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Generalized weak structures with associated neighborhoods

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Abstract: In this paper we would like to extend our earlier concept of generalized topologies with associating function. The main idea is to treat the notion of neighborhood in a manner closer to the one used in the area of weak modal logics than to the typical topological interpretation. In particular, we do not insist that a point x has to belong to each of its neighborhoods. Moreover, it may be beyond *any* of its neighborhoods. We connect these general assumptions with the idea of interior (and open set). The whole paper appears as an attempt to reconstruct some basic topological notions in a novel and specific framework. Additionally, we prove some theorems about generalized weak structures.

Key words: Associated neighborhoods, generalized weak structures, modal logic

1. Introduction

In topological spaces, the notion of *open neighborhood* is connected with some strict requirements. For example, we assume that if open set A is a neighborhood of $x \in X$ (where X is our universal set) and $A \subseteq B$ (where B is another open set), then B is a neighborhood of x too. Moreover, the family of open neighborhoods of x is closed under binary intersections. In general, it may be described as a *filter*.

Recent decades have brought many studies on various generalizations of the initial notion of topological space. For example, mathematicians all over the world investigated generalized topologies (see [3]), supra-topologies (see [9], infra-topologies (see [2] and [10]), minimal structures (see [11]), weak structures (see [4]) and, ultimately, generalized weak structures (see [1]). In fact, the latter have been already introduced by Lim in 1966 (in [8]) but not under this name. This author made an interesting attempt to reconstruct topology on the ground of the most elementary assumptions.

Some authors analyze these weak structures in other contexts. For example, there are neutrosophic minimal structures (see [5]) or generalized intuitionistic topological spaces (see [13]). Clearly, in these cases classical sets are replaced with non-classical sets that are used to model uncertainty. The same can be said about neutrosophic biminimal structures analyzed e.g. in [6]

Clearly, the notion of open neighborhood in these generalized structures (if it appears at all) is weaker than in classical topologies. Some of the primary requirements are dropped. However, it seems that in each case we assume that x belongs to each of its neighborhoods.

But the notion of neighborhood is present also in the world of modal logics. So-called *neighborhood semantics* for modal logics (see [11]) is based on the assumption that each point of the universe (that is, each *possible world*) has its own family of neighborhoods. However, this notion in this framework is understood in a

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very general sense. In particular, if A is a neighborhood of x, then it does not mean that $x \in A$. Moreover, empty set can be placed among these neighborhoods. Finally, the whole family can be empty.

Another interesting thing is the notion of interior. If $\mathfrak{g}$ is a generalized weak structure (which means that $\mathfrak{g}$ is just an arbitrary family of sets) then we define $Int(A)$ as $\bigcup\{B \in \mathfrak{g}; B \subseteq A\}$. If so, then this definition is equivalent with the following one: $x \in Int(A) \Leftrightarrow$ there exists $B \in \mathfrak{g}$ such that $B \subseteq A$ and $x \in B$. Clearly, B can be interpreted as a $\mathfrak{g}$-open neighborhood of x.

Our idea is: let us use the second definition of interior but with the assumption that the notion of neighborhood is more general than usually assumed. In particular, we do not want to assume that $x \in B$. It means that x is in some way *connected* or *associated* with its neighborhood B but *maybe* it does not belong to this set. We could say that x has a kind of *access* to its neighborhoods.

This approach is closer to the one known from modal neighborhood semantics. However, our model is not the most general one. This is because we still assume that there is some distinguished family $\mathfrak{g}$ such that if $x \in B \in \mathfrak{g}$, then B is treated as a neighborhood of x. In our future work we would like to discard even this condition and to reformulate the whole concept only in terms of arbitrary neighborhoods.

However, in the present paper we still differ between those points which are in $\bigcup\mathfrak{g}$ and those which are beyond this union. This is because we do not assume that $X \in \mathfrak{g}$. We have already presented an outline of this approach in [14] where our distinguished family $\mathfrak{g}$ had the properties of Császár's generalized topology (i.e. a family closed under arbitrary unions). We proved some theorems and described many features of such structures (equipped with associating function that linked points from $X \setminus \bigcup\mathfrak{g}$ with their $\mathfrak{g}$-open neighborhoods). We analyzed sequences, nets and generalized nets in that framework. However, it appeared that the vast majority of theorems (but not all of them) were true without the assumption about closure of $\mathfrak{g}$ under unions. Hence, we would like to generalize these results in a significant way, adding some new reflections too.

We think that this framework may be used as a convenient semantical model for various non-normal modal logics. The same can be said about the planned more general approach which we have signalized above.

2. Preliminary notions

In this section we introduce some basic notions. The first subsection contains earlier results which are mostly taken from other authors. The second subsection deals with our own concepts.

2.1. Generalized weak structures

First, let us start from the concept of generalized weak structure.

Definition 2.1. [1] Assume that X is a non-empty universal set and $\emptyset \neq \mathfrak{g} \subseteq P(X)$ (that is, $\mathfrak{g}$ is a non-empty class of subsets of X). Then we say that $\mathfrak{g}$ is a *generalized weak structure* (GWS) on X. Any element of $\mathfrak{g}$ is called $\mathfrak{g}$*-open* set. Any complement of $\mathfrak{g}$-open set is called $\mathfrak{g}$*-closed*.

Definition 2.2. Assume that $\mathfrak{g}$ is a GWS on non-empty X. We say that $\mathfrak{g}$ is a:

1. *Weak structure* if $\emptyset \in \mathfrak{g}$.
2. *Minimal structure* if $\emptyset, X \in \mathfrak{g}$.
3. *Infra-topological space* if $\emptyset, X \in \mathfrak{g}$ and $\mathfrak{g}$ is closed under finite intersections.
4. *Generalized topological space* (in the sense of Császár) if $\mathfrak{g}$ is closed under arbitrary unions. In particular, it means that $\emptyset \in \mathfrak{g}$.

5. *Supra-topological space* if $\mathfrak{g}$ is a generalized topological space and $X \in \mathfrak{g}$.
6. *Topological space* if $\mathfrak{g}$ is an infra-topological space closed under arbitrary unions.
7. *Anti-weak structure* if $\emptyset \notin \mathfrak{g}$.
8. *Anti-minimal structure* if $\emptyset, X \notin \mathfrak{g}$.
9. *Anti-topological space* (see [15]) if $\emptyset, X \notin \mathfrak{g}$, any non-trivial finite intersection of subsets from $\mathfrak{g}$ is beyond $\mathfrak{g}$ and any non-trivial union of subsets from $\mathfrak{g}$ is beyond $\mathfrak{g}$. Attention: by "non-trivial intersection (union)" we mean the one that involves at least two different sets.

Definition 2.3. Assume that $\mathfrak{g}$ is a GWS on X and $A \subseteq X$. We define $\mathfrak{g}$-interior and $\mathfrak{g}$-closure of A in the following way (where $-C$ is a complement of C, i.e. C^c):

1. $\mathfrak{g}Int(A) = \bigcup\{B; B \subseteq A, B \in \mathfrak{g}\}$.
2. $\mathfrak{g}Cl(A) = \bigcap\{C; A \subseteq C, -C \in \mathfrak{g}\}$.

Now we may list some properties of $\mathfrak{g}$-interior. We shall omit the proofs because they have been already presented in [1].

Lemma 2.1. *Let $\mathfrak{g}$ be a GWS on X and $A, B \subseteq X$. Then the following statements are true:*

1. $\mathfrak{g}Int(A) \subseteq A$.
2. *If $A \in \mathfrak{g}$, then $\mathfrak{g}Int(A) = A$.*
3. *If $A \subseteq B$, then $\mathfrak{g}Int(A) \subseteq \mathfrak{g}Int(B)$ (*monotonicity*).*
4. $\mathfrak{g}Int(\mathfrak{g}Int(A)) = \mathfrak{g}Int(A)$ *(*idempotence*).*

Remark 2.1. *Note that the converse of Lemma 2.1 (2) need not be true. However, is $\mathfrak{g}$ is closed under unions then it becomes true. This is because in this case the union of all $\mathfrak{g}$-open sets contained in A becomes $\mathfrak{g}$-open too. But if this union is equal to A, then A must be $\mathfrak{g}$-open.*

The next two lemmas are not difficult but they were not present in [1].

Lemma 2.2. *Let $\mathfrak{g}$ be a GWS on X and $\{A_i\}_{i \in J \neq \emptyset}$ be a family of sets. Then $\mathfrak{g}Int(\bigcap_{i \in J} A_i) \subseteq \bigcap_{i \in J} \mathfrak{g}Int(A_i)$.*

Proof. Of course $\bigcap_{i \in J} A_i \subseteq A_k$ for any $k \in J$. From the monotonicity of interior we infer that $\mathfrak{g}Int(\bigcap_{i \in J} A_i) \subseteq \mathfrak{g}Int(A_k)$. However, this is true for any $k \in J$, so we may write that $\mathfrak{g}Int(\bigcap_{i \in J} A_i) \subseteq \bigcap_{i \in J} Int(A_i)$. □

Remark 2.2. *Clearly, the converse of lemma above need not to be true. The reader can consider $X = \{a, b, c, d\}$ with anti-topology $\mathfrak{g} = \{\{a, c\}, \{b\}, \{c, d\}\}$. Take $A = \{a, b, c\}$ and $B = \{b, c, d\}$. Then $\mathfrak{g}Int(A) = \{a, b, c\} = A$ and $\mathfrak{g}Int(B) = \{b, c, d\} = B$. Then $\mathfrak{g}Int(A) \cap \mathfrak{g}Int(A) = \{b, c\} \nsubseteq \mathfrak{g}Int(A \cap B) = \mathfrak{g}Int(\{b\}) = \{b\}$.*

Lemma 2.3. *Let $\mathfrak{g}$ be a GWS on X and $\{A_i\}_{i \in J \neq \emptyset}$ be a family of sets. Then $\bigcup_{i \in J} \mathfrak{g}Int(A_i) \subseteq \mathfrak{g}Int(\bigcup_{i \in J} A_i)$.*

Proof. Assume that $x \in \bigcup_{i \in J} \mathfrak{g}Int(A_i)$. It means that there is some $k \in J$ and some $B \in \mathfrak{g}$ such that $B \subseteq A_k$ and $x \in B$. Clearly, $B \subseteq \bigcup_{i \in J} A_i$. However, B is $\mathfrak{g}$-open, hence $x \in B \subseteq \mathfrak{g}Int(\bigcup_{i \in J} A_i)$. □

Remark 2.3. *It is well known that the converse of the lemma above need not to be true even if our $\mathfrak{g}$ is a topological space.*

Now we may list some properties of $\mathfrak{g}$-closure.

Lemma 2.4. *Let $\mathfrak{g}$ be a GWS on X and $A, B \subseteq X$. Then the following statements are true:*

1. $A \subseteq \mathfrak{g}Cl(A)$.
2. *If* $-A \in \mathfrak{g}$, *then* $\mathfrak{g}Cl(A) = A$.
3. *If* $A \subseteq B$, *then* $\mathfrak{g}Cl(A) \subseteq \mathfrak{g}Cl(B)$ *(*monotonicity*).*
4. $\mathfrak{g}Cl(\mathfrak{g}Cl(A)) = \mathfrak{g}Cl(A)$ *(*idempotence*).*

Remark 2.4. *Note that the converse of Lemma 2.4 (2) may not be true. However, if $\mathfrak{g}$ is closed under unions, then it becomes true. Using de Morgan laws, we have that $\mathfrak{g}Cl(A) = \bigcap\{C; A \subseteq C, -C \in \mathfrak{g}\} = \bigcup\{-C; A \subseteq C, -C \in \mathfrak{g}\}$. But the last union belongs to $\mathfrak{g}$ because of the assumption about closure of $\mathfrak{g}$ under unions.*

The following two lemmas deal with unions and intersections of $\mathfrak{g}$-closures.

Lemma 2.5. *Let $\mathfrak{g}$ be a GWS on X and $\{A_i\}_{i \in J \neq \emptyset}$ be a family of sets. Then $\bigcup_{i \in J} \mathfrak{g}Cl(A_i) \subseteq \mathfrak{g}Cl(\bigcup_{i \in J} A_i)$.*

Proof. For any $k \in J$, $A_k \subseteq \bigcup_{i \in J} A_i$. But then $\mathfrak{g}Cl(A_k) \subseteq \mathfrak{g}Cl(\bigcup_{i \in J} A_i)$. However, this is true for any $k \in J$, so we may write that $\bigcup_{i \in J} \mathfrak{g}Cl(A_i) \subseteq \mathfrak{g}Cl(\bigcup_{i \in J} A_i)$. □

Remark 2.5. *In general, the converse of the lemma above is not true. The reader is encouraged to find a simple counter-example.*

Lemma 2.6. *Let $\mathfrak{g}$ be a GWS on X and $\{A_i\}_{i \in J \neq \emptyset}$ be a family of sets. Then $\mathfrak{g}Cl(\bigcap_{i \in J} A_i) \subseteq \bigcap_{i \in J} \mathfrak{g}Cl(A_i)$.*

Proof. Assume that $x \in \mathfrak{g}Cl(\bigcap_{i \in J} A_i)$. Hence, $x \in B$ for any $\mathfrak{g}$-closed $B \subseteq X$ such that $\bigcap_{i \in J} A_i \subseteq B$. Assume now that $x \notin \bigcap_{i \in J} \mathfrak{g}Cl(A_i)$. Hence there is some $k \in J$ such that $x \notin \mathfrak{g}Cl(A_k)$. Thus, there is $\mathfrak{g}$-closed set C such that $A_k \subseteq C$ and $x \notin C$. However, $\bigcap_{i \in J} A_i \subseteq A_k \subseteq C$, hence $x \in C$. This is contradiction. □

Remark 2.6. *Assume for a moment that $\mathfrak{g}$ is a topological space. Then the last proof could be reformulated in the following way. First, for any $k \in J$, $A_k \subseteq \mathfrak{g}Cl(A_k)$, so $\bigcap_{i \in J} A_i \subseteq \bigcap_{i \in J} \mathfrak{g}Cl(A_i)$. Then, by virtue of idempotency of closure, $\mathfrak{g}Cl(\bigcap_{i \in J} A_i) \subseteq \mathfrak{g}Cl(\bigcap_{i \in J} \mathfrak{g}Cl(A_i))$. But in topological space, $\mathfrak{g}Cl(A_i)$ is $\mathfrak{g}$-closed for any $i \in J$ and, moreover, any intersection of $\mathfrak{g}$-closed sets is $\mathfrak{g}$-closed too. Thus, $\mathfrak{g}Cl(\bigcap_{i \in J} \mathfrak{g}Cl(A_i)) = \bigcap_{i \in J} \mathfrak{g}Cl(A_i)$, so $\mathfrak{g}Cl(\bigcap_{i \in J} A_i) \subseteq \bigcap_{i \in J} \mathfrak{g}Cl(A_i)$.*

Remark 2.7. *Again, it is well known that the converse of the lemma above need not to be true even if our $\mathfrak{g}$ is a topological space.*

We may list some relationships between $\mathfrak{g}$-interior and $\mathfrak{g}$-closure (together with equivalent definitions of both terms).

Lemma 2.7. *[1] Let $\mathfrak{g}$ be a GWS on X, $A \subseteq X$ and $x \in X$. Then the following statements are true:*

1. $-\mathfrak{g}Int(A) = \mathfrak{g}Cl(-A)$.

2. $\mathfrak{g}Int(-A) = -\mathfrak{g}Cl(A)$.

3. $x \in \mathfrak{g}Int(A) \Leftrightarrow$ *there is* $B \in \mathfrak{g}$ *such that* $x \in U \subseteq A$.

4. $x \in \mathfrak{g}Cl(A) \Leftrightarrow B \cap A \neq \emptyset$ *for all* $B \in \mathfrak{g}$ *such that* $x \in B$.

Remark 2.8. *Note that while* $x \in \mathfrak{g}Int(A)$ *implies that* $x \in \bigcup \mathfrak{g}$, *this is not true in case of* $\mathfrak{g}Cl(A)$. *If* $x \in \mathfrak{g}Cl(A)$, *then it is possible that* $x \in X \setminus \bigcup \mathfrak{g}$. *Clearly, in this case* $x \notin B$ *for any* $B \in \mathfrak{g}$. *However, then it is vacuously true that "if* $B \in \mathfrak{g}$ *and* $x \in B$, *then* $B \cap A \neq \emptyset$*".*

2.2. Associated neighborhoods

Let us introduce the idea of a generalized weak structure with associated neighborhoods.

Definition 2.4. We define *GWS with associated neighborhoods* as a triple $(X, \mathfrak{g}, \mathcal{N})$ where X is a non-empty universal set, $\mathfrak{g}$ is a GWS on X and $\mathcal{N}$ is a function. We assume that $\mathcal{N} : X \to P(P(X))$ and:

1. If $x \in \bigcup \mathfrak{g}$, then $A \in \mathcal{N}(x) \Leftrightarrow A \in \mathfrak{g}$ and $x \in A$.
2. If $x \in X \setminus \bigcup \mathfrak{g}$, then if $A \in \mathcal{N}(x)$, then $A \in \mathfrak{g}$.

We shall write $\mathcal{N}_x$ instead of $\mathcal{N}(x)$.

Definition 2.5. Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods and $A \in \mathfrak{g}$. Then we define $A^* = \{x \in X; A \in \mathcal{N}_x\}$.

Example 2.1. *(compare [14]).*

Assume that $X = \mathbb{N}$, $\mathfrak{g} = \{\{1\}, \{1,3\}, \{1,3,5\}, \{1,3,5,7\}, ...\}$. *Note that this GWS is close to the idea of generalized topology in the sense of Császár. However, we do not assume that* $\emptyset \in \mathfrak{g}$. *Hence, it would be better to say that* $\mathfrak{g}$ *is an example of* σ*-structure in the sense of Min.*

If $n \in 2\mathbb{N}+1$, *then we assume that* $A \in \mathcal{N}_n \Leftrightarrow A \in \mathfrak{g}$ *and* $n \in A$.

Now let us define $f : 2\mathbb{N} \to 2\mathbb{N}+1$ *in the following way:* $f(x) = \max\{m; m \in 2\mathbb{N}+1, m < x\}$. *If* $n \in 2\mathbb{N}$, *then we assume that* $A \in \mathcal{N}_n \Leftrightarrow A \in \mathcal{N}_{f(n)}$. *Thus we obtained* $(X, \mathfrak{g}, \mathcal{N})$.

Now $\mathcal{N}_3 = \{\{1,3\}, \{1,3,5\}, \{1,3,5,7\}, ...\}$ *and* $\mathcal{N}_8 = \mathcal{N}_{f(8)} = \mathcal{N}_7 = \{\{1,3,5,7\}, \{1,3,5,7,9\}, \{1,3,5,7,9,11\}, ...\}$.

Now let us define another family of associated neighborhoods, namely $\mathcal{M}$. *In case of odd numbers, let us define* $\mathcal{M}_n$ *just like* $\mathcal{N}_n$. *Now assume that if* $n \in 2\mathbb{N}$, *then* $A \in \mathcal{M}_n \Leftrightarrow A \in \mathfrak{g}$ *and* $A \notin \mathcal{M}_{f(n)+2}$. *Thus we obtained* $(X, \mathfrak{g}, \mathcal{M})$.

For example, $\mathcal{M}_8 = \{\{1\}, \{1,3\}, \{1,3,5,7\}\}$.

Of course in the example above we have some kind of regularity, both in the definition of GWS and neighborhoods. This feature can be considered as an advantage. However, the reader is aware that such regularities (like closure of $\mathfrak{g}$ under non-empty unions) are not necessary. The whole framework is very general.

Now let us introduce the notion analogous to interior in this environment.

Definition 2.6. Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Suppose that $x \in X$ and $A \subseteq X$. Then we say that $x \in \mathcal{N}Int(A) \Leftrightarrow$ there exists $B \in \mathcal{N}_x$ such that $B \subseteq A$.

3. Properties and examples

Let us prove some properties of $\mathcal{N}Int$ (and disprove some hypothetical properties).

Lemma 3.1. *Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods, $A, B \subseteq X$. Then the following properties hold:*

1. *If $A \subseteq B$, then $\mathcal{N}Int(A) \subseteq \mathcal{N}Int(B)$.*
2. *$\mathfrak{g}Int(A) \subseteq \mathcal{N}Int(A)$. In particular, if $A \in \mathfrak{g}$, then $A \subseteq \mathcal{N}Int(A)$.*
3. *$\mathcal{N}Int(A) \cap \bigcup \mathfrak{g} = \mathfrak{g}Int(A)$.*
4. *$\mathcal{N}Int(\mathcal{N}Int(A)) = \mathcal{N}Int(A)$.*
5. *If $A \in \mathfrak{g}$, then $A^* \subseteq \mathcal{N}Int(A)$.*

Proof.

1. Assume that $x \in \mathcal{N}Int(A)$. Then there is $G \in \mathcal{N}_x$ such that $G \subseteq A$. However, $A \subseteq B$, hence $G \subseteq A \subseteq B$. But this means that $x \in \mathcal{N}Int(B)$.
2. Assume that $x \in \mathfrak{g}Int(A)$. It means that there is some $B \in \mathfrak{g}$ such that $x \in B \subseteq A$. But by the very definition of $\mathcal{N}$ function, it means that $B \in \mathcal{N}_x$. Then $x \in \mathcal{N}Int(A)$. Now, if $A \in \mathfrak{g}$ and $x \in A$, then $A \in \mathcal{N}_x$ and $A \subseteq A$, so $x \in \mathcal{N}Int(A)$. Alternatively, we could use the first inclusion and the fact that if $A \in \mathfrak{g}$, then $\mathfrak{g}Int(A) = A$.
3. $(\subseteq)$. Let $x \in \mathcal{N}Int(A) \cap \bigcup \mathfrak{g}$. Then there is $B \in \mathcal{N}_x$ such that $B \subseteq A$. But $B \in \mathfrak{g}$ and $x \in \bigcup \mathfrak{g}$, hence $x \in B$. This means that $x \in \mathfrak{g}Int(A)$.
   $(\supseteq)$. Let $x \in \mathfrak{g}Int(A)$. Then $x \in \mathcal{N}Int(A)$ (from (2)). Clearly, $x \in \bigcup \mathfrak{g}$. Hence $x \in \mathcal{N}Int(A) \cap \bigcup \mathfrak{g}$.
4. $(\subseteq)$. Let $x \in \mathcal{N}Int(\mathcal{N}Int(A))$. This means that there is $G \in \mathcal{N}_x$ such that $G \subseteq \mathcal{N}Int(A)$. Clearly, $G \in \mathfrak{g}$. Assume now that $x \notin \mathcal{N}Int(A)$. Thus, for any $K \in \mathcal{N}_x$, $K \nsubseteq A$. In particular, $G \nsubseteq A$. Then there is some $y \in G$ such that $y \notin A$. But $y \in \mathcal{N}Int(A)$, so there is $H \in \mathcal{N}_y$ such that $H \subseteq A$. However, $y \in \bigcup \mathfrak{g}$. Then $y \in H$ (again, recall the definition of $\mathcal{N}$). Consequently, $y \in A$. Now we can say that there is some neighborhood of x (namely, G) such that $G \subseteq A$. This means that $x \in \mathcal{N}Int(A)$.
   $(\supseteq)$. Let $x \in \mathcal{N}Int(A)$. Then there is $G \in \mathcal{N}_x$ such that $G \subseteq A$. Clearly, $G \in \mathfrak{g}$. Suppose that $x \notin \mathcal{N}Int(\mathcal{N}Int(A))$. Then for any $K \in \mathcal{N}_x$, $K \nsubseteq \mathcal{N}Int(A)$. In particular, $G \nsubseteq \mathcal{N}Int(A)$. Then there is some $y \in G$ such that for any $H \in \mathcal{N}_y$, $H \nsubseteq A$. However, $G \in \mathcal{N}_y$. This is contradiction.
5. Let $x \in A^*$. It means that $A \in \mathcal{N}_x$. But $A \in \mathfrak{g}$ and $A \subseteq A$, hence $x \in \mathcal{N}Int(A)$.

□

Now let us define two specific subsets.

Definition 3.1. Let $(X, \mathfrak{g}, \mathcal{N})$ be a GWS with associated neighborhoods. Then we define $\Phi = \{x \in X; \emptyset \in \mathcal{N}_x\}$ and $\mho = \{x \in X; \mathcal{N}_x = \emptyset\}$.

Theorem 3.1. *Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Then $\mathcal{N}Int(X) = X \Leftrightarrow \mho = \emptyset$.*

Proof. $(\Rightarrow)$. Suppose that $\mho \neq \emptyset$. It means that there is some $x \in X$ such that $\mathcal{N}_x = \emptyset$. Clearly, $x \notin \mathcal{N}Int(X)$ (because there is no any neighborhood in $\mathcal{N}_x$; in particular, there is no any $A \in \mathcal{N}_x$ such that $A \subseteq X$). But now $\mathcal{N}Int(X) \neq X$ and this is contradiction.

$(\Leftarrow)$. Assume that there is $x \in X$ such that $x \notin \mathcal{N}Int(X)$. Hence, for any $A \in \mathcal{N}_x$, $A \nsubseteq X$. Clearly, this is possible if and only if there are no elements in $\mathcal{N}_x$. Hence $\mho \neq \emptyset$ because $x \in \mho$. But this is contradiction. □

Theorem 3.2. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Then* $\mathcal{N}Int(\emptyset) = \emptyset \Leftrightarrow \Phi = \emptyset$.

Proof. $(\Rightarrow)$. Suppose that $\Phi \neq \emptyset$. Then there is $x \in X$ such that $\emptyset \in \mathcal{N}_x$. Then $x \in \mathcal{N}Int(\emptyset)$. Hence $\mathcal{N}Int(\emptyset) \neq \emptyset$ and this is contradiction.

$(\Leftarrow)$. Assume that $\mathcal{N}Int(\emptyset) \neq \emptyset$. Hence there is $x \in X$ for which there is $A \in \mathcal{N}_x$ such that $A \subseteq \emptyset$. But then A must be empty. If so, then $\Phi \neq \emptyset$ because $x \in \Phi$. This is contradiction. □

The last two theorems suggest us to define two significant classes:

Definition 3.2. Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. We say that this triple is:

1. *Normal* if and only if $\mho = \emptyset$.
2. *Sensible* if and only if $\Phi = \emptyset$.

The following two lemmas are simple:

Lemma 3.2. *Let* $(X, \mathfrak{g}, \mathcal{N})$ *be a GWS with associated neighborhoods. If* $\Phi \neq \emptyset$ *(which means that the structure is not sensible), then* $\mathfrak{g}$ *is a weak structure.*

Proof. If $\Phi \neq \emptyset$, then there is some $x \in \Phi$ such that $\emptyset \in \mathcal{N}_x$. In particular, it means that $\emptyset \in \mathfrak{g}$, hence $\mathfrak{g}$ is a weak structure. □

Remark 3.1. *The converse of above lemma need not to be true. For example, let* $X = \{a, b\}$, $\mathfrak{g} = \{\emptyset, \{a\}\}$, $\mathcal{N}_b = \{\{a\}\}$. *Now,* $\emptyset \notin \mathcal{N}_b$. *Of course* $\emptyset \notin \mathcal{N}_a = \{\{a\}\}$.

Lemma 3.3. *Let* $(X, \mathfrak{g}, \mathcal{N})$ *be a GWS with associated neighborhoods. Then* $\Phi \subseteq X \setminus \bigcup \mathfrak{g}$.

Proof. Assume on the contrary that $x \in \Phi \cap \bigcup \mathfrak{g}$. In particular, it means that $x \in \bigcup \mathfrak{g}$. Now, if $\emptyset \in \mathcal{N}_x$, then $\emptyset \in \mathfrak{g}$ and $x \in \emptyset$. While the first condition is possible (when $\mathfrak{g}$ is a weak structure), then the second one is never satisfied. □

In the next theorem we deal again with relationships between $\bigcup \mathfrak{g}$ and $X \setminus \bigcup \mathfrak{g}$.

Theorem 3.3. *Let* $(X, \mathfrak{g}, \mathcal{N})$ *be a GWS with associated neighborhoods. Suppose that there is some* $B \subseteq X$ *such that for any* $A \in \mathfrak{g}$ *the following relationship holds: if* $A \neq \emptyset$, *then* $A \nsubseteq B$. *Then* $\mathcal{N}Int(B) = \emptyset$ *or* $\mathcal{N}Int(B) \subseteq \Phi$.

Proof. Assume that $\mathcal{N}Int(B) \neq \emptyset$ and $\mathcal{N}Int(B) \nsubseteq \Phi$. Hence there is some $x \in \mathcal{N}Int(B)$ such that $\emptyset \notin \mathcal{N}_x$. But this means that $\mathcal{N}_x$ contains only non-empty $\mathfrak{g}$-open sets. However, we assumed that there are no such sets contained in B. □

Remark 3.2. *Note that in the preceding theorem we did not have to assume that* $B \subseteq \bigcup \mathfrak{g}$*. Our assumption was weaker.*

Lemma 3.4. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Suppose that there is at least one* $B \subseteq X$ *such that* $\mathcal{N}Int(B) = \emptyset$*. Then our structure is sensible.*

Proof. Suppose that $\Phi \neq \emptyset$. Then there is some $x \in \Phi$. Hence, there is $\emptyset \in \mathcal{N}_x$. Of course, $\emptyset \subseteq B$. But then $x \in \mathcal{N}Int(B)$. □

We may prove two important theorems: one about unions and one about intersections.

Theorem 3.4. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Suppose that* $J \neq \emptyset$ *and* $\{A_i\}_{i \in J}$ *is a family of subsets of* X*. Then* $\bigcup_{i \in J} \mathcal{N}Int(A_i) \subseteq \mathcal{N}Int(\bigcup_{i \in J} A_i)$*.*

Proof. Let $x \in \bigcup_{i \in J} \mathcal{N}Int(A_i)$. Hence, there is $k \in J$ such that $x \in \mathcal{N}Int(A_k)$. Then there is $B \in \mathcal{N}_x$ such that $B \subseteq A_k$. But then $B \subseteq A_k \subseteq \bigcup_{i \in J} A_i$. Thus $x \in \mathcal{N}Int(\bigcup_{i \in J} A_i)$. □

Remark 3.3. *The converse need not to be true. Consider the following case. Let* $X = \{a, b, c, d, e\}$ *and* $\mathfrak{g} = \{\{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$*. Let* $\mathcal{N}_e = \{\{a, c\}\}$*,* $A = \{a, b\}$ *and* $B = \{b, c\}$*. Now* $e \notin \mathcal{N}Int(A)$ *(because* $\{a, c\} \nsubseteq \{a, b\}$*). Moreover,* $e \notin \mathcal{N}Int(B)$ *(because* $\{a, c\} \nsubseteq \{b, c\}$*). However,* $e \in \mathcal{N}Int(A \cup B)$ *because* $\{a, c\} \subseteq \{a, b, c\}$*.*

Theorem 3.5. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Suppose that* $J \neq \emptyset$ *and* $\{A_i\}_{i \in J}$ *is a family of subsets of* X*. Then* $\mathcal{N}Int(\bigcap_{i \in J} A_i) \subseteq \bigcap_{i \in J} \mathcal{N}Int(A_i)$*.*

Proof. Let $x \in \mathcal{N}Int(\bigcap_{i \in J} A_i)$. Hence, there is $B \in \mathcal{N}_x$ such that $B \subseteq \bigcap_{i \in J} A_i$. Hence, for any $i \in J$, $B \subseteq A_i$. But this means that $x \in \mathcal{N}Int(A_i)$ for any $i \in J$. Thus, $x \in \bigcap_{i \in J} \mathcal{N}Int(A_i)$. □

Remark 3.4. *Again, the converse need not to be true. Consider* $X = \{a, b, c, d, e\}$ *and* $\mathfrak{g} = \{\{a\}, \{c\}, \{a, b, d\}\}$*. Assume that* $\mathcal{N}_e = \{\{a\}, \{c\}\}$*. Let* $A = \{a, b\}$ *and* $B = \{b, c\}$*. Clearly,* $e \in \mathcal{N}Int(A)$ *because* $\{a\} \subseteq A$*. Analogously,* $e \in \mathcal{N}Int(B)$ *because* $\{c\} \subseteq B$*. However,* $A \cap B = \{b\}$*, so* $e \notin \mathcal{N}Int(A \cap B)$*.*

Lemma 3.5. *It is possible to establish such GWS with associated neighborhoods that for some* $A \subseteq X$ *the following statements will be true:*

1. $\mathcal{N}Int(A) \nsubseteq A$*.*
2. $A \nsubseteq \mathcal{N}Int(A)$*.*

Proof.

1. For example, let $X = \{a, b, c, d\}$, $\mathfrak{g} = \{\{a\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{N}_d = \{\{a\}, \{b, c\}\}$. Then $\bigcup \mathfrak{g} = \{a, b, c\}$. Consider $A = \{a, b\}$. Now $d \in \mathcal{N}Int(A)$ because $\{a\} \in \mathcal{N}_d$ and $\{a\} \subseteq A$. However, $d \notin A$.
2. Take the same GWS and $\mathcal{N}$ as in the previous example. Consider $B = \{c, d\}$. Now let us think about the element d. Clearly, $d \in B$. However, no neighborhood of d is contained in B. Hence, $d \notin \mathcal{N}Int(B)$.

□

Remark 3.5. *One could say that the notion of "interior" is not appropriate here because it is possible that $A \nsubseteq \mathcal{N}Int(A)$ while it would be natural to assume that "interior of a set" is contained in this set. We can understand such an objection. However, we appealed to the fact that even if our $x \in X \setminus \bigcup \mathfrak{g}$ does not belong to A, then maybe there is still some neighborhood B of x such that $B \subseteq A$. This neighborhood is "representative" for x and gives it some "entry" to A.*

Lemma 3.6. *Suppose that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Assume that $\mathfrak{g}$ is closed under arbitrary unions. Let $A \subseteq \mathcal{N}Int(A) \subseteq \bigcup \mathfrak{g}$. Then $A \in \mathfrak{g}$.*

Proof. First, for any $x \in A$ there is $B \in \mathcal{N}_x$ such that $B \subseteq A$. Of course $B \in \mathfrak{g}$. But $x \in \bigcup \mathfrak{g}$, hence $x \in B$. Thus, $x \in \mathfrak{g}Int(A)$. Thus, $A \subseteq \mathfrak{g}Int(A)$. On the other hand, $\mathfrak{g}Int(A) \subseteq A$ (this is always true). Hence, $A = \mathfrak{g}Int(A)$. However, $\mathfrak{g}$ is closed under unions. Hence, $\mathfrak{g}Int(A) \in \mathfrak{g}$. But then $A \in \mathfrak{g}$. □

Remark 3.6. *Note that the lemma above allows us to say that for* any $(X, \mathfrak{g}, \mathcal{N})$ *the following holds: if $A \subseteq \mathcal{N}Int(A)$, then $A = \mathfrak{g}Int(A)$.*

Now we may distinguish three classes of sets.

Definition 3.3. Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Suppose that $A \subseteq X$. We say that A is:

1. $\mathcal{N}$-open $\Leftrightarrow$ $A = \mathcal{N}Int(A)$.
2. $d\mathcal{N}$-open $\Leftrightarrow$ $\mathcal{N}Int(A) \subseteq A$ (*down $\mathcal{N}$-open*).
3. $u\mathcal{N}$-open $\Leftrightarrow$ $A \subseteq \mathcal{N}Int(A)$ (*upper $\mathcal{N}$-open*).

We prove the following lemma which makes $\mathcal{N}$-open sets more understandable.

Lemma 3.7. *Let $(X, \mathfrak{g}, \mathcal{N})$ be a GWS with associated neighborhoods. Let $A \in \mathfrak{g}$. Then $A \cup \mathcal{N}Int(A)$ is $\mathcal{N}$-open.*

Proof. Let $G = A \cup \mathcal{N}Int(A)$. We would like to prove that $\mathcal{N}Int(G) = G$. If $A = \mathcal{N}Int(A)$, then $G = A \cup A = A = \mathcal{N}Int(A)$ and we are immediately ready. If not, then let us discuss the following reasoning.

($\subseteq$). Let $x \in \mathcal{N}Int(G)$. It means that $x \in \mathcal{N}Int(A \cup \mathcal{N}Int(A))$. Hence, there is some $B \in \mathcal{N}_x$ such that $B \subseteq A \cup \mathcal{N}Int(A)$. However, $B \in \mathfrak{g}$, hence $B \subseteq \bigcup \mathfrak{g}$. But (as we already know from Lemma 3.1) $\mathcal{N}Int(A) \cap \bigcup \mathfrak{g} = \mathfrak{g}Int(A) = A$. Thus, $B \subseteq A$. But now we can say that $x \in \mathcal{N}Int(A)$. Hence, $x \in G$.

($\supseteq$). Let $x \in G$. It means that $x \in A$ or $x \in \mathcal{N}Int(A)$. If $x \in A$, then $A \in \mathcal{N}_x$ (because $A \in \mathfrak{g}$), hence $x \in \mathcal{N}Int(G)$ (because $A \subseteq G$). If $x \in \mathcal{N}Int(A)$, then there is $H \in \mathcal{N}_x$ such that $H \subseteq A \subseteq G$. But then $x \in \mathcal{N}Int(G)$. □

Now we have the following two theorems.

Theorem 3.6. *Suppose that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Let $x \in X$. If $\mathcal{N}_x \neq \emptyset$, then there is $\mathcal{N}$-open set $G \subseteq X$ such that $x \in G$.*

Proof. We know that $\mathcal{N}_x \neq \emptyset$. Hence, there is at least one $A \in \mathcal{N}_x$. In particular, $A \in \mathfrak{g}$. If $A = \mathcal{N}Int(A)$, then we are ready and $A = G$. If not, then we may use Lemma 3.7 to say that $G = A \cup \mathcal{N}Int(A)$. □

Theorem 3.7. *Suppose that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Let $x \in X$. Assume that there exists u$\mathcal{N}$-open set $G \subseteq X$ such that $x \in G$. Then $\mathcal{N}_x \neq \emptyset$.*

Proof. Assume on the contrary that $\mathcal{N}_x = \emptyset$. We know that $G \subseteq \mathcal{N}Int(G)$ (because G is u$\mathcal{N}$-open). Hence, $x \in \mathcal{N}Int(G)$. But then there is $H \in \mathcal{N}_x$ such that $H \subseteq G$. Then $\mathcal{N}_x \neq \emptyset$ and this is contradiction. □

4. Closure and $\mathcal{N}$-closed sets

If we used our associated neighborhoods to define certain kind of interior (namely, $\mathcal{N}$-interior), then it is natural to establish the notion of closure.

Definition 4.1. Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Suppose that $x \in X$ and $A \subseteq X$. Then we say that $x \in \mathcal{N}Cl(A) \Leftrightarrow$ for each $B \in \mathcal{N}_x$, $G \cap A \neq \emptyset$.

We have the following lemma:

Lemma 4.1. *Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Then the following properties hold:*

1. *If $A \subseteq B$, then $\mathcal{N}Cl(A) \subseteq \mathcal{N}Cl(B)$.*
2. *$\mathcal{N}Cl(A) \subseteq \mathfrak{g}Cl(A)$.*
3. *$\mathfrak{g}Cl(A) \cap \bigcup \mathfrak{g} \subseteq \mathcal{N}Cl(A)$.*
4. *$\mathcal{N}Cl(\mathcal{N}Cl(A)) = \mathcal{N}Cl(A)$.*

Proof.

1. Let $x \in \mathcal{N}Cl(A)$. Then for any $G \in \mathcal{N}_x$, $G \cap A \neq \emptyset$. Hence, for any $G \in \mathcal{N}_x$, there is some $y \in G \cap A$. But then $y \in A \subseteq B$, so $y \in B$. This means that for any $G \in \mathcal{N}_x$, $G \cap B \neq \emptyset$ and thus $x \in \mathcal{N}Cl(B)$.
2. Let $x \in \mathcal{N}Cl(A)$. Hence, for any $B \in \mathcal{N}_x$, $B \cap A \neq \emptyset$. But for each $B \in \mathcal{N}_x$ it is true that $B \in \mathfrak{g}$. But then $B \cap A \neq \emptyset$ for all $B \in \mathfrak{g}$ such that $x \in B$. Hence, $x \in \mathfrak{g}Cl(A)$ (by means of Lemma 2.7 (4)). Note that this is trivially true even if $x \in X \setminus \bigcup \mathfrak{g}$ which means that the set of $\mathfrak{g}$-open sets to which x belongs is empty.
3. Let $x \in \mathfrak{g}Cl(A) \cap \bigcup \mathfrak{g}$. In particular, it means that for any $B \in \mathfrak{g}$ such that $x \in B$, $B \cap A \neq \emptyset$. However, $x \in \bigcup \mathfrak{g}$, so $\mathcal{N}_x$ consists exactly of B of this form. Hence, we can say that for any $B \in \mathcal{N}_x$, $B \cap A \neq \emptyset$. But this means that $x \in \mathcal{N}Cl(A)$.
4. $\subseteq$. Let $x \in \mathcal{N}Cl(\mathcal{N}Cl(A))$. Let $B \in \mathcal{N}_x$. Clearly, $B \cap \mathcal{N}Cl(A) \neq \emptyset$. Hence, there is some $y \in B \cap \mathcal{N}Cl(A)$. But $y \in \mathcal{N}Cl(A)$, so for any $C \in \mathcal{N}_y$, $C \cap A \neq \emptyset$. However, $y \in B$ and $B \in \mathfrak{g}$, so $B \in \mathcal{N}_y$. Thus $B \cap A \neq \emptyset$. This is true for any such B, hence $x \in \mathcal{N}Cl(A)$.
   $\supseteq$. Assume that $x \in \mathcal{N}Cl(A)$. Let $B \in \mathcal{N}_x$. Clearly, $B \cap A \neq \emptyset$. However, $A \subseteq \mathfrak{g}Cl(A)$, hence $B \cap \mathfrak{g}Cl(A) \neq \emptyset$. But $B \in \mathfrak{g}$, so $B \subseteq \bigcup \mathfrak{g}$. Hence, $B \cap \bigcup \mathfrak{g} \cap \mathfrak{g}Cl(A) \neq \emptyset$. But this implies that $B \cap \mathcal{N}Cl(A) \neq \emptyset$ (because $\mathfrak{g}Cl(A) \cap \bigcup \mathfrak{g} \subseteq \mathcal{N}Cl(A)$). Hence $x \in \mathcal{N}Cl(\mathcal{N}Cl(A))$.

□

The following theorems are analogous to Th. 3.1 and Th. 3.2.

Theorem 4.1. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Then* $\mathcal{N}Cl(\emptyset) = \emptyset \Leftrightarrow \mho = \emptyset$.

Proof. $(\Rightarrow)$. Assume that $\mho \neq \emptyset$. Hence there is some $x \in X$ such that $\mathcal{N}_x = \emptyset$. But then it is vacuously true that $x \in \mathcal{N}Cl(\emptyset)$. Hence $\mathcal{N}Cl(\emptyset) \neq \emptyset$ and this is contradiction.

$(\Leftarrow)$. Assume that $\mathcal{N}Cl(\emptyset) \neq \emptyset$. Hence there is $x \in \mathcal{N}Cl(\emptyset)$. Moreover, $\mathcal{N}_x \neq \emptyset$. Hence there is some $B \in \mathcal{N}_x$. But then $\emptyset \cap B \neq \emptyset$. Again, contradiction. □

Theorem 4.2. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Then* $\mathcal{N}Cl(X) = X \Leftrightarrow \Phi = \emptyset$.

Proof. $(\Rightarrow)$. Since $\mathcal{N}Cl(X) = X$, then it is not possible that there exists $x \in X$ such that $\emptyset \in \mathcal{N}_x$. It would mean that $\emptyset \cap X \neq \emptyset$.

$(\Leftarrow)$. Assume that there is $x \in X$ such that $\emptyset \in \mathcal{N}_x$. But then $\emptyset \cap X = \emptyset$. However, this implies that $x \notin \mathcal{N}Cl(X)$. Thus $\mathcal{N}Cl(X) \neq X$. □

As for the unions and intersections we have the following two theorems. In general, their converses are not true (we encourage the reader to find proper counter-examples).

Theorem 4.3. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Suppose that* $J \neq \emptyset$ *and* $\{A_i\}_{i \in J}$ *is a family of subsets of* X. *Then* $\bigcup_{i \in J} \mathcal{N}Cl(A_i) \subseteq \mathcal{N}Cl(\bigcup_{i \in J} A_i)$.

Proof. Let $x \in \bigcup_{i \in J} \mathcal{N}Cl(A_i)$. Hence there is $k \in J$ such that $x \in \mathcal{N}Cl(A_k)$. Then for any $B \in \mathcal{N}_x$, $B \cap A_k \neq \emptyset$. But $A_k \subseteq \bigcup_{i \in J} A_i$. Hence $B \cap \bigcup_{i \in J} A_i \neq \emptyset$. Thus $x \in \mathcal{N}Cl(\bigcup_{i \in J} A_i)$. □

Theorem 4.4. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Suppose that* $J \neq \emptyset$ *and* $\{A_i\}_{i \in J}$ *is a family of subsets of* X. *Ten* $\mathcal{N}Cl(\bigcap_{i \in J} A_i) \subseteq \bigcap_{i \in J} \mathcal{N}Cl(A_i)$.

Proof. Let $x \in \mathcal{N}Cl(\bigcap_{i \in J} A_i)$. Hence, for any $B \in \mathcal{N}_x$, $B \cap \bigcap_{i \in J} A_i \neq \emptyset$. Thus, for any $i \in J$, $B \cap A_i \neq \emptyset$. Hence, $x \in \mathcal{N}Cl(A_i)$ for each $i \in J$. But then $x \in \bigcap_{i \in J} \mathcal{N}Cl(A_i)$. □

Let us distinguish three classes of sets:

Definition 4.2. Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Suppose that $A \subseteq X$. We say that A is:

1. $\mathcal{N}$-closed $\Leftrightarrow$ $A = \mathcal{N}Cl(A)$.
2. $d\mathcal{N}$-closed $\Leftrightarrow$ $\mathcal{N}Cl(A) \subseteq A$ (*down* $\mathcal{N}$*-closed*).
3. $u\mathcal{N}$-closed $\Leftrightarrow$ $A \subseteq \mathcal{N}Cl(A)$ (*upper* $\mathcal{N}$*-closed*).

Now let us prove a theorem which describes the relationship between $\mathcal{N}$-open and $\mathcal{N}$-closed sets.

Theorem 4.5. *Assume that* $(X, \mathfrak{g}, \mathcal{N})$ *is a GWS with associated neighborhoods. Assume that* $A \subseteq X$ *is* $\mathcal{N}$*-open. Then* $-A$ *(that is,* $X \setminus A$*) is* $\mathcal{N}$*-closed.*

Proof. Since A is $\mathcal{N}$-open, we have that $\mathcal{N}Int(A) = A$. We would like to show that $\mathcal{N}Cl(-A) = -A$.

($\subseteq$). Let $x \in \mathcal{N}Cl(-A)$. It means that for any $B \in \mathcal{N}_x$, $B \cap -A \neq \emptyset$. Hence $B \nsubseteq A$. Suppose that $x \in A$. But it would mean that there is at least one $C \in \mathcal{N}_x$ such that $C \subseteq A$. However, we see that it is not possible. Hence $x \in -A$.

($\supseteq$). Let $x \in -A$. Hence, $x \notin A$. Thus $x \notin \mathcal{N}Int(A)$. Hence, there is no $C \in \mathcal{N}_x$ such that $C \subseteq A$. Thus, for any $B \in \mathcal{N}_x$, $B \nsubseteq A$. If so, then $B \cap -A \neq \emptyset$. Hence, $x \in \mathcal{N}Cl(-A)$. □

Lemma 4.2. *It is possible to establish such GWS with associated neighborhoods that for some $A \subseteq X$ the following statements will be true:*

1. $\mathcal{N}Cl(A) \nsubseteq A$.
2. $A \nsubseteq \mathcal{N}Cl(A)$.

Proof. We omit proofs and we leave them to the reader. As for 2., it is not difficult to find such a GWS $(X, \mathfrak{g}, \mathcal{N})$ that there will be certain $A \subseteq X$ and some $x \in X$ such that $x \notin A$ but each $\mathfrak{g}$-neighborhood of x has non-empty intersection with A. □

5. About $\mathcal{E}$-open sets

In this section we would like to analyze some notions which rely on the notions presented in the preceding section.

Definition 5.1. Let $(X, \mathfrak{g}, \mathcal{N})$ be a GWS with associated neighborhoods. Assume that $x \in X$. We define $\mathcal{E}_x = \{A \subseteq X; A \text{ is } \mathcal{N}\text{-open and } x \in A\}$.

We may formulate the following theorem.

Theorem 5.1. *Let $(X, \mathfrak{g}, \mathcal{N})$ be a GWS with associated neighborhoods. Assume that $\mathfrak{g}$ is closed under arbitrary unions. Assume that $\mathcal{N}_y = \emptyset$ for any $y \in X \setminus \bigcup \mathfrak{g}$. Then for any $x \in X$, $\mathcal{N}_x = \mathcal{E}_x$.*

Proof. Assume that $x \in X \setminus \bigcup \mathfrak{g}$. Then $\mathcal{N}_x = \emptyset$. Suppose that $\mathcal{E}_x \neq \emptyset$, so there is some $A \in \mathcal{E}_x$. Clearly, $x \in A$ and $\mathcal{N}Int(A) = A$. Hence, $x \in \mathcal{N}Int(A)$, so there is $B \in \mathcal{N}_x$ such that $B \subseteq A$. But $\mathcal{N}_x = \emptyset$. We obtained contradiction. Hence, $\mathcal{N}_x = \emptyset = \mathcal{E}_x$.

Now suppose that $x \in \bigcup \mathfrak{g}$.

($\subseteq$). Let $A \in \mathcal{N}_x$. Clearly, $x \in A$ and $A \in \mathfrak{g}$. Hence, $A = \mathfrak{g}Int(A) \subseteq \mathcal{N}Int(A)$. Now assume that $y \in \mathcal{N}Int(A)$. Hence, $\mathcal{N}_y \neq \emptyset$. This implies that $y \in \bigcup \mathfrak{g}$ and there is $B \in \mathfrak{g}$ such that $y \in B \subseteq A$. Thus, $y \in A$ and $\mathcal{N}Int(A) = A$.

($\supseteq$). Let $A \in \mathcal{E}_x$. Then $x \in A$ and $\mathcal{N}Int(A) = A$. Suppose that $A \notin \mathcal{N}_x$. Then $x \notin A$ (this is an immediate contradiction) or $A \notin \mathfrak{g}$. Let us consider the second statement. Suppose that $A \nsubseteq \bigcup \mathfrak{g}$. Hence, $A \cap (X \setminus \bigcup \mathfrak{g}) \neq \emptyset$. Thus, there is some $y \in A \cap (X \setminus \bigcup \mathfrak{g})$. In particular, $y \in A$, hence $y \in \mathcal{N}Int(A)$. But then $\mathcal{N}_y \neq \emptyset$ and this is contradiction. Hence we see that $A = \mathcal{N}Int(A) \subseteq \bigcup \mathfrak{g}$. Now we use Lemma 3.6 to conclude that $A \in \mathfrak{g}$. □

Let us define another notion of interior which relies on the idea of $\mathcal{E}_x$-open set.

Definition 5.2. Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods. Suppose that $A \subseteq X$. Then we say that $x \in \mathcal{E}Int(A) \Leftrightarrow$ there exists $B \in \mathcal{E}_x$ such that $B \subseteq A$. We say that $x \in \mathcal{E}Cl(A) \Leftrightarrow$ for any $C \in \mathcal{N}_x$, $C \cap A \neq \emptyset$. We say that A is $\mathcal{E}$-open if $\mathcal{E}Int(A) = A$ and $\mathcal{E}$-closed if $\mathcal{E}Cl(A) = A$.

We have the lemma below:

Lemma 5.1. *Assume that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods and $A \subseteq X$. Then the following properties are true:*

1. $\mathcal{E}Int(A) \subseteq A$.
2. $\mathcal{E}Int(\mathcal{E}Int(A)) = \mathcal{E}Int(A)$.

Proof.

1. Let $x \in \mathcal{E}Int(A)$. Hence, there is $B \in \mathcal{E}_x$ such that $B \subseteq A$. But, by the very definition of $\mathcal{E}_x$, $x \in B$ and thus $x \in A$.
2. $(\subseteq)$. This is obvious because of the previous point.
   $(\supseteq)$. Assume that $x \in \mathcal{E}Int(A)$. Hence, there is $B \in \mathcal{E}_x$ such that $B \subseteq A$. Suppose that $x \notin \mathcal{E}Int(\mathcal{E}Int(A))$. Hence, for any $C \in \mathcal{E}_x$, $C \nsubseteq \mathcal{E}Int(A)$. In particular, $B \nsubseteq \mathcal{E}Int(A)$. Then there is $y \in B$ such that $y \notin \mathcal{E}Int(A)$. Hence, for any $D \in \mathcal{E}_y$, $D \nsubseteq A$. However, B is $\mathcal{N}$-open and $y \in B$, so $B \in \mathcal{E}_y$. This is contradiction.

□

We may prove the following theorem.

Theorem 5.2. *Suppose that $(X, \mathfrak{g}, \mathcal{N})$ is a GWS with associated neighborhoods and $\mathcal{E}$ is a collection of all $\mathcal{E}$-open sets contained in X. Then $\mathcal{E}$ is closed under arbitrary unions. In particular, $\emptyset \in \mathcal{E}$. If $(X, \mathfrak{g}, \mathcal{N})$ is normal, then $X \in \mathcal{E}$.*

Proof. First, let us show that $\emptyset$ is $\mathcal{E}$-open. Clearly, $x \in \mathcal{E}Int(\emptyset)$ if and only if there is $B \in \mathcal{E}_x$ such that $B \subseteq \emptyset$. However, if $B \subseteq \emptyset$, then $B = \emptyset$. But this is not possible since $x \in B$. Thus, $\mathcal{E}Int(\emptyset) = \emptyset$.

Now let $J \neq \emptyset$ and assume that for any $i \in J$, A_i is $\mathcal{E}$-open. Let us prove that $\mathcal{E}Int(\bigcup_{i \in J} A_i) = \bigcup A_i$.

$(\subseteq)$. Let $x \in \mathcal{E}Int(\bigcup_{i \in J} A_i)$. Hence there is $B \in \mathcal{E}_x$ such that $B \subseteq \bigcup_{i \in J} A_i$. However, $x \in B$, so $x \in \bigcup_{i \in J} A_i$.

$(\supseteq)$. Let $x \in \bigcup_{i \in J} A_i$. Hence, there is A_k such that $x \in A_k$. But A_k is $\mathcal{E}$-open, so $x \in \mathcal{E}Int(A_k)$. Thus, there is $B \in \mathcal{E}_x$ such that $B \subseteq A_k \subseteq \bigcup_{i \in J} A_i$. Then $x \in \mathcal{E}Int(\bigcup_{i \in J} A_i)$.

Now assume that $(X, \mathfrak{g}, \mathcal{N})$ is normal. Hence, $\mathcal{N}_x \neq \emptyset$ for any $x \in X$. But then $\mathcal{E}_x \neq \emptyset$. Hence, $\mathcal{E}Int(X)$ is the set of all $x \in X$ such that there is $B \in \mathcal{E}_x$ such that $B \subseteq X$. But this means that $\mathcal{E}Int(X) = X$. □

6. Conclusion and future work

In this paper we analyzed the specific concept of generalized weak structure equipped with associated neighborhoods. The idea was to assume that those points which are beyond $\bigcup \mathfrak{g}$ still can have $\mathfrak{g}$-open neighborhoods. As a consequence, it became possible to speak about interiors defined by means of such neighborhoods. Clearly, the usage of word "neighborhood" is much more general here than its standard meaning. As we have already pointed out, it reminds neighborhood semantics for weak modal logics. In fact, this was our aim.

The paper contains seminal results. The whole project should be continued. We think that there are several possible ways:

1. To reconstruct basic topological notions (like separation axioms, compactness, density, nowhere density, rarity, connectedness, continuity etc.) in terms of $\mathcal{N}$- and $\mathcal{E}$-open sets. Also the analogues of α-, β-, b-, pre-, semi- or regular open sets should be studied in this context.

2. To impose additional conditions on $\mathfrak{g}$. As we could see, in some theorems it was necessary to assume that $\mathfrak{g}$ is closed under arbitrary unions.

3. To impose additional conditions on $\mathcal{N}$. For example, we can assume that for some (or for all) $x \in X$, the following holds: if $A \in \mathcal{N}_x$ and $A \subseteq B \in \mathfrak{g}$, then $B \in \mathcal{N}_x$.

4. To assume that there is a function $\mathsf{f} : X \setminus \bigcup \mathfrak{g} \to \bigcup \mathfrak{g}$ such that: if $A \in \mathfrak{g}$ and $\mathsf{f}(x) \in A$, then $A \in \mathcal{N}_x$. It would mean that x inherits $\mathfrak{g}$-open neighborhoods from its "twin" point $\mathsf{f}(x)$.

5. To assume that there is no any distinguished GWS $\mathfrak{g}$. This would mean that we consider the following structure: $(X, \mathcal{N})$, where $\mathcal{N} : X \to P(P(X))$ and $x \in \mathcal{N}Int(A) \Leftrightarrow$ there is $B \in \mathcal{N}_x$ such that $B \subseteq A$.

Of course we could mix all the options mentioned above. Moreover, we would like to use our tools in formal logic. It would be especially interesting to use our notion of $\mathcal{N}$-interior to define logical value of formulas of the form $\Box\varphi$ (that is, necessity of φ). In topological semantics, $V(\varphi)$ in a given topological model is identified with $Int(V(\varphi))$. We could replace topological interior with $\mathcal{N}$-interior to obtain a new and more general framework.

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