Applications of Hyperideals in Characterizations of Left Regular LA-semihyperrings

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Abstract: The main objective of this paper is to investigate the class of left regular LA-semihyperrings with respect to their hyperideals. Then, produce the interesting characterizations of left regular LA-semihyperrings with respect to their hyperideals. In this connection, we prove that right (left, two sided, interior, bi-, generalized bi-, quasi) hyperideals are coincide in a left regular LA-semihyperrings having pure left identity, these hyperideals are normally not coincide in other classes of regularities of LA-semihyperrings.

Key words: Left invertive law, left regular, hyperideals, LA-semihyperrings, left identity.

1. Introduction

A new algebraic structure called left almost semigroup (for short, LA-semigroup) [13], initiated by Kazim and Naseeruddin in 1972. This structure is additionally called as Abel-Grassmann’s Groupoid (for short, AG-Groupoid) by Protic and Stevanovic [14]. That algebraic structure is non commutative and non associative, lying in middle of groupoid and commutative semigroup possess numerous applications in the theory of flocks [39]. Mushtaq and Kamran named an AG-Groupoid with weak associative law [11] as AG*-Groupoid. The generality of an AG-Groupoid having left identity was called an AG**-Groupoid. Protic and Stevanovic have also introduced a useful technique for confirmation of AG-Groupoid, AG**-Groupoid and AG*-Groupoid in [12]. Khan and Asif [15, 16] characterized intra-regular and regular LA-semigroup with respect to their fuzzy ideals in 2010. Khan and et al. [17] characterize right regular LA-semigroup with respect to their fuzzy ideals. Yousafzai et al. [19] characterize weakly regular LA-semigroup by their smallest fuzzy ideals. Further, Sezer [18] apply the idea of soft sets to LA-semigroup and produce characterization of intra-regular, completely regular, regular, quasi-regular and weakly regular LA-semigroup. Currently, much researchers explored numerous characterizations of LA-semigroup (see, [20–22]). Moreover, few researchers have examine the concept of LA-semirings, that is a generalization of LA-rings [23]. Massouros and Yaqoob [25] studies the right and left almost groups and Rehman et al. [26] explore the idea of neutrosophic LA-rings and studies several types of ideals neutrosophic LA-rings.

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In 1934 is the first occasion when the idea of algebraic hyperstructure was floated for the first time 
by a French Mathematician Marty [27]. Hyperstructures have a distinct advantage over classical algebraic 
structures because the application of binary operation in hyperstructure produce a set, if this set is restricted to 
a singleton element, it effectively generalizes researchers to investigate these hyperstructures in different branches 
of mathematics. Various books have been written on hyperstructures, (see [10, 28]). Some authors explored 
different features of semihypergroups, like Davvaz et al. [4], Drbohlav et al. [5], Gutan [6], Hedayati [7], Hila 
et al. [8], Leoreanu [9] and Onipchuk [3]. Currently, Hila and Dine [1], initiate the idea of AG-hypergroupoids 
that is a generalization of semihypergroups, semigroups and AG-Groupoids.

Rehman et al. [29], introduce the concept of LA-hyperrings and characterize LA-hyperrings with respect 
to their hyperideals. In 2020, Hu et al. [30] apply the idea of neutrosophic set to LA-hypergroups. The idea 
of LA-semihypergroups was introduced by Hila and Dine [31] and an LA-semihypergroup is lies in middle of 
hypergroupoid and commutative semihypergroup. Yaqoob et al. in [32] give the characterizations intra-regular 
LA-semihypergroups using right and left hyperideals. Gulistan et al. [33] studies the class of regular LA-
semihypergroups with respect to \((\in_\Gamma, \in_\Gamma \vee q_\Delta)\)-cubic hyperideals. Moreover, Khan et al. [34] explore few 
characteristics of fuzzy right and left hyperideals in intra-regular and regular LA-semihypergroups.

In current paper, we focused in left regular class of LA-semihypergroups. We gave few interesting char-
acterizations of left regular LA-semihypergroups by the properties of their hyperideals. Further more, we prove 
that right (left, two sided, inerior, bi-, generalized bi-, quasi) hyperideals are coincide in a left regular LA-
semihypergroups having left identity, these hyperideals are normally not coincide in other classes of regularities 
of LA-semihyperings.

2. Preliminaries

This section contains few definitions and results that are helpful in upcoming work. A mapping \(\circ : H \times H \rightarrow P^*(H)\) is knows as hyperoperation on the set \(H\), where \(H\) is nonempty set and \(P^*(H) = P(H) \setminus \{\emptyset\}\) represents the all nonempty subsets of \(H\). An ordered pair \((H, \circ)\) is known as hypergroupoid, where \(H \neq \emptyset\) and "\(\circ\)" is hyperoperation.

If \(\emptyset \neq A, B \subseteq H\), then \(A \circ B = \bigcup_{p \in A, q \in B} p \circ q\), \(p \circ A = \{p\} \circ A\) and \(p \circ B = \{p\} \circ B\).

A hypergroupoid \((H, \circ)\) is known as LA-semihyperring [1], if is satisfies, \((p \circ q) \circ r = (r \circ q) \circ p, \forall p, q, r \in H\). This is called a left invertive law. For a nonempty subset \(X, Y, Z\) of an LA-semihyperring \(H\), means that \((X \circ Y) \circ Z = (Z \circ Y) \circ X\).

A hyperstructure \((R, +, \circ)\) is known as an LA-semihyperring [37], if it satisfies:

(i) \((R, +)\) is an LA-semihypergroup;
(ii) \((R, \circ)\) is an LA-semihypergroup;
(iii) \(\xi_1 \circ (\xi_2 + \xi_3) = (\xi_1 \circ \xi_2) + (\xi_1 \circ \xi_3)\) and \((\xi_2 + \xi_3) \circ \xi_1 = (\xi_2 \circ \xi_1) + (\xi_3 \circ \xi_1)\) for all \(\xi_1, \xi_2, \xi_3 \in R\).
Example 2.1. Let $R = \{\xi_1, \xi_2, \xi_3\}$ with the binary hyperoperation defined below:

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It is easy to see that $R$ is an LA-semihyperring. It is also notice that $\xi_1$ is right identity but $\xi_1$ is not left identity.

It is a common reality that if an LA-semihyperring carries a pure right identity (Pure-RI), then it become a pure identity (Pure-I) and an LA-semihyperring having pure identity (Pure-I) is coincide with commutative semi hypergroup with pure identity (Pure-I). The example below shows that if an LA-semihyperring $R$ carries a right identity, then it not become a left identity.

Example 2.2. Let $R = \{\xi_1, \xi_2, \xi_3\}$ with the binary hyperoperations $+$ and $\circ$ defined below:

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Clearly $R$ satisfies left invertive law, so it is an LA-semihyperring and it is simple to observe that $\xi_1$ is the identity of $R$ but $R$ is neither commutative and nor associative.

In upcoming work, we call $R$ is an LA-semihyperring rather than $(R, +, \circ)$ and we write $pq$ instead of $p \circ q \forall p, q \in R$.

The below listed concets are occurred in [37], will apply in this research. In every LA-semihyperring $R$, the medial law $(pq)(rs) = (pr)(qs)$ holds $\forall p, q, r, s \in R$. A member $e \in R$ is known as left identity (resp., Pure-LI) if $r \in e$ (resp., $r = e$) $\forall r \in R$. Each LA-semihyperring $R$ having a Pure-LI $e$ satisfies the following two law: $p(qr) = q(pr)$ and $(pq)(rs) = (sr)(qp), \forall p, q, s, r \in R$. The second law is known as paramedical law. A member $r \in R$ having left identity (resp., Pure-LI) $e$ is known as left invertible (resp., pure left invertible) if there exists $r' \in R$ such that $e \in r'a$ (resp., $e = r'a$). An LA-semihyperring $R$ is known as left invertible (resp., pure left invertible) if each member of $R$ is left invertible (resp., pure left invertible). An LA-semihyperring $R$ having Pure-LI $e$ become a left identity, but conversly it is invalid generally, see in [35].

The given law holds in an LA-semihyperrings $R$, $(LM)(NO) = (LN)(MO)$ for each nonempty subsets $L, M, N, O$ of $R$. If $R$ carries a Pure-LI $e$, then $R$ also satisfies, $(LM)(NO) = (ON)(ML)$ and $(LMN) = M(LN)$ for each nonempty subsets $L, M, N, O$ of $R$.

Suppose $R$ is an LA-semihyperring and $\emptyset \neq I \subseteq R$ such that $I + I \subseteq I$. Then:

(i) $I$ is known as right hyperideal [37] of $R$ if $IR \subseteq I$;
(ii) $I$ is known as left hyperideal [37] of $R$ if $RI \subseteq I$;
(iii) $I$ is known as hyperideal [37] of $R$ if $IR \subseteq I$ and $RI \subseteq I$;
(iv) $I$ is known as quasi-hyperideal [37] of $R$ if $RI \cap IR \subseteq I$;
(v) $I$ is known as generalized bi-hyperideal [37] of $R$ if $(IR)I \subseteq I$.

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(vi) \(I\) is known as bi-hyperideal \([37]\) of \(R\) if \(II \subseteq I\) and \((IR)I \subseteq I\).

(vii) \(I\) is known as interior hyperideal \([37]\) of \(R\) if \(II \subseteq I\) and \((RI)R \subseteq I\).

**Lemma 2.1.** \([35]\) If \(P\) is left and \(Q\) is right hyperideal of an LA-semihyperring \(R\), then \(P \cap Q\) is quasi-hyperideal of \(R\).

**Lemma 2.2.** An LA-semihyperring \(R\) containing left identity \(e\) satisfies, \(R \circ R = R\).

*Proof.* Suppose \(R\) is an LA-semihyperring with left identity \(e\). Then any \(r \in R \implies r \in e \circ r \subseteq R \circ R\), therefore \(R \subseteq R \circ R\). Thus \(R = R \circ R\). \(\Box\)

**Corollary 2.1.** An LA-semihyperring \(R\) containing Pure-LI satisfies, \(R = e \circ R = R \circ e\) and \(R \circ R = R\).

**Lemma 2.3.** If an LA-semihyperring \(R\) contains Pure-LI, then the given conditions hold.

(i) \(RI = I\) for each left hyperideal \(I\) of \(R\).

(ii) \(JR = R\) for each right hyperideal \(J\) of \(R\).

*Proof.* It is simple. \(\Box\)

**Lemma 2.4.** If \(B\) is a bi-hyperideal of an LA-semihyperring \(R\) having Pure-LI, then \((rB)s\) is also a bi-hyperideal of \(R\), for any \(r, s \in R\).

*Proof.* Suppose \(B\) is a bi-hyperideal of \(R\). Then \(B + B \subseteq B, BB \subseteq B,\) and \((BR)B \subseteq B\). Thus

\[
((rB)s)R)((rB)s) = ((Rs)(rB))((rB)s) = (((rB)s)(rB))(Rs)
= (((Br)(rB))((sB)s)) = ((Br)(rB)((sB)s))
= ((Br)(sB))(rR)s \subseteq ((Br)(sB))R
= ((Br)s)(eR) = ((Br)s)((sB)R)
= ((erB)((RB)s) = (rB)((RB)s)
= (s(RB))(Br) = ((es)(RB))(Br)
= ((BR)(se))(Br) = ((BR)B)((se)r)
\subseteq B((se)r) = (se)(Br) = (rB)(es) = (rB)s.
\]

Hence \((rB)s\) is a bi-hyperideal of \(R\). \(\Box\)

**Lemma 2.5.** In an LA-semihyperring \(R\) with Pure-LI, each idempotent quasi-hyperideal is a bi-hyperideal of \(R\).

*Proof.* Suppose \(Q\) is an idempotent quasi-hyperideal of \(R\), then obviously \(Q\) is an LA-subsemihyperring. Thus

\[
(QR)Q \subseteq (QR)R \subseteq (RR)Q = RQ, \text{ and}
\]

\[
(QR)Q \subseteq (RR)(QQ) = (QQ)(RR) = QR.
\]

Therefore, \((QR)Q \subseteq QR \cap RQ \subseteq Q\). Thus \(Q\) is a bi-hyperideal of \(R\). \(\Box\)
Lemma 2.6. If $L$ and $M$ are quasi-hyperideal of an LA-semihyperring $R$ having Pure-LI, where $L$ is idempotent, then $LM$ is a bi-hyperideal of $R$.

Proof. Suppose $L$ and $M$ is a quasi-hyperideals of $R$, and $L$ is an idempotent. Thus by using Lemma 2.5, we have


Lemma 2.7. A subset $P$ of an LA-semihyperring $R$ having Pure-LI is right hyperideal of $R \iff P$ is an interior hyperideal of $R$.

Proof. Suppose $P$ is a right hyperideal of $R$, then clearly $P$ is an hyperideal of $R$, so is an interior hyperideal of $R$.

Conversely, suppose $P$ is an interior hyperideal of $R$. Thus

\[PR = P(RR) = R(PR) = (RR)(PR) = (RP)(RR) = (RP)R \subseteq P.\]

Lemma 2.8. If $P$ is right hyperideal or $Q$ is left hyperideal of an LA-semihyperring $R$ having Pure-LI then $P \cup RP$ and $Q \cup QR$ are hyperideals of $R$.

Proof. Suppose $P$ is right hyperideal of $R$. Then $P + P \subseteq$ and $PR \subseteq P$. Thus

\[(P \cup RP)R = PR \cup (RP)R \subseteq P \cup (RP)(RR) = P \cup R(PR) = P \cup P(RR) = P \cup PR = P \subseteq (P \cup RP), \text{ also}\]

\[R(P \cup RP) = RP \cup R(RP) = RP \cup (RR)(RP) = RP \cup (PR)(RR) \subseteq RP \cup P = P \cup RP.\]

Hence $(P \cup RP)$ is an hyperideal of $R$. In a similar manner $(Q \cup RQ)$ is also an hyperideal of $R$.

Definition 2.1. If $I$ and $J$ are hyperideals of an LA-semihyperring $R$, such that $I^2 \subseteq J$ implies $I \subseteq J$, then $J$ is called semiprime.

Theorem 2.1. If an LA-semihyperring $R$ contains a Pure-LI, then the given conditions are identical.

(i) If $I$ and $J$ are hyperideals of $R$, then $I^2 \subseteq J$ implies $I \subseteq J$.

(ii) If $A$ is right and $J$ is hyperideal of $R$ then $A^2 \subseteq J$ implies $A \subseteq J$.

(iii) If $B$ is left and $J$ is a hyperideal of $R$ then $B^2 \subseteq J$ implies $B \subseteq J$.

Proof. Suppose $B$ is a left hyperideal of $R$ and $B^2 \subseteq J$, then by Lemma 2.8, $B \cup BR$ is an hyperideal of $R$, therefore by hypothesis (i), $(B \cup BR)^2 \subseteq J$ which implies $(B \cup BR) \subseteq J$ which further implies that $B \subseteq J$. (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are simple.
**Theorem 2.2.** An hyperideal $J$ of an LA-semihyperring $R$ with Pure-LI is semiprime $\iff a^2 \subseteq J$ implies $a \in J$.

**Proof.** Let $J$ be a semiprime hyperideal of $R$ and $a^2 \subseteq J$. Since $Ra^2$ is a left hyperideal of $R$ and $a^2 \subseteq Ra^2$, also $a^2 \subseteq J$, therefore $a^2 \subseteq Ra^2 \subseteq J$. Thus $Ra^2 = R(aa) = (RR)(aa) = (Ra)(Ra) = (Ra)^2$. Therefore $(Ra)^2 \subseteq J$, but $J$ is semiprime so $Ra \subseteq J$. Since $a \in Ra$, therefore $a \in J$.

Conversely, suppose $I$ is an hyperideal of $R$ and let $I^2 \subseteq J$ and $a \in I$ implies that $a^2 \subseteq I^2$, which implies that $a^2 \subseteq J$ which further implies that $a \in J$. Therefore $I^2 \subseteq J$ implies $I \subseteq J$. Hence $J$ is semiprime. □

3. Left Regular LA-semihyperring

In current section, the notion of left regular LA-semihyperrings is defined and few of its properties are studies.

**Definition 3.1.** A member $a$ of an LA-semihyperring $R$ is known as left regular if there exists $r \in R$ such that $a \in ra^2$, and $R$ is known as left regular if each member of $R$ is left regular.

**Example 3.1.** In Example 2.2, we can show that there exists $r \in R$ such that $a \in ra^2 \forall a \in R$. Therefore, $R$ is left regular LA-semihyperring.

Note that in a left regular LA-semihyperring $R$ and an LA-semihyperring $R$ with left identity, $R^2 = R$.

**Lemma 3.1.** If $B$ is bi- (generalized bi-) hyperideal of a left regular LA-semihyperring $R$ then $(BR)B = B$.

**Proof.** Suppose $B$ is bi- (generalized bi-) hyperideal of $R$, then $(BR)B \subseteq B$. Let $b \in B$, since $R$ is left regular so there exists an element $r \in R$ such that $b \in rb^2$. Thus

$\begin{align*}
 b \in rb^2 & = b(rb) = (rb^2)(rb) = (br)(b^2r) = b^2((br)r)b \\
 & = ((br)(br))b = (b((br)r)b \in B((BR)R))B \subseteq (BR)B.
\end{align*}$

Hence $(BR)B = B$. □

**Lemma 3.2.** Suppose $A$ and $B$ are any hyperideals of left regular LA-semihyperring $R$, then $A \cap B = AB$.

**Proof.** Suppose $A$ and $B$ are any two hyperideals of $R$, then clearly $AB \subseteq A \cap B$. Let $a \in A \cap B$, then $a \in A$ and $a \in B$. Since $R$ is left regular, so there exists an element $r \in R$ such that $a \in ra^2$. Thus

$\begin{align*}
 a \in ra^2 & = a(ra) = (ra^2)(ra) = (ar)(a^2r) = a^2((ar)r) \\
 & = (((ar)r)a)a \in (((AR)R)A)B \subseteq ((AR)A)B \subseteq AB.
\end{align*}$

Hence $A \cap B = AB$. □

**Lemma 3.3.** Suppose $A$ and $B$ are any hyperideals of left regular LA-semihyperring $R$, then $AB = BA$.

**Proof.** It obtains from Lemma 3.2. □

**Lemma 3.4.** In an left regular $R$ with having left identity, every left, right and hyperideals are idempotent.

**Proof.** It is simple.
Lemma 3.5. A nonempty subset $A$ of a left regular LA-semihyperring $R$ having Pure-LI is a left hyperideal of $R$ if and only if it is a right hyperideal of $R$.

Proof. It is simple. \qed

Lemma 3.6. In a left regular LA-semihyperring $R$ having Pure-LI, $PQ = P \cap Q$, for every hyperideals $P$ and $Q$ in $R$.

Proof. Suppose $P$ and $Q$ are any hyperideals of $R$, then obviously $PQ \subseteq P \cap Q$. Since $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$, then $(P \cap Q)^2 \subseteq PQ$, also $P \cap Q$ is an hyperideal of $R$, so using Lemma 3.4, we have $P \cap Q = (P \cap Q)^2 \subseteq PQ$. Hence $PQ = P \cap Q$. \qed

4. Characterization Problems

The current section contains the characterizations left regular LA-semihyperrings with respect to (left, right) hyperideals, bi-(generalized bi-) hyperideals, interior hyperideals and quasi-hyperideals.

Theorem 4.1. For an left regular LA-semihyperring $R$ having Pure-LI the given conditions are identical.

(i) $J$ is left hyperideal of $R$.

(ii) $J$ is right hyperideal of $R$.

(iii) $J$ is hyperideal of $R$.

(iv) $J$ is bi-hyperideal of $R$.

(v) $J$ is generalized bi-hyperideal of $R$.

(vi) $J$ is interior hyperideal of $R$.

(vii) $J$ is quasi-hyperideal of $R$.

(viii) $JR = J$ and $RJ = J$.

Proof. (i) $\implies$ (viii) Suppose $J$ is left hyperideal of $R$. So by Lemma 2.3, $RJ = J$. Now let $a \in J$ and $b \in R$. As $R$ is left regular, so there exists $r \in R$ such that $a \in ra^2$. Thus

$$ab \subseteq (ra^2)b = (a(ra))b = (b(ra))a \in (R(RJ))J \subseteq (RJ)J \subseteq RJ,$$

which implies that $J$ is right hyperideal of $R$, again by Lemma 2.3, $JR = R$.

(viii) $\implies$ (vii)

Suppose $JR = J$ and $RJ = J$ then $JR \cap RJ = J$, which clearly implies that $J$ is quasi-hyperideal of $R$.

(vii) $\implies$ (vi)

Suppose $J$ is quasi-hyperideal of $R$. Now let $(ba)b \subseteq (RJ)R$, since $R$ is left regular so there exists $r, s \in R$ such that $b \in (ra^2)$ and $a \in (sa^2)$. Thus

$$(ba)b \subseteq ((ba)(sa))b = (a(b(sa)))b = (b(b(sa)))a \subseteq RJ,$$

and

$$(ba)a \subseteq (ba)(sa^2) = (a^2s)(ab) = a((a^2)b) \subseteq JR.$$

Therefore $(ba)b \subseteq JR \cap RJ \subseteq J$. Hence $J$ is an interior hyperideal of $R$.\[67\]
Suppose $J$ is an interior hyperideal of $R$ and $(ab)a \subseteq (JR)J$, since $R$ is left regular so there exists $r \in R$ such that $a \in (ra^2)$. Thus

$$(ab)a \subseteq (ab)(ra^2) = (ab)(a(ra)) = ((ra)a)(ba) \subseteq (RJ)R \subseteq J.$$  

(v) $\Rightarrow$ (iv)

Suppose $J$ is generalize bi-hyperideal of $R$. Let $a \in J$, and since $R$ is left regular so there exists $r \in R$ such that $a \in (ra^2)$. Thus

$$aa \subseteq (ra^2)a = (a(ra))a \subseteq (JR)J \subseteq J.$$ 

Hence $J$ is bi-hyperideal of $R$.

(iv) $\Rightarrow$ (iii)

Suppose $J$ is any bi-hyperideal of $R$ and let $ab \subseteq JR$. Since $R$ is left regular, so there exists $r \in R$ such that $a \in (ra^2)$. Thus

$$ab \subseteq (ra^2)b = (a(ra))b = (b(ra))a = (b(r(ra)))a = (b(a(r(ra))))a \subseteq (JR)J \subseteq J,$$

and

$$ba \subseteq b(ra^2) = (eb)(ra^2) = (a^2r)(be) = ((be)r)(aa) = (aa)(r(be))a = ((r(be))(ra^2))a \subseteq (JR)J \subseteq J.$$ 

Hence $J$ is an hyperideal of $R$.

(iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are simple.

**Theorem 4.2.** An LA-semihyperring $R$ having Pure-LI is left regular $\iff$ each bi- (interior, left, right, two-sided) hyperideals of $R$ are idempotent.

**Proof.** Suppose $B$ is a bi-hyperideal of $R$. Let $a \in B$, as $R$ is left regular so there exists $r \in R$ such that $b \in rb^2$. Thus

$$b \in rb^2 = (er)(bb) = (bb)(re) = ((re)b)b = ((re)(rb^2))b = (rr)(eb^2)b$$

$$= ((rr)(bb))b = ((bb)(rr))b = ((rr)b)b = ((rr)(rb^2))b = (rr)(rb^2)b$$

$$= ((r^2(b(rb)))b = ((b(r^2(rb)))b \subseteq ((BR)B)B \subseteq BB.$$ 

Hence $B^2 = B$.

Conversely, since $a \in Ra$ is a bi-hyperideal of $R$ and by hypothesis $Ra$ is idempotent. Thus

$$a \in (Ra)(Ra) = (RR)(aa) = Ra^2.$$ 

Hence $R$ is left regular.
**Definition 4.1.** A (resp., right, left) hyperideal $P$ of an LA-semihyperring $R$ is known as semiprime if $a^2 \subseteq P$ implies $a \in P$, for any $a \in R$.

**Theorem 4.3.** In an LA-semihyperring $R$ having Pure-LI, the given statements are identical.

(i) $S$ is left regular.

(ii) Every hyperideal of $R$ is semiprime.

(iii) Each right hyperideal of $R$ is semiprime.

(iv) Each left hyperideal of $R$ is semiprime.

**Proof.** (i) $\implies$ (iv)

Suppose $S$ is left regular, so by Theorem 4.1 and Lemma 3.4, each left hyperideal of $S$ is semiprime.

(iv) $\implies$ (iii)

Suppose $A$ is right and $I$ is any hyperideal of $S$ such that $I^2 \subseteq A$. Then clearly $I^2 \subseteq A \cup RA$. Now by Lemma 2.8, $A \cup RA$ is a hyperideal of $R$, so is left hyperideal. Then by (iv), we have $I \subseteq A \cup RA$. Now, $RA = (RR)A = (AR)R \subseteq AR \subseteq A$, therefore $I \subseteq A \cup RA \subseteq A$. Hence $A$ is semiprime.

(iii) $\implies$ (ii).is obvious.

Now (ii) $\implies$ (i)

Since $a^2R$ is a right hyperideal of $R$ containing $a^2$ and clearly it is a hyperideal so by hypothesis (ii), $a^2R$ is semiprime Thus by Theorem 2.2, $a \in a^2R$. Therefore

$$a \in a^2R = (aa)(RR) = (RR)(aa) = Ra^2.$$ 

Hence $R$ is left regular.

**Theorem 4.4.** In an LA-semihyperring $R$ having Pure-LI, the given statements are identical.

(i) $R$ is left regular.

(ii) $P \cap Q = PQ$, for every semiprime right hyperideal $P$ and every left hyperideal $Q$ of $R$

**Proof.** (i) $\implies$ (ii): Suppose $R$ is left regular. Let $P$ be right and $Q$ be left hyperideals of $R$, so by Theorem 4.1 $P$ and $Q$ become hyperideals of $R$, therefore by Lemma 3.6, $P \cap Q \subseteq PQ$. Now let $a \in P \cap Q$, implies that $a \in P$ and $a \in Q$. As $R$ is left regular, so there exists $r \in R$ such that $a \in ra^2$. Thus

$$a \in ra^2 = a(ra) \in P(RQ) \subseteq PQ.$$ 

Therefore $P \cap Q \subseteq PQ$. Thus by Theorem 4.3, $P$ is semiprime.

(ii) $\implies$ (i): Let $P \cap Q = PQ$ for each right hyperideal $P$, which is semiprime and every left hyperideal $Q$ of $R$. Since $a^2 \subseteq a^2R$, where $a^2R$ is a right hyperideal of $R$, so is semiprime implies that $a \in a^2R$. Obviously $Ra$ is a left hyperideal of $R$ and $a \in Ra$. Thus

$$a \in (a^2R) \cap (Ra) \subseteq (a^2R)(Ra) = ((Ra)R)a^2 \subseteq (RR)a^2 \subseteq Ra^2.$$ 

Thus $R$ is a left regular.

**Theorem 4.5.** For an LA-semihyperring $R$ having Pure-LI, the given statements are identical.

(i) $R$ is left regular.
Proof. (i) $\implies$ (iii)

Let $P$ be any right and $Q$ be any left hyperideals of $R$. Let $a \in Q \cap P$, then $a \in Q$ and $a \in P$. As $R$ is left regular then there exists $r$ in $R$, such that $a \in ra^2$. Thus

\[
a \in ra^2 = a(ra) = (ra^2)(ra) = (ar)(a^2r) = ((a^2r)a) = ((ra)(ra))a = (ra)(ra) = ((ra)((ar)e))a \subseteq ((RQ)((PR)R))Q \subseteq (Q(PR))Q \subseteq (QP)Q,
\]

therefore, $Q \cap P \subseteq (QP)Q$. Also by Theorem 4.3, $Q$ is semiprime.

(iii) $\implies$ (ii)

Suppose $P$ is left and $Q$ is right hyperideals of $R$. Let $P$ be semiprime. Thus

\[
P \cap Q \subseteq (PQ)P \subseteq (PQ)R = (PR)(RR) = RQ(PR)Q = Q((PR)Q) = Q(RP) \subseteq QP.
\]

(ii) $\implies$ (i)

Since $b \in Rb$, which is left hyperideal of $R$, and $b^2 \subseteq b^2R$, that is semiprime right hyperideal of $R$, so by Theorem 2.2, $b \in b^2R$. Thus

\[
b \in (Rb) \cap (b^2R) \subseteq (Rb)(b^2R) \subseteq (RR)(b^2R) = R(b^2R) = b^2(RR) = (bb)(RR) = (RR)(bb) = Rb^2.
\]

Hence $R$ is left regular.

Lemma 4.1. Each LA-semihyperring $R$ having Pure-LI is left regular if $R$ is pure left (right) invertible.

Proof. Let $r \in R$, then there exists $r' \in R$ such that $r'r = r$. Thus

\[
e = er = e(er) = \left(r'r\right)(er) \in (Rr)(Rr) = (RR)(rr) = Rr^2.
\]

Thus $R$ is left regular. In a similar way, the case of pure right invertible hold.

Theorem 4.6. For a pure left (right) invertible LA-semihyperring $R$, the given statements are identical:

(i) $S$ is left regular;

(ii) $P \cap Q = PQ$, for each right $Q$ is left hyperideal of $R$.

Proof. (i) $\implies$ (ii) : It follows from Theorem 4.4.

Proof. (ii) $\implies$ (i) : It obtains from Lemma 4.1.

Theorem 4.7. The given conditions are identical on an LA-semihyperring $R$ left pure identity:

(i) $R$ is left regular.

(ii) $Q \cap P = QP$, for each semiprime right hyperideal $P$ and left hyperideal $Q$ of $R$.

Proof. (i) $\implies$ (ii) : It obtains by using Lemma 3.5 and Theorem 4.4.
Proof. (ii) ⇒ (i): It is simple. □

Theorem 4.8. The given statements are identical on a pure left (right) invertible LA-semihyperring \( R \):

(i) \( R \) is left regular.
(ii) \( P \cap Q = PQ \), for each \( P \) is right hyperideal and \( Q \) is left hyperideal of \( R \).
(iii) \( R(AA) = A \), for each \( A \) is quasi-hyperideal of \( R \).

Proof. (i) ⇒ (iii)
Suppose \( A \) is any quasi-hyperideal of \( R \). Let \( a \in R \), such that \( a \in A \). As \( R \) is left regular so there exists \( r \) in \( R \), such that \( a \in ra^2 \subseteq RA^2 = R(AA) \). Therefore \( A \subseteq R(AA) \). Now

\[
R(AA) = A(RA) \subseteq A(RR) = AR \quad \text{and} \quad R(AA) = (RR)(AA) = (AA)(RR) = (AA)R = (RA)A \subseteq (RR)A = RA.
\]

therefore, \( R(AA) \subseteq AR \cap RA \subset A \). Hence \( R(AA) = A \), for every quasi-hyperideal \( A \) of \( R \).

(iii) ⇒ (ii)
Suppose \( P \) is left and \( Q \) is right hyperideal of \( R \). So by Lemma 2.1, \( P \cap Q \) is a quasi-ideal of \( R \). Thus by hypothesis (iii), we have

\[
P \cap Q = R((P \cap Q)(P \cap Q)) \subseteq R(PQ) \subseteq P(RQ) \subseteq PQ.
\]

Also, \( PQ \subseteq PR \cap RQ \subseteq P \cap Q \). Hence \( P \cap Q = PQ \), for each right hyperideal \( P \) and left hyperideal \( Q \) of \( R \).

(iii) ⇒ (i): It obtains from Theorem 4.6. □

Theorem 4.9. The given conditions are identical on an LA-semihyperring \( R \) Pure-LI:

(i) \( R \) is left regular;
(ii) \( J = J^3 \), for \( J \) is left hyperideal of \( R \).

Proof. (i) ⇒ (ii) : Suppose \( J \) is left hyperideal of \( R \), then by Lemma 3.4, we have \( J^3 = (JJ)J = JJ \subseteq RJ \subseteq J \).

Furthermore, let \( j \in J \), then there exists \( r \in R \) such that \( j \in rj^2 \). Thus

\[
\begin{align*}
    j & \in rj^2 = j(rj) = (rj^2)(rj) = (j(rj))(rj) = ((jr)(jr))j \\
    & = (((jr)r)j)j \subseteq ((RJ)J)J \subseteq (JJ)J = J^3.
\end{align*}
\]

Hence \( J = J^3 \), for every left hyperideal \( J \) of \( R \).

(ii) ⇒ (i) : Suppose \( J \) is left hyperideal of \( R \) having Pure-LI such that \( J = J^3 \). As \( RJ \) is left hyperideal of \( R \) and \( j \in RJ \). Thus

\[
\begin{align*}
    j & \in RJ = ((Rj)(Rj))Rj = ((RR)(jj))Rj = (Rj^2)(Rj) \\
\end{align*}
\]

Thus \( R \) is left regular. □
Theorem 4.10. The given conditions are identical on an LA-semihyperring $R$ Pure-LI:

(i) $R$ is left regular;
(ii) $L = L^{n+1}$, where $n \in \mathbb{N}$.

Proof. By generalization of proof of Theorem 4.9, gives its proof.

Theorem 4.11. The set of all hyperideals $I_R$ of a left regular LA-semihyperring $R$ having Pure-LI, forms a semilattice structure.

Proof. Let $P, Q \in I_R$, as $P$ and $Q$ are hyperideals of $R$, then $(PQ)R = (PQ)(RR) = (PR)(QR) \subseteq PQ$. Also $R(PQ) = (RR)(PQ) = (RP)(RQ) \subseteq PQ$. Thus $PQ$ is an hyperideal of $R$. Hence $I_R$ is closed. Also by using Lemma 3.6, we have

\[ PQ = P \cap Q = Q \cap P = QP, \]

which implies that $I_R$ is commutative and commutativity gives so is associativity. So by using Lemma 3.4, $P^2 = P$, $\forall \ P \in I_R$. Thus $I_R$ is semilattice.

5. Conclusions
In this paper, the notion of left regular LA-semihyperring is defined and the basic properties of many hyperideals in terms of left regular LA-semihyperrings are discussed. The fundamental characterization of left regular LA-semihyperrings by the properties of their (right, left) hyperideals, bi- (generalized bi-) hyperideals, interior hyperideals and quasi-hyperideals are produced. In our future work, we shall characterized strongly-regular class of LA-semihyperrings with respect to their hyperideals.

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