Asia Mathematika
Volume: 6 Issue: 2 , (2022) Pages: 61 – 74
Available online at **www.asiamath.org**

Applications of Hyperideals in Characterizations of Left Regular LA-semihyperrings

Tauseef Asif[1], Murad Ul Islam Khan[1], Asghar Khan[2], Mohammed M. Al-Shamiri[3,4] and Mohammed M. Khalaf[5*]
[1]Department of Mathematics and Statistics, The University of Haripur, KP, Pakistan
[2]Department of Mathematics, Abdul Wali Khan University, Mardan, KP, Pakistan
[3]Department of Mathematics, Faculty of Science and Arts, King Khalid University, Muhayl Assir, Saudi Arabia
[4]Department of Mathematics and computer, Faculty of science, Ibb University,Ibb, Yemen
[5]Faculty of Engineering, Arab Academy for Science & Technology and Maritime Transport (AASTMT), Aswan Branch, Egypt

Received: 02 Aug 2022 • **Accepted:** 12 Aug 2022 • **Published Online:** 31 Aug 2022

Abstract: The main objective of this paper is to investigate the class of left regular LA-semihyperrings with respect to their hyperideals. Then, produce the intresting characterizations of left regular LA-semihyperrings with respect to their hyperideals. In this connection, we prove that right (left, two sided, inerior, bi-, generalized bi-, quasi) hyperideals are coincide in a left regular LA-semihyperrings having pure left identity, these hyperideals are normally not coincide in other classes of regularities of LA-semihyperrings.

Key words: Left invertive law, left regular, hyperideals, LA-semihyperrings, left idetity.

1. Introduction

A new algebraic structure called left almost semigroup (for short, LA-semigroup) [13], initiated by Kazim and Naseeruddin in 1972. This structure is additionally called as Abel-Grassmann's Groupoid (for short, AG-Groupoid) by Protic and Stevanovic [14]. That algebraic structure is non commutative and non associative, lying in middle of groupoid and commutative semigroup possess numerous applications in the theory of flocks [39]. Mushtaq and Kamran named an AG-Groupoid with weak associative law [11] as AG*-Groupoid. The generality of an AG-Groupoid having left identity was called an AG**-Groupoid. Protic and Stevanovic have also introduced a useful technique for confirmation of AG-Groupoid, AG**-Groupoid and AG*-Groupoid in [12]. Khan and Asif [15, 16] characterized intra-regular and regular LA-semigroup with respect to their fuzzy ideals in 2010. Khan and et al. [17] characterize right regular LA-semigroup with respect to their fuzzy ideals. Yousafzai et al. [19] characterize weakly regular LA-semigroup by their smallest fuzzy ideals. Further, Sezer [18] apply the idea of soft sets to LA-semigroup and produce characterization of intra-regular, completely regular, regular, quasi-regular and weakly regular LA-semigroup. Currently, much researchers explored numerious characterizations of LA-semigroup (see, [20–22]). Moreover, few researchers have examine the concept of LA-semirings, that is a generalization of LA-rings [23]. Massouros and Yaqoob [25] studies the right and left almost groups and Rehman et al. [26] explore the idea of neutrosophic LA-rings and studies several types of ideals neutrosophic LA-rings.

*Correspondence: khalfmohammed2003@aast.edu

In 1934 is the first occasion when the idea of algebraic hyperstructure was floated for the first time by a French Mathematician Marty [27]. Hyperstructures have a distinct advantage over classical algebraic structures because the application of binary operation in hyperstructure produce a set, if this set is restricted to a singleton element, it effectively generalizes researchers to investigate these hyperstructures in different branches of mathematics. Various books have been written on hyperstructures, (see [10, 28]). Some authors explored different features of semihypergroups, like Davvaz et al. [4], Drbohlav et al. [5], Gutan [6], Hedayati [7], Hila et al. [8], Leoreanu[9] and Onipchuk [3]. Currently, Hila and Dine [1], initiate the idea of AG-hypergroupoids that is a generalization of semihypergroups, semigroups and AG-Groupoids.

Rehman et al.[29], introduce the concept of LA-hyperrings and characterize LA-hyperrings with respect to their hyperideals. In 2020, Hu et al. [30] apply the idea of neutrosophic set to LA-hypergroups. The idea of LA-semihypergroups was introduced by Hila and Dine [31] and an LA-semihypergroup is lies in middle of hypergroupoid and commutative semihypergroup. Yaqoob et al. in [32] give the characterizations intra-regular LA-semihypergroups using right and left hyperideals. Gulistan et al. [33] studies the class of regular LA-semihypergroups with respect to $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic hyperideals. Moreover, Khan et al. [34] explore few characteristics of fuzzy right and left hyperideals in intra-regular and regular LA-semihypergroups.

In 2019, Nakkhasen and Pibaljommee [38], investigate.the intra-regular class of semihyperrings. Later on, Nawaz et al. [37] introduce the idea of left almost semihyperrings shortenly called LA-semihyperrings, that is a generality of LA-semirings. Currently, Nakkhasen [35] characterizes regular and weakly regular LA-semihyperrings with repect to their hyperideals. Furthermore, Nakkhasen [36], consider the intra-regular class of LA-semihyperrings and characterize intra-regular LA-semihyperrings using their hyperideals.

In current paper, we focused in left regular class of LA-semihyperrings. We gave few interesting characterizations of left regular LA-semihyperrings by the properties of their hyperideals. Further more, we prove that right (left, two sided, inerior, bi-, generalized bi-, quasi) hyperideals are coincide in a left regular LA-semihyperrings having left identity, these hyperideals are normally not coincide in other classes of regularities of LA-semihyperrings.

2. Preliminaries

This section contains few definitions and results that are helpful in upcoming work. A maping $\circ : H \times H \to P^*(H)$ is knows as hyperoperation on the set H, where H is nonempty set and $P^*(H) = P(H)\backslash\{\emptyset\}$ represents the all nonempty subsets of H. An ordered pair $(H, \circ)$ is known as hypergroupoid, where $H \neq \emptyset$ and "$\circ$" is hyperoperation.

If $\emptyset \neq A, B \subseteq H$, then $A \circ B = \bigcup\limits_{p \in A, q \in B} p \circ q$, $p \circ A = \{p\} \circ A$ and $p \circ B = \{p\} \circ B$.

A hypergroupoid $(H, \circ)$ is known as LA-semihyperring [1], if is satisfies, $(p \circ q) \circ r = (r \circ q) \circ p$, $\forall$ $p, q, r \in H$. This is called a left invertive law. For a nonempty subset $X, Y,$and Z of an LA-semihyperring H, means that $(X \circ Y) \circ Z = (Z \circ Y) \circ X$.

A hyperstructure $(R, +, \circ)$ is known as an LA-semihyperring [37], if it satisfies:

(i) $(R, +)$ is an LA-semihypergroup;

(ii) $(R, \circ)$ is an LA-semihypergroup;

(iii) $\xi_1 \circ (\xi_2 + \xi_3) = (\xi_1 \circ \xi_2) + (\xi_1 \circ \xi_3)$ and $(\xi_2 + \xi_3) \circ \xi_1 = (\xi_2 \circ \xi_1) + (\xi_3 \circ \xi_1)$ for all $\xi_1, \xi_2, \xi_3 \in R$.

Example 2.1. *Let* $R = \{\xi_1, \xi_2, \xi_3\}$ *with the binary hyperoperation defined below:*

$+$	ξ_1	ξ_2	ξ_3
ξ_1	ξ_1	$\{\xi_1, \xi_2, \xi_3\}$	$\{\xi_1, \xi_2, \xi_3\}$
ξ_2	$\{\xi_1, \xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$
ξ_3	$\{\xi_1, \xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$

$\circ$	ξ_1	ξ_2	ξ_3
ξ_1	$\{\xi_1, \xi_3\}$	ξ_3	$\{\xi_2, c\}$
ξ_2	$\{\xi_2, c\}$	ξ_3	ξ_3
ξ_3	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$

It is easy to see that R *is an LA-semihyperring. It is also notice that* ξ_1 *is right identity but* ξ_1 *is not left identity.*

It is a common reality that if an LA-semihyperring carries a pure right identity (Pure-RI), then it become a pure identity (Pure-I) and an LA-semihyperring having pure identity (Pure-I) is coincide with commutative semihypergroup with pure identity (Pure-I). The example below shows that if an LA-semihyperring R carries a right identity, then it not become a left identity.

Example 2.2. *Let* $R = \{\xi_1, \xi_2, \xi_3\}$ *with the binary hyperoperations* $+$ *and* $\circ$ *defined below:*

$+$	ξ_1	ξ_2	ξ_3
ξ_1	ξ_1	$\{\xi_1, \xi_2, \xi_3\}$	$\{\xi_1, \xi_2, \xi_3\}$
ξ_2	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$
ξ_3	$\{\xi_1, \xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$

$\circ$	ξ_1	ξ_2	ξ_3
ξ_1	$\{\xi_1, c\}$	ξ_2	$\{\xi_2, \xi_3\}$
ξ_2	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$
ξ_3	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$	$\{\xi_2, \xi_3\}$

Clearly R *satisfies left invertive law, so it is an LA-semihyperring and it is simple to observe that* ξ_1 *is the identity of* R *but* R *is neither commutative and nor associative .*

In upcoming work, we call R is an LA-semihyperring rather than $(R, +, \circ)$ and we write pq instead of $p \circ q \ \forall \ p, q \in R$.

The bellow listed conceps are occurred in [37], will apply in this research. In every LA-semihyperring R, the medial law $(pq)(rs) = (pr)(qs)$ holds $\forall \ p, q, r, s \in R$. A member $e \in R$ is known as left identity (resp., Pure-LI) if $r \in er$ (resp., $r = er$) $\forall \ r \in R$. Each LA-semihyperring R having a Pure-LI e satisfies the following two law: $p(qr) = q(pr)$ and $(pq)(rs) = (sr)(qp)$, $\forall \ p, q, s, r \in R$. The second law is known as paramedical law. A member $r \in R$ having left identity (resp., Pure-LI) e is known as left invertible (resp., pure left invertible) if there exists $r^{'} \in R$ such that $e \in r^{'}a$ (resp., $e = r^{'}a$). An LA-semihyperring R is known as left invertible (resp., pure left invertible) if each member of R is left invertible (resp., pure left invertible). An LA-semihyperring R having Pure-LI e become a left identity, but conversly it is invalid generally, see in [35].

The given law holds in an LA-semihyperrings R, $(LM)(NO) = (LN)(MO)$ for each nonempty subsets L, M, N, O of R . If R carries a Pure-LI e, then R also satisfies, $(LM)(NO) = (ON)(ML)$ and $L(MN) = M(LN)$ for each nonempty subsets L, M, N, O of R.

Suppose R is an LA-semihyperring and $\emptyset \neq I \subseteq R$ such that $I + I \subseteq I$. Then:

(i) I is known as right hyperideal [37] of R if $IR \subseteq I$;

(ii) I is known as left hyperideal [37] of R if $RI \subseteq I$;

(iii) I is known as hyperideal [37] of R if $IR \subseteq I$ and $RI \subseteq I$;

(iv) I is known as quasi-hyperideal [37] of R if $RI \cap IR \subseteq I$;

(v) I is known as generalized bi-hyperideal [37] of R if $(IR)I \subseteq I$.

(vi) I is known as bi-hyperideal [37] of R if $II \subseteq I$ and $(IR)I \subseteq I$.

(vii) I is known as interior hyperideal [37] of R if $II \subseteq I$ and $(RI)R \subseteq I$.

Lemma 2.1. *[35] If P is left and Q is right hyperideal of an LA-semihyperring R, then $P \cap Q$ is quasi-hyperideal of R.*

Lemma 2.2. *An LA-semihyperring R containing left identity e satisfies, $R \circ R = R$.*

Proof. Suppose R is an LA-semihyperring with left identity e. Then any $r \in R \implies r \in e \circ r \subseteq R \circ R$, therefore $R \subseteq R \circ R$. Thus $R = R \circ R$. □

Corollary 2.1. *An LA-semihyperring R containing Pure-LI satisfies, $R = e \circ R = R \circ e$ and $R \circ R = R$.*

Lemma 2.3. *If an LA-semihyperring R contains Pure-LI, then the given conditions hold.*

(i) $RI = I$ for each left hyperideal I of R.

(ii) $JR = R$ for each right hyperideal J of R.

Proof. It is simple. □

Lemma 2.4. *If B is a bi-hyperideal of an LA-semihyperring R having Pure-LI, then $(rB)s$ is a also bi-hyperideal of R, for any r, $s \in R$.*

Proof. Suppose B is a bi-hyperideal of R. Then $B + B \subseteq B$, $BB \subseteq B$, and $(BR)B \subseteq B$. Thus

$$\begin{aligned}
(((rB)s)R)((rB)s) &= ((Rs)(rB))((rB)s) = (((rB)s)(rB))(Rs) \\
&= (((rB)r)(sB))(Rs) = (((rB)r)R)((sB)s) \\
&= ((Rr)(rB))((sB)s) = ((Br)(rR))((sB)s) \\
&= ((Br)(sB))((rR)s) \subseteq ((Br)(sB))R \\
&= ((Br)(sB))(eR) = ((Br)e)((sB)R) \\
&= ((er)B)((RB)s) = (rB)((RB)s) \\
&= (s(RB))(Br) = ((es)(RB))(Br) \\
&= ((BR)(se))(Br) = ((BR)B)((se)r) \\
&\subseteq B((se)r) = (se)(Br) = (rB)(es) = (rB)s.
\end{aligned}$$

Hence $(rB)s$ is a bi-hyperideal of R. □

Lemma 2.5. *In an LA-semihyperring R with Pure-LI, each idempotent quasi-hyperideal is a bi-hyperideal of R.*

Proof. Suppose Q is an idempotent quasi-hyperideal of R, then obviously Q is an LA-subsemihyperring. Thus

$$(QR)Q \subseteq (QR)R \subseteq (RR)Q = RQ, \text{ and}$$

$$(QR)Q \subseteq (RR)(QQ) = (QQ)(RR) = QR.$$

Therefore, $(QR)Q \subseteq QR \cap RQ \subseteq Q$. Thus Q is a bi-hyperideal of R. □

Lemma 2.6. *If L and M are quasi-hyperideal of an LA-semihyperring R having Pure-LI, where L is idempotent, then LM is a bi-hyperideal of R.*

Proof. Suppose L and M is a quasi-hyperideals of R, and L is an idempotent. Thus by using Lemma 2.5, we have

$$\begin{aligned}((LM)R)(LM) &= ((RM)L)(LM) \subseteq ((RR)L)(LM) \\ &= (RL)(LM) = (ML)(LR) = ((LR)L)M \subseteq LM.\end{aligned}$$

□

Lemma 2.7. *A subset P of an LA-semihyperring R having Pure-LI is right hyperideal of $R \iff P$ is an interior hyperideal of R.*

Proof. Suppose P is a right hyperideal of R, then clearly P is an hyperideal of R, so is an interior hyperideal of R.

Conversely, suppose P is an interior hyperideal of R. Thus

$$PR = P(RR) = R(PR) = (RR)(PR) = (RP)(RR) = (RP)R \subseteq P.$$

□

Lemma 2.8. *If P is right hyperideal or Q is left hyperideal of an LA-semihyperring R having Pure-LI then $P \cup RP$ and $Q \cup QR$ are hyperideals of R.*

Proof. Suppose P is right hyperideal of R. Then $P + P \subseteq$ and $PR \subseteq P$. Thus

$$\begin{aligned}(P \cup RP)R &= PR \cup (RP)R \subseteq P \cup (RP)(RR) \\ &= P \cup (RR)(PR) = P \cup R(PR) \\ &= P \cup P(RR) = P \cup PR = P \subseteq (P \cup RP), \text{ also} \\ R(P \cup RP) &= RP \cup R(RP) = RP \cup (RR)(RP) \\ &= RP \cup (PR)(RR) \subseteq RP \cup P(RR) \\ &= RP \cup PR \subseteq RP \cup P = P \cup RP.\end{aligned}$$

Hence $(P \cup RP)$ is an hyperideal of R. In a similar manner $(Q \cup RQ)$ is also an hyperideal of R. □

Definition 2.1. If I and J are hyperideals of an LA-semihyperring R, such that $I^2 \subseteq J$ implies $I \subseteq J$, then J is called semiprime.

Theorem 2.1. *If an LA-semihyperring R contains a Pure-LI, then the given conditions are identical.*

(i) If I and J are hyperideals of R, then $I^2 \subseteq J$ implies $I \subseteq J$.

(ii) If A is right and J is hyperideal of R then $A^2 \subseteq J$ implies $A \subseteq J$.

(iii) If B is left and J is a hyperideal of R then $B^2 \subseteq J$ implies $B \subseteq J$.

Proof. Suppose B is a left hyperideal of R and $B^2 \subseteq J$, then by Lemma 2.8, $B \cup BR$ is an hyperideal of R, therefore by hypothesis (i), $(B \cup BR)^2 \subseteq J$ which implies $(B \cup BR) \subseteq J$ which further implies that $B \subseteq J$.

(iii) $\implies$ (ii) and (ii) $\implies$ (i) are simple. □

Theorem 2.2. *An hyperideal J of an LA-semihyperring R with Pure-LI is semiprime $\Longleftrightarrow$ $a^2 \subseteq J$ implies $a \in J$.*

Proof. Let J be a semiprime hyperideal of R and $a^2 \subseteq J$. Since Ra^2 is a left hyperideal of R and $a^2 \subseteq Ra^2$, also $a^2 \subseteq J$, therefore $a^2 \subseteq Ra^2 \subseteq J$. Thus $Ra^2 = R(aa) = (RR)(aa) = (Ra)(Ra) = (Ra)^2$. Therefore $(Ra)^2 \subseteq J$, but J is semiprime so $Ra \subseteq J$. Since $a \in Ra$, therefore $a \in J$.

Conversely, suppose I is an hyperideal of R and let $I^2 \subseteq J$ and $a \in I$ implies that $a^2 \subseteq I^2$, which implies that $a^2 \subseteq J$ which further implies that $a \in J$. Therefore $I^2 \subseteq J$ implies $I \subseteq J$. Hence J is semiprime. □

3. Left Regular LA-semihyperring

In current section, the notion of left regular LA-semihyperrings is defined and few of its properties are studies.

Definition 3.1. A member a of an LA-semihyperring R is known as left regular if there exists $r \in R$ such that $a \in ra^2$, and R is known as left regular if each member of R is left regular.

Example 3.1. *In Example 2.2, we can show that there exists $r \in R$ such that $a \in ra^2$ $\forall$ $a \in R$. Therefore, R is left regular LA-semihyperring.*

Note that in a left regular LA-semihyperring R and an LA-semihyperring R with left identity, $R^2 = R$.

Lemma 3.1. *If B is bi- (generalized bi-) hyperideal of a left regular LA-semihyperring R then $(BR)B = B$.*

Proof. Suppose B is bi- (generalized bi-) hyperideal of R, then $(BR)B \subseteq B$. Let $b \in B$, since R is left regular so there exists an element $r \in R$ such that $b \in rb^2$. Thus

$$\begin{aligned} b &\in rb^2 = b(rb) = (rb^2)(rb) = (br)(b^2r) = b^2((br)r) = (((br)r)b)b \\ &= ((br)(br))b = (b((br)r))b \in (B((BR)R))B \subseteq (BR)B. \end{aligned}$$

Hence $(BR)B = B$. □

Lemma 3.2. *Suppose A and B are any hyperideals of left regular LA-semihyperring R, then $A \cap B = AB$.*

Proof. Suppose A and B are any two hyperideals of R, then clearly $AB \subseteq A \cap B$. Let $a \in A \cap B$, then $a \in A$ and $a \in B$. Since R is left regular, so there exists an element $r \in R$ such that $a \in ra^2$. Thus

$$\begin{aligned} a &\in ra^2 = a(ra) = (ra^2)(ra) = (ar)(a^2r) = a^2((ar)r) \\ &= (((ar)r)a)a \in (((AR)R)A)B \subseteq ((AR)A)B \subseteq AB. \end{aligned}$$

Hence $A \cap B = AB$. □

Lemma 3.3. *Suppsose A and B are any hyperideals of left regular LA-semihyperring R, then $AB = BA$.*

Proof. It obtains from Lemma 3.2. □

Lemma 3.4. *In an left regular R with having left identity, every left, right and hyperideals are idempotent.*

Proof. It is simple. □

Lemma 3.5. *A nonempty subset A of a left regular LA-semihyperring R having Pure-LI is a left hyperideal of $R \iff$ it is a right hyperideal of R.*

Proof. It is simple. □

Lemma 3.6. *In a left regular LA-semihyperring R having Pure-LI $PQ = P \cap Q$, for every hyperideals P and Q in R.*

Proof. Suppose P and Q are any hyperideals of R, then obviously $PQ \subseteq P \cap Q$. Since $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$, then $(P \cap Q)^2 \subseteq PQ$, also $P \cap Q$ is an hyperideal of R, so using Lemma 3.4, we have $P \cap Q = (P \cap Q)^2 \subseteq PQ$. Hence $PQ = P \cap Q$. □

4. Characterization Probems

The current section contains the characterizations left regular LA-semihyperrings with respect to (left, right) hyperideals, bi-(genrealized bi-) hyperideals, interior hyperideals and quasi-hyperideals.

Theorem 4.1. *For an left regular LA-semihyperring R having Pure-LI the given conditions are identical.*

(i) J is left hyperideal of R.

(ii) J is right hyperideal of R.

(iii) J is hyperideal of R.

(iv) J is bi-hyperideal of R.

(v) J is generalize bi-hyperideal of R.

(vi) J is interior hyperideal of R.

(vii) J is quasi-hyperideal of R.

(viii) $JR = J$ and $RJ = J$.

Proof. (i) $\implies$ (viii) Suppose J is left hyperideal of R. So by Lemma 2.3, $RJ = J$. Now let $a \in J$ and $b \in R$. As R is left regular, so there exists $r \in R$ such that $a \in ra^2$. Thus

$$ab \subseteq (ra^2)b = (a(ra))b = (b(ra))a \in (R(RJ))J \subseteq (RJ)J \subseteq RJ,$$

which implies that J is right hyperideal of R, again by Lemma 2.3, $JR = R$.

(viii) $\implies$ (vii)

Suppose $JR = J$ and $RJ = J$ then $JR \cap RJ = J$, which clearly implies that J is quasi-hyperideal of R.

(vii) $\implies$ (vi)

Suppose J is quasi-hyperideal of R. Now let $(ba)b \subseteq (RJ)R$, since R is left regular so there exists $r, s \in R$ such that $b \in (ra^2)$ and $a \in (sa^2)$. Thus

$$(ba)b \subseteq \left(\left(b\left(sa^2\right)\right)b = \left(\left(b(a(sa)\right)\right)b = \left(a\left(b\left(sa\right)\right)\right)b = \left(b\left(b\left(sa\right)\right)\right)a \subseteq RJ, \text{ and}$$

$$(ba)a \subseteq (ba)(sa^2) = (a^2s)(ab) = a((a^2s)b) \subseteq JR.$$

Therefore $(ba)b \subseteq JR \cap RJ \subseteq J$. Hence J is an interior hyperideal of R.

(vi) $\Longrightarrow$ (v)

Suppose J is an interior hyperideal of R and $(ab)a \subseteq (JR)J$, since R is left regular so there exists $r \in R$ such that $a \in (ra^2)$. Thus

$$(ab)a \subseteq (ab)(ra^2) = (ab)(a(ra)) = ((ra)a)(ba) \subseteq (RJ)R \subseteq J.$$

(v) $\Longrightarrow$ (iv)

Suppose J is generalize bi-hyperideal of R. Let $a \in J$, and since R is left regular so there exists r in R such that $a \in (ra^2)$. Thus

$$aa \subseteq \left(ra^2\right) a = (a(ra))a \subseteq (JR)J \subseteq J.$$

Hence J is bi-hyperideal of R.

(iv) $\Longrightarrow$ (iii)

Suppose J is any bi-hyperideal of R and let $ab \subseteq JR$. Since R is left regular, so there exists r in R such that $a \in (ra^2)$. Thus

$$\begin{aligned} ab &\subseteq (ra^2)b = (a(ra))b = (b(ra))a = (b(r(ra^2)))a = (b(r(a(ra))))a \\ &= (b(a(r(ra))))a = (a(b(r(ra))))a \subseteq (JR)J \subseteq J, \text{ and} \end{aligned}$$

$$\begin{aligned} ba &\subseteq b(ra^2) = (eb)(ra^2) = (a^2r)(be) = ((be)r)(aa) \\ &= (aa)(r(be)) = ((r(be))a)a = ((r(be))(ra^2))a \\ &= ((r(be))(a(ra)))a = (a((r(be))(ra)))a \subseteq (JR)J \subseteq J. \end{aligned}$$

Hence J is an hyperideal of R.

(iii) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (i) are simple. □

Theorem 4.2. *An LA-semihyperring R having Pure-LI is left regular $\Longleftrightarrow$ each bi- (interior, left, right, two-sided) hyperideals of R are idempotent.*

Proof. Suppose B is a bi-hyperideal of R. Let $b \in B$, as R is left regular so there exists $r \in R$ such that $b \in rb^2$. Thus

$$\begin{aligned} b &\in rb^2 = (er)(bb) = (bb)(re) = ((re)b)b = ((re)(rb^2))b == ((rr)(eb^2))b \\ &= ((rr)(bb))b = ((bb)(rr))b = (((rr)b)\, b)b = (\left((rr)(rb^2)\right) b)b \\ &= (\left(r^2(b(rb))\right) b)b = (\left(b(r^2(rb))\right) b)b \subseteq ((BR)B)\, B \subseteq BB. \end{aligned}$$

Hence $B^2 = B$.

Conversely, since $a \in Ra$ is a bi-hyperideal of R and by hypothesis Ra is idempotent. Thus

$$a \in (Ra)(Ra) = (RR)(aa) = Ra^2.$$

Hence R is left regular. □

Definition 4.1. A (resp., right, left) hyperideal P of an LA-semihyperring R is known as semiprime if $a^2 \subseteq P$ implies $a \in P$, for any $a \in R$.

Theorem 4.3. *In an LA-semihyperring R having Pure-LI, the given statements are identical.*

(i) S is left regular.

(ii) Every hyperideal of R is semiprime.

(iii) Each right hyperideal of R is semiprime.

(iv) Each left hyperideal of R is semiprime.

Proof. (i) $\implies$ (iv)

Suppose S is left regular, so by Theorem 4.1 and Lemma 3.4, each left hyperideal of S is semiprime.

(iv) $\implies$ (iii)

Suppose A is right and I is any hyperideal of S such that $I^2 \subseteq A$. Then clearly $I^2 \subseteq A \cup RA$. Now by Lemma 2.8, $A \cup RA$ is a hyperideal of R, so is left hyperideal. Then by (iv), we have $I \subseteq A \cup RA$. Now, $RA = (RR)A = (AR)R \subseteq AR \subseteq A$, therefore $I \subseteq A \cup RA \subseteq A$. Hence A is semiprime.

(iii) $\implies$ (ii).is obvious.

Now (ii) $\implies$ (i)

Since a^2R is a right hyperideal of R containing a^2 and clearly it is a hyperideal so by hypothesis (ii), a^2R is semiprime Thus by Theorem 2.2, $a \in a^2R$. Therefore

$$a \in a^2R = (aa)(RR) = (RR)(aa) = Ra^2.$$

Hence R is left regular. □

Theorem 4.4. *In an LA-semihyperring R having Pure-LI, the given statements are identical.*

(i) R is left regular.

(ii) $P \cap Q = PQ$, for every semiprime right hyperideal P and every left hyperideal Q of R

Proof. (i) $\implies$ (ii): Suppose R is left regular. Let P be right and Q be left hyperideals of R, so by Theorem 4.1 P and Q become hyperideals of R, therefore by Lemma 3.6, $P \cap Q \subseteq PQ$. Now let $a \in P \cap Q$, implies that $a \in P$ and $a \in Q$. As R is left regular, so there exists $r \in R$ such that $a \in ra^2$. Thus

$$a \in ra^2 = a(ra) \in P(RQ) \subseteq PQ.$$

Therefore $P \cap Q \subseteq PQ$. Thus by Theorem 4.3, P is semiprime.

(ii) $\implies$ (i): Let $P \cap Q = PQ$ for each right hyperideal P, which is semiprime and every left hyperideal Q of R. Since $a^2 \subseteq a^2R$, where a^2R is a right hyperideal of R,so is semiprime implies that $a \in a^2R$. Obviously Ra is a left hyperideal of R and $a \in Ra$. Thus

$$a \in (a^2R) \cap (Ra) \subseteq (a^2R)(Ra) = ((Ra)R)\, a^2 \subseteq ((RR)R)a^2 \subseteq (RR)a^2 \subseteq Ra^2.$$

Thus R is a left regular. □

Theorem 4.5. *For an LA-semihyperring R having Pure-LI, the given statements are identical.*

(i) R is left regular.

(ii) $Q \cap P \subseteq QP$*, for each right hyperideal* P*, which is semiprime and each left hyperideal* Q *of* R*.*

(iii) $Q \cap P \subseteq (QP)Q$*, for each semiprime right hyperideal* P *and each left hyperideal* Q *of* R*.*

Proof. (i) $\Longrightarrow$ (iii)

Let P be any right and Q be any left hyperideals of R. Let $a \in Q \cap P$, then $a \in Q$ and $a \in P$. As R is left regular then there exists r in R, such that $a \in ra^2$. Thus

$$\begin{aligned} a &\in ra^2 = a(ra) = (ra^2)(ra) = (ar)(a^2r) = ((a^2r)r)a = ((rr)(aa))a = ((ra)(ra))a \\ &= ((ra)((er)a))a = ((ra)((ar)e))a \subseteq ((RQ)((PR)R))\,Q \subseteq (Q(PR))Q \subseteq (QP)Q, \end{aligned}$$

therefore, $Q \cap P \subseteq (QP)Q$. Also by Theorem 4.3, Q is semiprime.

(iii) $\Longrightarrow$ (ii)

Suppose P is left and Q is right hyperideals of R. Let P be semiprime. Thus

$$P \cap Q \subseteq (PQ)P \subseteq (PQ)R = (PQ)(RR) = (RR)(QP) = Q((RR)P) = Q(RP) \subseteq QP.$$

(ii) $\Longrightarrow$ (i)

Since $b \in Rb$, which is left hyperideal of R, and $b^2 \subseteq b^2R$, that is semiprime right hyperideal of R, so by Theorem 2.2, $b \in b^2R$. Thus

$$\begin{aligned} b &\in (Rb) \cap (b^2R) \subseteq (Rb)(b^2R) \subseteq (RR)(b^2R) \\ &= R(b^2R) = b^2(RR) = (bb)(RR) = (RR)(bb) = Rb^2. \end{aligned}$$

Hence R is left regular. □

Lemma 4.1. *Each LA-semihyperring* R *having Pure-LI is left regular if* R *is pure left (right) invertible.*

Proof. Let $r \in R$, then there exists $r' \in R$ such that $r'r = r$. Thus

$$e = er = e(er) = \left(r'r\right)(er) \in (Rr)\,(Rr) = (RR)\,(rr) = Rr^2.$$

Thus R is left regular. In a similar way, the case of pure right invertible hold. □

Theorem 4.6. *For a pure left (right) invertible LA-semihyperring* R*, the given statements are identical:*

(i) S *is left regular;*

(ii) $P \cap Q = PQ$*, for each* P *is right* Q *is left hyperideal of* R*.*

Proof. (i) $\Longrightarrow$ (ii) : It follows from Theorem 4.4. □

Proof. (ii) $\Longrightarrow$ (i) : It obtains from Lemma 4.1. □

Theorem 4.7. *The given conditions are identical on an LA-semihyperring* R *left pure identity:*

(i) R *is lef regular.*

(ii) $Q \cap P = QP$*, for each semiprime right hyperideal* P *and left hyperideal* Q *of* R*.*

Proof. (i) $\Longrightarrow$ (ii) : It obtains by using Lemma 3.5 and Theorem 4.4. □

Proof. (ii) $\Longrightarrow$ (i) : It is simple. □

Theorem 4.8. *The given statements are identical on a pure left (right) invertible LA-semihyperring R:*

(i) R is left regular.

(ii) $P \cap Q = PQ$, for each P is right hyperideal and Q is left hyperideal of R.

(iii) $R(AA) = A$, for each A is quasi-hyperideal of R.

Proof. (i) $\Longrightarrow$ (iii)

Suppose A is any quasi-hyperideal of R. Let $a \in R$, such that $a \in A$. As R is left regular so there exists r in R, such that $a \in ra^2 \subseteq RA^2 = R(AA)$. Therefore $A \subseteq R(AA)$. Now

$$\begin{aligned} R(AA) &= A(RA) \subseteq A(RR) = AR \text{ and} \\ R(AA) &= (RR)(AA) = (AA)(RR) \\ &= (AA)R = (RA)A \subseteq (RR)A = RA. \end{aligned}$$

therefore, $R(AA) \subseteq AR \cap RA \subseteq A$. Hence $R(AA) = A$, for every quasi-hyperideal A of R.

(iii) $\Longrightarrow$ (ii)

Supose P is left and Q is right hyperideal of R. So by Lemma 2.1, $P \cap Q$ is a quasi-ideal of R. Thus by hypothesis (iii), we have

$$P \cap Q = R((P \cap Q)(P \cap Q)) \subseteq R(PQ) \subseteq P(RQ) \subseteq PQ.$$

Also, $PQ \subseteq PR \cap RQ \subseteq P \cap Q$. Hence $P \cap Q = PQ$, for each right hyperideal P and left hyperideal Q of R.

(ii) $\Longrightarrow$ (i): It obtains from Theorem 4.6. □

Theorem 4.9. *The given conditions are identical on an LA-semihyperring R Pure-LI:*

(i) R is left regular;

(ii) $J = J^3$, for J is left hyperideal of R.

Proof. (i) $\Longrightarrow$ (ii) : Suppose J is left hyperideal of R, then by Lemma 3.4, we have $J^3 = (JJ)J = JJ \subseteq RJ \subseteq J$.

Furthermore, let $j \in J$, then there exists $r \in R$ such that $j \in rj^2$. Thus

$$\begin{aligned} j &\in rj^2 = j\,(rj) = \left(rj^2\right)(rj) = (j\,(rj))\,(rj) = ((rj)\,(rj))\,j = ((jr)\,(jr))\,j \\ &= (((jr)\,r)\,j)\,j = (((rr)\,j)\,j)\,j \subseteq ((RJ)\,J)\,J \subseteq (JJ)\,J = J^3. \end{aligned}$$

Hence $J = J^3$, for every left hyperideal J of R..

(ii) $\Longrightarrow$ (i) : Suppose J is left hyperideal of R having Pure-LI such that $J = J^3$. As Rj is left hyperideal of R and $j \in Rj$. Thus

$$\begin{aligned} j &\in Rj = ((Rj)\,(Rj))\,Rj = ((RR)\,(jj))\,Rj = \left(Rj^2\right)(Rj) \\ &= (jR)\left(j^2R\right) = j^2\,((jR)\,R) \subseteq (jj)(RR) = (RR)(jj) = Rj^2. \end{aligned}$$

Thus R is left regular. □

Theorem 4.10. *The given conditions are identical on an LA-semihyperring R Pure-LI:*

(i) R is left regular;

(ii) $L = L^{n+1}$, where $n \in N$.

Proof. By generalization of proof of Theorem 4.9, gives its proof. □

Theorem 4.11. *The set of all hyperideals I_R of a left regular LA-semihyperring R having Pure-LI, forms a semilattice structure.*

Proof. Let $P, Q \in I_R$, as P and Q are hyperideals of R, then $(PQ)R = (PQ)(RR) = (PR)(QR) \subseteq PQ$. Also $R(PQ) = (RR)(PQ) = (RP)(RQ) \subseteq PQ$. Thus PQ is an hyperideal of R. Hence I_R is closed. Also by using Lemma 3.6, we have

$$PQ = P \cap Q = Q \cap P = QP,$$

which implies that I_R is commutative and commutativity gives so is associativity. So by using Lemma 3.4, $P^2 = P$, $\forall\ P \in I_R$. Thus I_R is semilattice. □

5. Conclusions

In this paper, the notion of left regular LA-semihyperring is defined and the basic properties of many hyperideals in terms of left regular LA-semihyperrings are discussed. The fundamental characterization of left regular LA-semihyperrings by the properties of their (right, left) hyperideals, bi- (generalized bi-) hyperideals, interior hyperideals and quasi-hyperideals are produced. In our future work, we shell characterized stronly-regular class of LA-semihyperrings with repect to their hyperideals.

6. Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through General Research Projectunder grant number (R.G.P.1/208/41).

7. Availability of data and materials

No data were used to support this study.

References

[1] Hila, K. and Dine, J. On hyperideals in left almost semihypergroups. ISRN Algebra (2011), Article ID 953124.

[2] Marty, F. Sur une generalization de la notion de groupe. In 8th congres Math. Scandinaves. Stockholm (1934).

[3] Onipchuk, S. V. Regular semihypergroups. Mat. Sb., (1992), 183:43–54.

[4] Davvaz, B. Some results on congruences on semihypergroups. Bull. Malays. Math. Sci. Soc, (2000), 23:53–58.

[5] Drbohlav, K. Associativity semihypergroups and related problems. Atti Convegno su Ipergruppi, altre Strutture Multivoche e loro Applicazioni, (1985).

[6] Gutan, C. Simplifiable semihypergroups. In Proceedings of the Algebraic hyperstructures and applications, pages (1990), 103–111.

[7] Hedayati, H. On semihypergroups and regular relations. In Tarbiat Moallem University, 20th Seminar on Algebra, pages (2009), 92–94.

[8] Hila, K., Davvaz, B., and Naka, K. On quasi-hyperideals in semihypergroups. Commun. Algebra. (2011), 39:4183–4194.

[9] Leoreanu, V. About the simplifiable cyclic semihypergroups. Ital. J. Pure Appl. Math., (2000), 7:69–76.

[10] Corsini, P. Prolegomena of hypergroup theory. Aviani editor (1993).

[11] Mushtaq, Q. and Kamran, M. S. On LA-semigroups with weak associative law. Sci. Khy., (1989), 2:69–71.

[12] Protic, P. V. and Stevanovic, N. AG-test and some general properties of AbelGrassmann's groupoids. PU. M. A., (1985), 6:371–383.

[13] Kazim, M. A. and Neseeruddin, M. On almost semigroups, Rort. Math., 36 (1977), 41–47.

[14] Protic, P. V. and Stevanovic, N. AG-test and some general properties of Abel-Grassmann's groupoids, Pure Math. Appl., 4 (1995), 371–383.

[15] Khan, M. and Asif, T. Characterizations of intra-regular left almost semigroups by their fuzzy ideals, J. Math. Res., 2 (2010), 87–96.

[16] Khan, M. and Asif, T. Characterizations of regular left almost semigroups by their fuzzy ideals, J Adv Res Dyn Control Syst, 2 (2010), 59–75.

[17] Khan, M. , Jun, Y. B. and Yousafzai, F. Fuzzy ideals in right regular LA-semigroups, Hacet. J. Math. Stat., 44 (2015), 569–586.

[18] Sezer, A. S. Certain characterizations of LA-semigroups by soft sets, J. Intell. Fuzzy Syst., 27 (2014), 1035–1046. https://doi.org/10.3233/IFS-131064

[19] Yousafzai, F. , Iampam, A. and Tang, J. Study on smallest (fuzzy) ideals of LA-semigroups, Thai J. Math., 16 (2018), 549–561.

[20] Al-Qudah, Y., Yousafzai, F. , Khalaf , M. M. and Almousa, M. On (2, 2)-regular Non-associative Ordered Semigroups via Its Semilattices and Generated (Generalized Fuzzy) Ideals, Mathematics and Statistics 8(3) (2020), 353-362. DOI: 10.13189/ms.2020.080315

[21] Ahmad, I. , Rahman, S. and Iqbal, M. Amanullah, A note on left abelian distributive LA-semigroups, Punjap Univ. J. Math., 52 (2020), 47–63.

[22] Younas, I. , Mushtaq, Q. and Rafiq, A. Presentation of inverse LA-semigroups, Maejo Int. J. Sci. Technol., 14 (2020), 242–251.

[23] Shah, T. and Rehman, I. On LA-rings of finitely nonzero functions, Int. J. Contemp. Math. Sciences, 5 (2010), 209–222.

[24] Elmoasy, A. On rough fuzzy prime ideals in left almost semigroups, Int. J. Anal. Appl., 19 (2021), 455–464.

[25] Massouros, C. G. and Yaqoob, N. On theory of left/right almost groups and hypergroups with their relevant enumerations, Mathematics, 9 (2021), 1828. https://doi.org/10.3390/math9151828

[26] Rehman, I. , Razzaque, A. and Faraz, M. I. Neutrosophic set approach to study the characteristic behavior of left almost rings, Neutrosophic Sets Sy., 46 (2021), 24–36. https://doi.org/10.29020/nybg.ejpam.v14i3.4034

[27] Marty, F. Sur une generalization de la notion de group, 8th Congress Mathematics Scandinaves, Stockholm, 1934.

[28] Davvaz, B. and Fotea, V. L. Hypergroup Theory, Word Scientific, (2022), DOI: 10.1142/12645

[29] Rehman, I. , Yaqoob, N. and Nawaz, S. Hyperideals and hypersystems in LA-hyperrings, Songklanakarin J. Sci. Technol., 39 (2017), 651–657. https://doi.org/10.14456/sjst-psu.2017.80

[30] Hu, M. , Smarandache, F. and Zhang, X. On neutrosophic extended triplet LA-hypergroups and strong pure LA-semihypergroups, Symmetry, 12 (2020), 163. https://doi.org/10.3390/sym12010163

[31] Hila, K. and Dine, J. On hyperideals in left almost semihypergroups, International Scholarly Research Notices, 2011 (2011), 953124. https://doi.org/10.5402/2011/953124

[32] Yaqoob, N. , Corsini, P. and Yousafzai, F. On intra-regular left almost semihypergroups with pure left identity, J. Math., 2013 (2013), 510790. https://doi.org/10.1155/2013/510790

[33] Gulistan, M. , Khan, M. , Yaqoob, N. and Shahzad, M. Structural properties of cubic sets in regular LA-semihypergroups, Fuzzy Inform. Eng., 9 (2017), 93–116. https://doi.org/10.1016/j.fiae.2017.03.005

[34] Khan, A. , Farooq, M. , Izhar, M. and Davvaz, B. Fuzzy hyperideals of left almost semihypergroups, Int. J. Anal. Appl., 15 (2017), 155–171.

[35] Nakkhasen, W. Left almost semihyperrings characterized by their hyperideals, AIMS Math., 6 (2021), 13222–13234. https://doi.org/10.3934/math.2021764

[36] Nakkhasen, W. Characterizations of intra-regular LA-semihyperrings in terms of their hyperideals, AIMS Math., 7(4) (2022), 5844-5859. https://doi.org/10.3934/math.2022324

[37] Nawaz, S. , Rehman, I. and Gulistan, M. On left almost semihyperrings, Int. J. Anal. Appl., 16 (2018), 528–541. https://doi.org/10.28924/2291-8639-16-2018-528

[38] Nakkhasen, W. and Pibaljommee, B. Intra-regular semihyperrings, J. Discret. Math. Sci. C., 22 (2019), 1019–1034. https://doi.org/10.1080/09720529.2019.1649818

[39] Naseeruddin, N. Some studies in almost semigroups and flocks. Ph.D., thesis, Aligarh Muslim University, Aligarh, India, 1970.