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The Schröder-Bernstein Problem for Objects in Grothendieck Categories

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Abstract: We show that the Schröder-Bernstein problem has a positive solution for pseudo-injective objects in any Grothendieck category. Later we obtain that the Schröder-Bernstein property is satisfied for pseudo pure-injective objects in a finitely accessible additive category. We also give an application to exactly definable additive categories.

Key words: Schröder-Bernstein problem, pseudo-injective object, Grothendieck category, pseudo pure-injective object, finitely accessible additive category, exactly definable additive category

1. Introduction

Let $\mathcal{C}$ be any additive category and $\mathcal{X}$ a class of objects in $\mathcal{C}$ closed under isomorphisms and direct summands. An $\mathcal{X}$-*preenvelope* of an object M in $\mathcal{X}$ is a morphism $u : M \to X$ with $X \in \mathcal{X}$ such that any other morphism $g : M \to X'$ with $X' \in \mathcal{X}$ factor through u, in other words, $\mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}_{\mathcal{C}}(M, X') \to 0$ is exact. A preenvelope $u : M \to X$ is called an $\mathcal{X}$-*envelope* if any endomorphism $h : X \to X$ such that $hu = u$ is an isomorphism. $\mathcal{X}$-*cover* is defined in a dual manner (see [11]). When $\mathcal{X}$ is the class of all injective objects, the $\mathcal{X}$-envelope $u : M \to X$ is called an *injective envelope*. We should note that in any Grothendieck category $\mathcal{C}$, every object M has an injective envelope which will be denoted by $E(M)$ and M is an essential subobject of $E(M)$ (see [10]). Recall that A is an *essential subobject* of B in any additive category means that $A \cap C$ is nonzero, whenever C is a nonzero subobject of B and it is denoted by $A \subseteq_e B$.

Let R be any ring with identity and let M, N be two right R-modules. M is called *pseudo-N-injective* if for every monomorphisms $f : Y \to N$ and $g : Y \to M$, where Y is any right R-module, there exists a homomorphism $\varphi : N \to M$ such that $\varphi f = g$ (see [2]). We can generalize this definition to any additive category $\mathcal{C}$: Let M and N be two objects in an additive category $\mathcal{C}$. M is called *pseudo-N-injective* if for every monomorphisms $f : Y \to N$ and $g : Y \to M$, where Y is any object in $\mathcal{C}$, there exists a morphism $\varphi : N \to M$ such that $\varphi f = g$. It is easy to see that if M is a pseudo-N-injective object and A is a subobject of N, then M is also pseudo-A-injective. If M is pseudo-M-injective, then M is called *pseudo-injective*. Here we would like to prove the following easy observation:

Lemma 1.1. *Let $\mathcal{C}$ be an abelian category and M a pseudo-injective object in $\mathcal{C}$. Then $\Delta = \{f \in \mathrm{End}_{\mathcal{C}}(M) \mid \mathrm{Ker} f \subseteq_e M\} \subseteq J(\mathrm{End}_{\mathcal{C}}(M))$, where $J(\mathrm{End}_{\mathcal{C}}(M))$ is the Jacobson radical of $\mathrm{End}_{\mathcal{C}}(M)$.*

Proof. Let $f : M \to M$ be any endomorphism and $\mathrm{Ker} f \subseteq_e M$. Since $\mathrm{Ker} f \cap \mathrm{Ker}(1 - f) = 0$, $1 - f : M \to M$ is a monomorphism. Considering the identity morphism $1_M : M \to M$, we have an endomorphism $\varphi : M \to M$

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such that $\varphi(1-f) = 1_M$. This means that $1-f$ splits hence $(1-f)(M)$ is a direct summand of M. On the other hand, $\mathrm{Ker}f \subseteq (1-f)(M) \subseteq M$ and $\mathrm{Ker}f \subseteq_e M$ imply that $(1-f)(M) \subseteq_e M$. Namely, $(1-f)(M) = M$. Therefore $1-f$ is an isomorphism. Since Δ is an ideal of $\mathrm{End}_{\mathcal{C}}(M)$, for all $g \in \mathrm{End}_{\mathcal{C}}(M)$, $fg \in \Delta$. Now $1-fg$ is an isomorphism for all $g \in \mathrm{End}_{\mathcal{C}}(M)$ from the above proof. So, $f \in J(\mathrm{End}_{\mathcal{C}}(M))$. □

[7] is a perfect work to introduce Schröder-Bernstein problem. In this paper [7, Theorem 3.1], among the others, they prove that if M and N are automorphism-invariant modules, which are equivalently pseudo-injective modules by [6] and if $f: M \to N$ and $g: N \to M$ are two monomorphisms, then $M \cong N$. In other words they showed that the Schröder-Bernstein problem has a positive solution for any pseudo-injective modules.

In our paper we will generalize this result to any Grothendieck category. Later we give applications to exactly definable additive categories and to finitely accessible additive categories. We should note that, from the duality principle on abelian categories, dual results can be obtained in this work easily on Grothendieck categories.

2. The Schröder-Bernstein problem for pseudo-injective objects in Grothendieck categories

Theorem 2.1. *Let $\mathcal{C}$ be a Grothendieck category and let M, N be objects in $\mathcal{C}$ with $M \subseteq N$, both injective. Assume there is a monomorphism $\varphi: N \to M$. Then $M \cong N$.*

Proof. We follow the proof of [1, Theorem]. Now since M is injective in $\mathcal{C}$, $N = H \oplus M$. Then $N = H \oplus M \supseteq H \oplus \varphi(N) = H \oplus \varphi(H) \oplus \varphi(M) \supseteq H \oplus \varphi(H) \oplus \varphi^2(H) \oplus \varphi^2(M) \supseteq \cdots$. Say $P = H \oplus \varphi(H) \oplus \varphi^2(H) \oplus \cdots$. Thus $P \cap M = \varphi(H) \oplus \varphi^2(H) \oplus \cdots$, namely $\varphi(P) = P \cap M$.

Let Q be a maximal essential extension of $P \cap M$ in M. Since M is injective, Q is injective. Hence $M = Q \oplus K$ for some subobject K of M. Also, $N = H \oplus (Q \oplus K) = (H \oplus Q) \oplus K$. So, $H \oplus Q$ is injective. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "\varphi"'] & H \oplus Q \arrow[dl, "\overline{\varphi}"] \\ & Q & \end{tikzcd}$$

to construct $\overline{\varphi}: H \oplus Q \to Q$. Note that $\overline{\varphi}$ is an isomorphism. Finally, the morphism $\overline{\varphi} \oplus 1 : (H \oplus Q) \oplus K \to Q \oplus K$ is the required isomorphism. □

Corollary 2.1. ([1, Corollary 1]) *Let $\mathcal{C}$ be a Grothendieck category and let M and N be any two objects which are isomorphic to subobjects of each other. Then the injective envelopes $E(M)$ and $E(N)$ are isomorphic.*

Proof. If $\varphi: M \to N$ is a monomorphism, then any morphism $\overline{\varphi}: E(M) \to E(N)$ which extends φ is a monomorphism. Similarly, if we have the monomorphism $\psi: N \to M$ then we obtain a monomorphism $\overline{\psi}: E(M) \to E(N)$ extending ψ. By Theorem 2.1, $E(M) \cong E(N)$. □

Theorem 2.2. *Let $\mathcal{C}$ be a Grothendieck category. Let M, N be pseudo-injective objects in $\mathcal{C}$ and let $f: M \to N$ and $g: N \to M$ be monomorphisms. Then $M \cong N$.*

Proof. An easy adaptation of the proof of [7, Theorem 3.1]. □

Remark 2.1. *Theorem 2.2 can olso be proved via the techniques in the proof of [4, Proposition 2.1] and using Gabriel-Popescu Theorem.*

3. Applications

In this section we give applications of Schröder-Bernstein Problem to exactly definable additive categories and finitely accessible additive categories. Following Crawley- Boevey [3], Krause [8] and Prest [9], we recall some terminology on exactly definable additive categories and finitely accessible additive categories.

An additive category $\mathcal{A}$ is called *exactly definable* if it is equivalent to the category $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ of exact contravariant additive functors from $\mathcal{C}$ to the category Ab of abelian groups for some skeletally small abelian category $\mathcal{C}$. The category of unitary modules over a ring with enough idempotents, the category of torsion abelian groups [9, Example 10.2], and the category of torsion-free abelian groups [9, Example 10.5] are exactly definable additive categories.

Purity may be defined in exactly definable additive categories. Let $\mathcal{A}$ be an exactly definable additive category. An object M of $\mathcal{A}$ is called *pure-injective* if for every set I the summation morphism $M^{(I)} \to M$ factors through the canonical morphism $M^{(I)} \to M^I$ [8]. A sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathcal{A}$ is called *pure exact* if it induces an exact sequence of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Z, Q) \to \mathrm{Hom}_{\mathcal{A}}(Y, Q) \to \mathrm{Hom}_{\mathcal{A}}(X, Q) \to 0$$

for every pure-injective object Q of $\mathcal{A}$. This implies that f and g form a kernel-cokernel pair, that f is a monomorphism and g an epimorphism. In such a pure exact sequence f is called a *pure-monomorphism* and g a *pure-epimorphism.*

Let $\mathcal{A}$ be an exactly definable additive category. Then there exist a locally coherent Grothendieck category $\mathcal{D}(\mathcal{A})$ (uniquely determinated up to equivalence) and a fully faithful functor $T : \mathcal{A} \to \mathcal{D}(\mathcal{A})$ (naturally isomorphic to the inclusion functor), which induces an equivalence between $\mathcal{A}$ and the full subcategory of fp-injective objects of $\mathcal{D}(\mathcal{A})$. Moreover, a sequence in $\mathcal{A}$ is pure exact if and only if T takes it into an exact sequence in $\mathcal{D}(\mathcal{A})$ and the above equivalence restricts to equivalence between pure-injective objects of $\mathcal{A}$ and injective objects of $\mathcal{D}(\mathcal{A})$ [8, Theorem 2.8].

An additive category $\mathcal{A}$ is called *finitely accessible* if it has direct limits, the class of finitely presented objects $\mathcal{A}_0$ is skeletally small and every object is a direct limit of finitely presented objects. The category of unitary modules over a ring with enough idempotents, the category of torsion abelian groups [9, Example 10.2] and the category of torsion-free abelian groups [9, Example 10.5] are typical examples of finitely accessible additive categories.

By a *sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in the finitely accessible additive category $\mathcal{A}$ we main a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ such that $gf = 0$. The sequence is called *pure exact* provided it induces an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(F, X) \to \mathrm{Hom}_{\mathcal{A}}(F, Y) \to \mathrm{Hom}_{\mathcal{A}}(F, Z) \to 0$$

for all finitely presented objects F of $\mathcal{A}_0$. In this case, f is called a *pure-monomorphism* and g a *pure-epimorphism*. An object M of $\mathcal{A}$ is called *pure-injective* if every pure exact sequence in $\mathcal{A}$ with the first term M splits.

Let $\mathcal{E} = \mathrm{Func}(\mathcal{A}_0^{\mathrm{op}}, \mathrm{Ab})$ be the category of all contravariant (additive) functors from $\mathcal{A}_0$ to the category of abelian groups. Then, the category $\mathcal{E}$ is a locally finitely presented Grothendieck category. A contravariant functor F in $\mathcal{E}$ is *flat* if it is a direct limit of finitely generated projective objects. Using Yoneda's Lemma, it is well known that $\mathcal{A}$ is equivalent to the full subcategory $\mathcal{F} = \mathrm{Flat}(\mathcal{A}_0^{\mathrm{op}}, \mathrm{Ab})$ of all flat objects in $\mathcal{E}$ by the full and faithful functor $H : \mathcal{A} \to \mathcal{E}$ defined as $H(A) = \mathrm{Hom}(-, \mathrm{A})$. Moreover, a sequence in $\mathcal{F}$ is pure exact if and only if H takes it into an exact sequence $\mathcal{E}$.

In this section we assume the full subcategory $\mathcal{F}$ of $\mathcal{E}$ which is equivalent to finitely accessible additive category $\mathcal{A}$ is closed under homomorphic images as a subcategory in $\mathcal{E}$.

Every finitely accessible additive category with products is exactly definable [3, 3.3]. In general, exactly definable additive categories need not be finitely accessible. For example, the category of divisible abelian groups is exactly definable, but not finitely accessible [9, Example 10.3].

Definition 3.1. Let M, N be objects in a finitely accessible additive category $\mathcal{A}$. M is called *pseudo pure-N-injective* if every pure-monomorphisms $f : Y \to N$ and $g : Y \to M$ in $\mathcal{A}$ there exists a homomorphism $\varphi : N \to M$ such that $\varphi f = g$. If M is pseudo pure-M-injective, then M is called *pseudo pure-injective.*

Recall that an object M of a Grothendieck category $\mathcal{C}$ is *fp-injective* if it has the injectivity property with respect to any short exact sequence $0 \to A \to B \to C \to 0$ with C finitely presented (see [5]).

Proposition 3.1. *Let M be an object in an exactly definable additive category $\mathcal{A}$ such that every homomorphic image of M is finitely presented in $\mathcal{D}(\mathcal{A})$. Then, M is pseudo-injective in $\mathcal{D}(\mathcal{A})$.*

Proof. Let $f : Y \to M$ and $g : Y \to M$ be monomorphisms in $\mathcal{D}(\mathcal{A})$. Since M is fp-injective in $\mathcal{D}(\mathcal{A})$ and every homomorphic image of M is finitely presented in $\mathcal{D}(\mathcal{A})$ also, there exists a homomorphism $\varphi : M \to M$ such that $\varphi f = g$. □

Theorem 3.1. *Let $\mathcal{A}$ be an exactly definable additive category. Let M, N be two objects in $\mathcal{A}$ such that every homomorphic image of M and N are finitely presented in $\mathcal{D}(\mathcal{A})$ and, let $f : M \to N$ and $g : N \to M$ be pure-monomorphisms in $\mathcal{A}$. Then $M \cong N$.*

Proof. Consider the fully faithful functor $T : \mathcal{A} \to \mathcal{D}(\mathcal{A})$. By Proposition 3.1, $T(M)$ and $T(N)$ are pseudo-injective objects in $\mathcal{D}(\mathcal{A})$. Also $T(f) : T(M) \to T(N)$, $T(g) : T(N) \to T(M)$ are monomorphisms in $\mathcal{D}(\mathcal{A})$. Then by Theorem 2.2, $T(M) \cong T(N)$. Suppose $\Psi : T(M) \to T(N)$ be any isomorphism in $\mathcal{D}(\mathcal{A})$. Then there exists a morphism $\Phi : T(N) \to T(M)$ such that $\Psi\Phi = 1_{T(N)}$ and $\Phi\Psi = 1_{T(M)}$. Since T is full, there exist morphisms $\alpha : N \to M$ and $\beta : M \to N$ in $\mathcal{A}$ such that $T(\alpha) = \Phi$ and $T(\beta) = \Psi$. Now $T(\alpha\beta) = T(\alpha)T(\beta) = 1_{T(M)} = T(1_M)$ and $T(\beta\alpha) = T(\beta)T(\alpha) = 1_{T(N)} = T(1_N)$. Since T is faithful, $\alpha\beta = 1_M$ and $\beta\alpha = 1_N$. This means that $M \cong N$. □

Proposition 3.2. *Let M be an object in a finitely accessible addditive category $\mathcal{A}$. If M is pseudo pure-injective in $\mathcal{A}$ then it is pseudo-injective in $\mathcal{E}$.*

Proof. Assume M is pseudo pure-injective in $\mathcal{A}$ and, $f : Y \to M$ and $g : Y \to M$ are monomorphisms in $\mathcal{E}$. Since $\mathcal{A}$ is closed under homomorphic images as a subcategory in its functor category, Y is flat in $\mathcal{E}$. Hence, f and g are pure-monomorphisms in $\mathcal{A}$. Since M is pseudo pure-injective in $\mathcal{A}$, there exists $\varphi : M \to M$ such that $\varphi f = g$ in $\mathcal{A}$ and so does in $\mathcal{E}$. Thus, M is pseudo-injective in $\mathcal{E}$. □

Theorem 3.2. *Let $\mathcal{A}$ be a finitely accessible additive category. Let M, N be two pseudo pure-injective objects and let $f : M \to N$ and $g : N \to M$ are pure-monomorphisms in $\mathcal{A}$. Then $M \cong N$.*

Proof. Assume M, N be two pseudo pure-injective objects and let $f : M \to N$ and $g : N \to M$ are pure-monomorphisms in $\mathcal{A}$. Then, by Proposition 3.2, M and N are pseudo-injective objects and $f : M \to N$ and $g : N \to M$ are monomorphisms in $\mathcal{E}$. Since $\mathcal{E}$ is a Grothendieck category, apply Theorem 2.2 to conclude. □

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