense set. Hence, subset of \( \mu_N \) strongly nowhere dense set is \( \mu_N \) strongly nowhere dense set. \( \square \)

**Proposition 3.13.** A neutrosophic set is \( \mu_N \) strongly nowhere dense set if and only \( \mu_N \text{Cl}(\mu_N \text{Int}(\overline{P} \cap \overline{P})) = 1_N \).

**Proof.** Suppose \( P \) is \( \mu_N \) strongly nowhere dense then \( \mu_N \text{Int}(\mu_N \text{Cl}(P \cap P)) = 0_N \). Now, \( \mu_N \text{Cl}(\mu_N \text{Int}(\overline{P} \cap \overline{P})) = \mu_N \text{Cl}(\mu_N \text{Cl}(P \cap P)) = 0_N \). Conversely we assume that \( \mu_N \text{Cl}(\mu_N \text{Int}(\overline{P} \cap \overline{P})) = 1_N \). On considering, \( \mu_N \text{Int}(\mu_N \text{Cl}(P \cap P)) = 0_N \). Hence it is \( \mu_N \) strongly nowhere dense set. \( \square \)

**Proposition 3.14.** If \( P \) is \( \mu_N \) strongly nowhere dense set then \( \mu_N \text{Int}(P \cap P) = 0_N \).

**Proof.** Suppose \( P \) is \( \mu_N \) strongly nowhere dense then \( \mu_N \text{Int}(\mu_N \text{Cl}(P \cap P)) = 0_N \). Now, \( \mu_N \text{Int}(P \cap P) \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(P \cap P)) = 0_N \). Hence, \( \mu_N \text{Int}(P \cap P) = 0_N \). \( \square \)

4. \( \mu_N \) strongly first category sets in \( \mu_N \) TS:

**Definition 4.1.** A neutrosophic set is said to be \( \mu_N \) strongly first category set in \( \mu_N \) TS if \( \delta = \bigcup_{i=1}^{\infty} \delta_i \) where \( \delta_i \)'s are \( \mu_N \) strongly nowhere dense sets. The left out sets are called as \( \mu_N \) strongly second category sets. The complement of \( \mu_N \) strongly first category sets are called \( \mu_N \) strongly residual sets.

**Example 4.1.** Let \((X, \mu_N)\) be a \( \mu_N \) TS where \( X = \{a, b\} \) and we define neutrosophic sets \( L_1 = \{<0.6, 0.4, 0.8><0.8, 0.6, 0.9>\}, L_2 = \{<0.6, 0.3, 0.8><0.9, 0.2, 0.7>\}, L_3 = \{<0.5, 0.4, 0.9><0.7, 0.8, 0.9>\}, L_4 = \{<0.4, 0.6, 0.9><0.6, 0.8, 0.9>\}, L_5 = \{<0.3, 0.7, 0.9><0.5, 0.9, 0.9>\} \) and \( \mu_N = \{0_N, L_1, L_2, L_3, L_4\} \) be a \( \mu_N \) TS. Here \( \mu_N \) strongly first category set is \( L_2 = \{<0.6, 0.3, 0.8><0.9, 0.2, 0.7>\} \). The \( \mu_N \) strongly second category sets are \( 0_N, 1_N, L_1, L_3, L_4 \) and the \( \mu_N \) strongly residual set is \( \overline{L_2} \).
Proposition 4.1. If \( P \) is a \( \mu_N \) first category set then \( P \) is \( \mu_N \) strongly first category set.

Proof. Let \( P \) be a \( \mu_N \) first category set in a \( \mu_N \) TS. Then \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) nowhere dense sets in \( \mu_N \) TS. By making use of the fact “Every \( \mu_N \) nowhere dense set is \( \mu_N \) strongly nowhere dense set” we deduce that \( P_i \)'s are \( \mu_N \) strongly nowhere dense sets and hence \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) strongly nowhere dense sets. Therefore, \( P \) is \( \mu_N \) strongly first category set.

Remark 4.1. Every \( \mu_N \) strongly first category sets need not be \( \mu_N \) first category set. It is exemplified below.

Example 4.2. Let \( X = \{ a \} \) and \( \mu_N = \{ 0_N, A, C, E \} \) be a \( \mu_N \) TS where \( 0_N = \{ < 0, 1, 1 > \} \), \( A = \{ < 0.7, 0.8, 0.9 > \} \), \( B = \{ < 0.3, 0.4, 0.6 > \} \), \( C = \{ < 0.9, 0.7, 0.6 > \} \), \( 1_N = \{ < 1, 0, 0 > \} \). Here the \( \mu_N \) first category set is \( 0_N = \{ < 0, 1, 1 > \} \) and the \( \mu_N \) strongly first category set is \( C = \{ < 0.9, 0.7, 0.6 > \} \). From this clearly we deduce that the \( \mu_N \) strongly first category sets need not be \( \mu_N \) first category set.

Proposition 4.2. If \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) closed sets with \( \mu_N \text{Int} P = 0_N \) then \( P \) is a \( \mu_N \) strongly first category set.

Proof. Suppose \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) closed sets with \( \mu_N \text{Int} P = 0_N \) in a \( \mu_N \) TS \((X, \mu_N)\). Then by the fact, “If \( P \) is \( \mu_N \) closed in \( \mu_N \) TS with \( \mu_N \text{Int} (P_i) = 0_N \) then \( P \) is \( \mu_N \) strongly nowhere dense set”.

By making use of this theorem we deduce that \( P \) is \( \mu_N \) closed in \( \mu_N \) TS with \( \mu_N \text{Int} (P_i) = 0_N \). Thus, \( P \) is \( \mu_N \) strongly nowhere dense set and then we have \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) strongly nowhere dense sets. Thereupon \( P \) is a \( \mu_N \) strongly first category set.

Theorem 4.1. If \( P = \bigcup_{i=1}^{\infty} P_i \) where \( \mu_N \text{Int}(\mu_N \text{Fr}(P_i)) = 0_N \) then \( P \) is \( \mu_N \) strongly first category set.

Proof. Assume that \( P = \bigcup_{i=1}^{\infty} P_i \) where \( \mu_N \text{Int}(\mu_N \text{Fr}(P_i)) = 0_N \). By theorem, “If \( \mu_N \text{Int}(\mu_N \text{Fr}(P)) = 0_N \) for a \( \mu_N \) open set in \( \mu_N \) TS then \( P \) is \( \mu_N \) strongly nowhere dense set.” By making use of this we obtain that \( P_i \)'s are \( \mu_N \) strongly nowhere dense sets. Therefore, \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) strongly nowhere dense sets and hence \( P \) is \( \mu_N \) strongly first category set.

Theorem 4.2. If \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) open sets in \( \mu_N \) TS then \( P \) is \( \mu_N \) strongly first category set.

Proof. Given that \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) open sets in \( \mu_N \) TS and also \( \mu_N \) dense set in \( \mu_N \) TS. By theorem, “If \( P \) is a \( \mu_N \) open set in \( \mu_N \) TS and \( P \) is also \( \mu_N \) dense set in \((X, \mu_N)\), then \( P \) is a \( \mu_N \) strongly nowhere dense set in \((X, \mu_N)\). ” By making use of this theorem we obtain that \( P_i \)'s are \( \mu_N \) strongly nowhere dense sets and \( P = \bigcup_{i=1}^{\infty} P_i \), where \( P_i \)'s are \( \mu_N \) strongly nowhere dense sets. Thereupon we get \( P \) is \( \mu_N \) strongly first category set.

Theorem 4.3. Every subset of \( \mu_N \) strongly first category set is \( \mu_N \) strongly first category set.

Proof. Let \( P \) be a \( \mu_N \) strongly first category set. Then \( P = \bigcup_{i=1}^{\infty} P_i \) where \( P_i \)'s are \( \mu_N \) strongly nowhere dense sets. Suppose \( \zeta \subseteq P = \bigcup_{i=1}^{\infty} P_i \). From this we deduce that \( \zeta \subseteq \bigcup_{i=1}^{\infty} P_i \) which implies us that \( \zeta \subseteq P \) for some \( \mu_N \) strongly nowhere dense sets. By using proposition 3.14 we obtain that \( \zeta \) is \( \mu_N \) strongly first category set.

Remark 4.2. Superset of \( \mu_N \) strongly first category set need not be \( \mu_N \) strongly first category set. This can be explained in the below example.
Example 4.3. Let $\mu_N = \{0_N, \tau_a, \tau_b\}$ where $0_N = \{<0,0.1>, \tau_a = \{<0.1,0.4,0.6>, \tau_b = \{<0.2,0.3,0.5>, \tau_c = \{<0.6,0.6,0.1>, \tau_d = \{<0.5,0.7,0.2>. \text{ Here, } \tau_e = \{<0.6,0.6,0.1> \text{ is } \mu_N \text{ strongly first category set but } \tau_f = \{<0.6,0.6,0.1> \text{ but } \tau_g = \{<0.5,0.7,0.2> \text{ is not } \mu_N \text{ strongly first category set.}

5. $\mu_N$ Strongly Baire Space:

Definition 5.1. A $\mu_N$ TS is called $\mu_N$ strongly Baire space if $\mu_N Cl(\bigcup_{i=1}^{\infty}P_i) = 1_N$ where $P_i$’s are $\mu_N$ strongly nowhere dense sets.

Example 5.1. Let $X = \{a\} \text{ and } \mu_N = \{0_N, A, C, E\} \text{ be a } \mu_N \text{ TS where } 0_N = \{<0,1,1>, A = \{<0.7,0.8,0.9>, B = \{<0.3,0.4,0.6>, C = \{<0.9,0.7,0.6>, 1_N = \{<1,0,0>. \text{ Here the } \mu_N \text{ first category set is } 0_N = \{<0,1,1> \text{ and the } \mu_N \text{ strongly first category sets are } C = \{<0.9,0.7,0.6> \text{ and } 1_N = \{<1,0,0>. \text{ Then } \mu_N Cl(1_N) = 1_N \text{. Hence } (X, \mu_N) \text{ is a } \mu_N \text{ strongly Baire space.}

Theorem 5.1. Let $(X, \mu_N)$ be a $\mu_N$ TS. Then the following statements are parallel in nature.

1. $(X, \mu_N)$ is a $\mu_N$ strongly Baire space.
2. $\mu_N Cl(P) = 1_N$, for every $\mu_N$ strongly first category set.
3. $\mu_N Int(P) = 0_N$, for every $\mu_N$ residual sets.

Proof. (i) $\Rightarrow$ (ii). Let $P$ be a $\mu_N$ strongly first category set in $(X, \mu_N)$. Then $P = \bigcup_{i=1}^{\infty}P_i$ where $P_i$’s are $\mu_N$ strongly nowhere dense sets. Since $(X, \mu_N)$ is a $\mu_N$ strong Baire space we get $\mu_N Cl(\bigcup_{i=1}^{\infty}P_i) = 1_N$. Hence, $\mu_N Cl(P) = 1_N$.

(ii) $\Rightarrow$ (iii). Let $P$ be a $\mu_N$ strongly residual set in $(X, \mu_N)$. Then we retrieve that $P$ is a $\mu_N$ strongly first category set in $(X, \mu_N)$. From (ii) we obtain that $\mu_N Cl(P) = 1_N \Rightarrow \mu_N Int(P) = 1_N$. Hence, $\mu_N Int(P) = 0_N$.

(iii) $\Rightarrow$ (i). Let $P$ be a $\mu_N$ strongly first category set in $(X, \mu_N)$. Then $P = \bigcup_{i=1}^{\infty}P_i$ where $P_i$’s are a $\mu_N$ strongly nowhere dense sets. We have if $P$ is a $\mu_N$ strongly first category set then $P$ is a $\mu_N$ strongly residual set in $(X, \mu_N)$. Now by making use of (iii) we obtain that $\mu_N Int(P) = 0_N$ which gives us that $\mu_N Cl(P) = 0_N$. Therefore we get $\mu_N Cl(P) = 1_N$ and hence $\mu_N Cl(\bigcup_{i=1}^{\infty}P_i) = 1_N$ where $P_i$’s are a $\mu_N$ strongly nowhere dense sets. Hence, $(X, \mu_N)$ is a $\mu_N$ strongly Baire space.

Theorem 5.2. If $\{P_i\}, i = 1 \text{ to } \infty$ is $\mu_N$ open set and $\mu_N$ dense set in $\mu_N$ TS then $(X, \mu_N)$ is a $\mu_N$ strongly Baire space.

Proof. We know that, “If $\zeta$ is $\mu_N$ open set and $\mu_N$ dense then $\zeta$ is $\mu_N$ strongly nowhere dense sets”. By making use of this fact we obtain that $P_i$’s are $\mu_N$ strongly nowhere dense sets in $(X, \mu_N)$. Let $\xi = \bigcup_{i=1}^{\infty}\xi_i$ then $\xi_i$’s are $\mu_N$ strongly first category sets. Now, $\mu_N Cl(\xi_i) = \mu_N Cl(\bigcup_{i=1}^{\infty}\xi_i) \supseteq \bigcup_{i=1}^{\infty}\mu_N Cl(\xi_i) = 1_N$. Hence, $(X, \mu_N)$ is a $\mu_N$ strongly Baire space.

References