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A Generalized Version of Holditch Theorem under Homothetic Motions in $\mathbb{C}_p$

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Abstract: In this study, we give the Cauchy-length formula for the homothetic motions in generalized complex plane $\mathbb{C}_p$ and we express a geometric interpretation of this formula when this length is constant in $\mathbb{C}_p$. Then, we calculate the area formula of the non linear points for the homothetic motions in $\mathbb{C}_p$ and we express this area formula with respect to the Cauchy length formula. Moreover, for non linear three points we give new version of Holditch theorem during the homothetic motions in $\mathbb{C}_p$. Consequently, we obtain some conclusions. Therefore, Holditch theorem in this study is the most general theorem including all the studies for planar motions so far.

Key words: The generalized complex plane, homothetic motion, Cauchy length formula, Holditch theorem

1. Introduction

Mechanics, a subbranch of physics, is the science which studies motion and equilibrium. Mechanics can be divided into three chapters: Kinematics, Dynamics and Statics. Kinematics, one of the subbranch of mechanics, is the science that studies the way the geometric properties of material systems change over time. The formation of kinematics belongs to Ampère (1775-1836), who founded this science and named it [1]. Kinematics tries to determine how the object motions, what trajectory it goes on, what its location, velocity and acceleration are at any moment, without taking into account the forces acting on the object. Dynamics is the science that studies motion by considering the forces that create and change the motions of material systems. The fundamental quantities in kinematics are time and length when in dynamics, there are three fundamental quantities: time, length and mass. Therefore, kinematics becomes a science between dynamics and geometry.

In different spaces and dimensions, kinematics was studied by many scientists. In Euclidean and complex planes the planar motions with one parameter were expressed by Müller [2]. Then, in Lorentzian plane the same motions were given by Ergin [3] and Görmez [4]. After that, for hyperbolic planes the planar motions were expressed by Yüce and Kuruoğlu [5]. In addition to that, these motions for Galilean planes were obtained Akar and Yüce [6]. The Holditch theorem given by Holditch [7] is one of the remarkable expressions of kinematics. The most important part of this classic Holditch theorem given by Holditch is that the area of the trajectory during the motion is independent of the drawn curve. Therefore, thanks to this feature, many scientists started to study this theorem. On the other hand, Steiner expressed the area of the trajectory in terms of Steiner points for one-parameter planar motions [8, 9]. Later, many scientists generalized Holditch theorem and the area formula in different ways and perspectives [2, 10–26].

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2. Preliminaries

Any generalized complex number is expressed $Z = (z, w)$ or $Z = z + iw$ and the number system consisting of these numbers includes ordinary (when $p + q^2/4$ is negative), dual (zero) and double (positive) numbers where $i^2 = (q, p)$, $(i^2 = iq + p)$ and $z, w, q, p \in \mathbb{R}$ [27, 29, 31]. In this study, we assume that $q = 0$ and $i^2 = p \in \mathbb{R}$ $(-\infty < p < \infty)$. Therefore, we study in generalized complex number system

$$\mathbb{C}_p = \left\{ z + iw \, : \, z, w \in \mathbb{R}, \; i^2 = p \in \mathbb{R} \right\}.$$

Some operations defined on this system are as follows. If we take two numbers $Z_1 = z_1 + iw_1, Z_2 = z_2 + iw_2 \in \mathbb{C}_p$ therefore, we define that the addition of this numbers as

$$Z_1 \pm Z_2 = (z_1 + iw_1) \pm (z_2 + iw_2) = (z_1 \pm z_2) + i(w_1 \pm w_2).$$

In addition to that, the product in system $\mathbb{C}_p$ is

$$M^p(Z_1, Z_2) = (z_1 z_2 + p w_1 w_2) + i(z_1 w_2 + z_2 w_1)$$

[27–29]. In addition to that, we suppose that $\boldsymbol{z}_1 = z_1 + iw_1, \boldsymbol{z}_2 = z_2 + iw_2 \in \mathbb{C}_p$ are position vectors of Z_1, Z_2. Therefore, the scalar product can be expressed as

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle_p = \operatorname{Re}\left(M^p\left(\boldsymbol{z}_1, \bar{\boldsymbol{z}}_2\right)\right) = \operatorname{Re}\left(M^p\left(\bar{\boldsymbol{z}}_1, \boldsymbol{z}_2\right)\right) = z_1 w_1 - p z_2 w_2$$

[27]. In addition, the $p-$magnitude of $Z = z + iw \in \mathbb{C}_p$ is

$$|Z|_p = \sqrt{\left|M^p(Z, \overline{Z})\right|} = \sqrt{|z^2 - pw^2|}$$

where " $-$ " is the ordinary complex conjugation, [27]. On the other hand, the unit circle is defined as $|Z|_p = 1$ in $\mathbb{C}_p$. Therefore, the unit circles in this plane are given for the special cases of p in Figure 1.

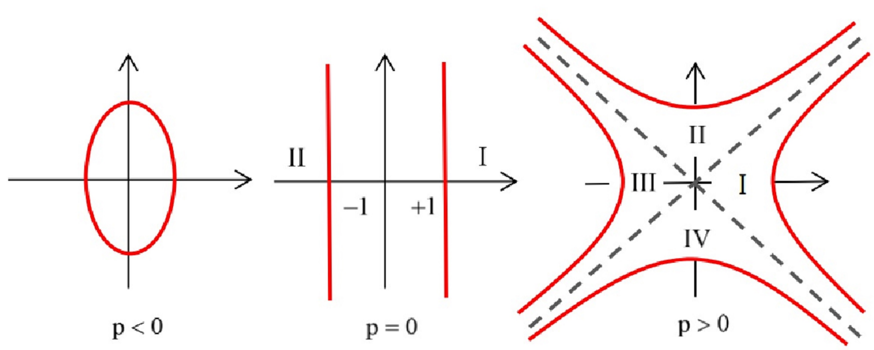

Figure 1. The Unit Circle in $\mathbb{C}_\mathrm{p}$

Moreover, in $\mathbb{C}_p$ any circle can be characterized by the equation

$$\left| (z-a)^2 - p(w-b)^2 \right| = r^2$$

where the center and radius of this circle $M(a,b)$ and r, respectively [27].

Now, we symbolise the number $Z = z + iw$ in $\mathbb{C}_p$ with $\overrightarrow{OT}$. Therefore, the angles γ_p are expressed by the inverse of tangent function as

$$\gamma_p = \begin{cases} \frac{1}{\sqrt{|\mathbf{p}|}}\tan^{-1}\left(\alpha\sqrt{|p|}\right), & p<0 \\ \alpha, & p=0 \\ \frac{1}{\sqrt{\mathbf{p}}}\tan^{-1}\left(\alpha\sqrt{p}\right), & p>0 \ (branch\, I, III) \end{cases}$$

where $\alpha \equiv w/z$ (see Figure 2).

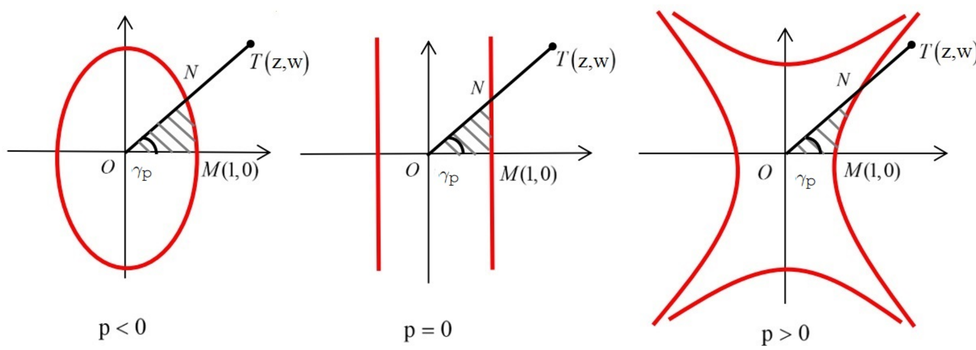

Figure 2. Elliptic, Parabolic and Hyperbolic Angles

We suppose that the intersection of unit circle with OT is any point N, L is the orthogonal projection of N on OM, and the tangent of the unit circle at M is QM (see Figure 3). Therefore, the $p-$trigonometric functions ($sinp$, $cosp$ and $tanp$) can be given as

$$\sin p\gamma_p = \begin{cases} \frac{1}{\sqrt{|\mathbf{p}|}}\sin\left(\gamma_p\sqrt{|p|}\right), & p<0 \\ \gamma_p, & p=0 \\ \frac{1}{\sqrt{\mathbf{p}}}\sinh\left(\gamma_p\sqrt{p}\right), & p>0 \end{cases}$$

$$\cos p\gamma_p = \begin{cases} \cos\left(\gamma_p\sqrt{|p|}\right), & p<0 \\ 1, & p=0 \\ \cosh\left(\gamma_p\sqrt{p}\right), & p>0 \end{cases}$$

where these functions are given for branch I when $p \geq 0$ and the ratio $\frac{QM}{OM} = \frac{NL}{OL}$ gives

$$\tan p\gamma_p = \frac{\sin \mathrm{p}\gamma_\mathrm{p}}{\cos \mathrm{p}\gamma_\mathrm{p}}.$$

On the other hand, the Maclaurin expansions of these function for branch I can be expressed $\cos p\gamma_p = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n}{(2n)!}\gamma_p^{2n}$ and $\sin p\gamma_p = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n}{(2n+1)!}\gamma_p^{2n+1}$. The Euler Formula in $\mathbb{C}_p$ is expressed as $e^{i\gamma_p} = \cos p\gamma_p + i \sin p\gamma_p$

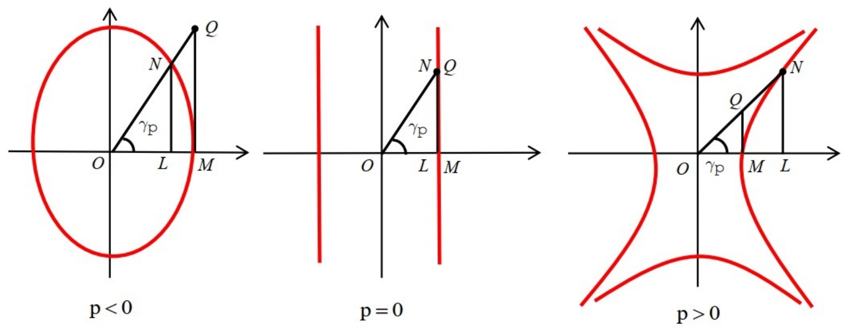

Figure 3. $\gamma_{\rm p}$ for the special cases of p

where $i^2 = p$. Moreover, the exponential forms of Z in $\mathbb{C}_p$ is

$$Z = r_p(cosp\gamma_p + i\sin p\gamma_p) = r_p e^{i\gamma_p}$$

where $r_p = |Z|_p$ [27]. Then, we can write that the $p-$rotation matrix is

$$A(\gamma_p) = \begin{bmatrix} \cos p\gamma_p & p\sin p\gamma_p \\ \sin p\gamma_p & \cos p\gamma_p \end{bmatrix}.$$

[27].

Now, we express the homothetic planar motions with one parameter in $\mathbb{C}_p$. The homothetic motions in p-complex plane, the subset of $\mathbb{C}_p$, was expressed as

$$\mathbb{C}_J = \left\{z + Jw : x, y \in R, J^2 = p, p \in \{-1, 0, 1\}\right\}$$

by Gürses et. al [30]. Analogously, with the help of that study the homothetic planar motions with one parameter in $\mathbb{C}_p$ can be obtained as follows.

We suppose that $\mathbb{K}_p, \mathbb{K}_p'$ are the moving and fixed planes in $\mathbb{C}_p$, respectively and the vectors $\mathbf{x} = x_1 + ix_2$ and $\mathbf{x}' = x_1' + ix_2'$ are the position vectors of any point X according to the these planes, respectively. Therefore, the homothetic motions in $\mathbb{C}_p$ is characterized by the equation

$$\mathbf{x}' = (h\mathbf{x} - \mathbf{u})\, e^{i\gamma_p}$$

where $\overrightarrow{\mathrm{OO}'} = \mathbf{u}$ ($\mathbf{u}' = -\mathbf{u}e^{i\gamma_p}$), γ_p is the $p-$rotation angle of this homothetic motion, and h is the homothetic scale in $\mathbb{C}_p$. In that case, the relative and absolute velocity vectors of X can be calculated as

$$\mathbf{V}_r{}' = \mathbf{V}_r e^{i\gamma_p} = h\dot{\mathbf{x}}e^{i\gamma_p} \tag{1}$$

and

$$\mathbf{V}_a{}' = \mathbf{V}_a e^{i\gamma_p} = \left(\dot{h} + i\dot{\gamma}_p h\right)\mathbf{x}e^{i\gamma_p} - \left(\dot{\mathbf{u}} + i\dot{\gamma}_p\mathbf{u}\right)e^{i\gamma_p} + h\dot{\mathbf{x}}e^{i\gamma_p}, \tag{2}$$

respectively. Therefore, considering the equations (1) and (2) the guide velocity vector can be given as

$$\mathbf{V}_f{}' = \mathbf{V}_f e^{i\gamma_p} = \left(\dot{h} + i\dot{\gamma}_p h\right) \mathbf{x} e^{i\gamma_p} + \dot{\mathbf{u}}'.$$

Theorem 2.1. *The relationship between velocity vectors of the homothetic motions* $\mathbb{K}_p/\mathbb{K}'_p$ *in* $\mathbb{C}_p$ *is*

$$\boldsymbol{V}_a = \boldsymbol{V}_f + \boldsymbol{V}_r.$$

On the other hand, there are points (defined as pole points) during this homothetic motions that seem to be fixed both in $\mathbb{K}_p$ and $\mathbb{K}'_p$ in $\mathbb{C}_p$. Therefore, we consider that the pole points of the homothetic motions $\mathbb{K}_p/\mathbb{K}'_p$ are $Q = (q_1, q_2) \in \mathbb{C}_p$ and the components of these pole points can be expressed as

$$\begin{aligned} q_1 &= \tfrac{dh(du_1 + \mathrm{p}u_2 d\gamma_\mathrm{p}) - \mathrm{p}h(du_2 + u_1 d\gamma_\mathrm{p})d\gamma_\mathrm{p}}{dh^2 - \mathrm{p}h^2 d\gamma_\mathrm{p}{}^2}, \\ q_2 &= \tfrac{dh(du_2 + u_1 d\gamma_\mathrm{p}) - h(du_1 + \mathrm{p}u_2 d\gamma_\mathrm{p})d\gamma_\mathrm{p}}{dh^2 - \mathrm{p}h^2 d\gamma_\mathrm{p}{}^2}. \end{aligned}$$

where $\mathbf{V}_f = 0$. Let the position vector of the pole point Q be $\mathbf{q}$. Moreover, the guide velocity vector of the fixed point X in $\mathbb{K}_p$ can be expressed with the help of the pole points as

$$\mathbf{d}_x{}' = (dh + ihd\gamma_p)\,(\mathbf{x} - \mathbf{q})\, e^{i\gamma_p}.$$

In addition to that, for the homothetic motions in $\mathbb{C}_p$ the following proposition can be expressed.

Proposition 2.1. *We consider that two any generalized complex vectors be* $\boldsymbol{u} = (u_1, u_2)$ *and* $\boldsymbol{v} = (v_1, v_2)$ *in* $\mathbb{C}_p$*. Therefore, the equations*

$$\begin{aligned} &i)\ \ [\boldsymbol{u}e^{i\gamma_p}, \boldsymbol{v}e^{i\gamma_p}] = [\boldsymbol{u}, \boldsymbol{v}] \\ &ii)\ [\boldsymbol{u}, (dh + ihd\gamma_p)\,\boldsymbol{v}] = [\boldsymbol{u}, \boldsymbol{v}]dh + \tfrac{1}{2}\,[\boldsymbol{u}\bar{\boldsymbol{v}} + \bar{\boldsymbol{u}}\boldsymbol{v}]\,hd\gamma_p, \end{aligned}$$

are hold where h *is homothetic scale and*

$$[\boldsymbol{u}, \boldsymbol{v}] = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$$

[16].

In this study, we assume the open motions in branch I restricted to time interval $[t_1, t_2]$ in $\mathbb{C}_p$.

3. Main Theorems and Results

In this section, we obtain the Cauchy length formula for the homothetic motions in $\mathbb{C}_p$ and we give a geometric interpretation of this formula when this length is constant. Then, we calculate the area formula of the non linear points for the homothetic motions in $\mathbb{C}_p$ and we express this area formula with respect to the Cauchy length formula. Moreover, for non linear three points we give new version of Holditch theorem during the homothetic motions. Consequently, we obtain some conclusions.

Now, we assume that $\mathbb{C}_p$ is the generalized complex plane and $\mathbb{K}_p/\mathbb{K}'_p$ is the homothetic motion in $\mathbb{C}_p$. Moreover, we consider that any point in $\mathbb{C}_p$ is $X = (x_1, x_2)$ and g is any straight line through point X in the branch I in $\mathbb{C}_p$. In this case, the Hesse form of the line g with regard to the moving generalized complex plane $\mathbb{K}_p$ can be written as $h = x_1 \cos p\psi_p - px_2 \sin p\psi_p$ where the Hesse coordinates are (h, ψ_p), the distance between the origin point O and the straight line g is $h = h(\psi_\mathrm{p})$ and the contact point of the straight line g and the envelope curve (g) is the point X. In addition to that, the angle ψ_p in $\mathbb{C}_p$ is the angle in the positive direction made by the perpendicular descending from O to the straight line g and the principal axis of the moving plane $\mathbb{K}_p$. Similarly, the straight line g can be expressed with regard to $\mathbb{K}'_p$ as

$$k' = x'_1 \cos p\psi'_p - px'_2 \sin p\psi'_p \tag{3}$$

where the point $X' = (x'_1, x'_2)$ is the representation of X according to the fixed plane $\mathbb{K}'_p$ and the angle ψ'_p in $\mathbb{C}_p$ is the angle in the positive direction made by the perpendicular descending from the origin point O to the straight line g and the principal axis of the moving plane $\mathbb{K}'_p$. In addition to that, the relationship between the angles ψ_p and ψ'_p is $\psi'_p = \gamma_p + \psi_p$ where the angle γ_p is the rotation angle of the homothetic motion in $\mathbb{C}_p$.

Theorem 3.1. *The Cauchy length formula under the homothetic motions in $\mathbb{C}_p$ is obtained*

$$L' = \frac{1}{\sqrt{|p|}} \left| phk\delta_p - A\cos p\psi_p + pB \sin p\psi_p \right| \tag{4}$$

where $\delta_p = \int_{t_1}^{t_2} d\gamma_p$, $A = \int_{t_1}^{t_2} (pu_1 - \ddot{u}_1) d\gamma_p$ *and* $B = \int_{t_1}^{t_2} (pu_2 - \ddot{u}_2) d\gamma_p$, h *is homothetic scale and* ψ_p *is the angle in the positive direction made by the perpendicular descending from* O *to the straight line* g *and the principal axis of* $\mathbb{K}_p$.

Proof. We consider the equation (3) and write the components x'_1 and x'_2 in equation (3). Therefore, we get

$$k' = hk - u_1 \cos p\psi_p + pu_2 \sin p\psi_p$$

where h is the homothetic scale in $\mathbb{C}_p$ and taking the differential we obtain

$$\begin{aligned} \dot{k}' &= h\dot{k} - \dot{u}_1 \cos p\psi_p + p\dot{u}_2 \sin p\psi_p, \\ \ddot{k}' &= h\ddot{k} - \ddot{u}_1 \cos p\psi_p + p\ddot{u}_2 \sin p\psi_p. \end{aligned} \tag{5}$$

On the other hand, we know that the Cauchy length formula in $\mathbb{C}_p$ as

$$L' = \frac{1}{\sqrt{|\mathrm{p}|}} \int_{t_1}^{t_2} \left| \mathrm{p}k' - \ddot{k}' \right| d\gamma_\mathrm{p} \tag{6}$$

by the help of [14] where $d\psi'_\mathrm{p} = d\gamma_\mathrm{p}$. If we use the equation (5) and (6) then, for the homothetic planar motions in $\mathbb{C}_p$ the Cauchy length formula can be obtained

$$L' = \frac{1}{\sqrt{|p|}} \left| phk\delta_p - A\cos p\psi_p + pB \sin p\psi_p \right|$$

where $\delta_p = \int_{t_1}^{t_2} d\gamma_p$, $A = \int_{t_1}^{t_2} (\mathrm{p}u_1 - \ddot{u}_1) d\gamma_\mathrm{p}$ and $B = \int_{t_1}^{t_2} (\mathrm{p}u_2 - \ddot{u}_2) d\gamma_\mathrm{p}$. It should be emphasized here that the expression in the absolute value in equation (6) is positive for $p > 0$ and negative for $p < 0$ $(p \neq 0)$. □

Conclusion 3.1. *We suppose that the fixed straight lines which have Hesse coordinates* (h, ψ_p) *and the enveloping trajectories of these lines have the same Cauchy length* $L' = d$ *during the homothetic motions in* $\mathbb{C}_p$*. Therefore, all of these lines are tangent to the cycles with radius* $\frac{d}{\sqrt{|p|}h\delta_p}$ *and center* $S_G = \left(\frac{A}{ph\delta_p}, \frac{B}{ph\delta_p}\right)$ *in* $\mathbb{K}_p$ *where* $A = \int\limits_{t_1}^{t_2} (pu_1 - \ddot{u}_1) d\gamma_p$ *and* $B = \int\limits_{t_1}^{t_2} (pu_2 - \ddot{u}_2) d\gamma_p$*.*

Proof. We consider that the Cauchy length formula under the homothetic planar motions in $\mathbb{C}_p$ given in the equation (4) is constant. Therefore, we can write $L' = d = constant$ and obtain the function k as

$$k = \frac{d}{\sqrt{|p|}h\delta_p} + \frac{A\cos p\psi_p}{ph\delta_p} - \frac{B\sin p\psi_p}{h\delta_p}. \tag{7}$$

On the other hand, we can write the straight line g as

$$k = r + a\cos p\psi_p - pb\sin p\psi_p \tag{8}$$

where the point (a, b) is the point at distance r from the point X on the straight line g. Therefore, considering the equation (7) and (8) we obtain

$$r = \frac{d}{\sqrt{|p|}h\delta_p}, \quad a = \frac{A\cos p\psi_p}{ph\delta_p}, \quad b = \frac{B\sin p\psi_p}{h\delta_p}.$$

Consequently, all the constant straight lines g of homothetic motion in $\mathbb{C}_\mathrm{p}$, which draw envelope trajectories with the same length, lie on the cycle with center $S_G = \left(\frac{A}{ph\delta_p}, \frac{B}{ph\delta_p}\right)$ and radius $r = \frac{d}{\sqrt{|p|}h\delta_p}$ where $A = \int\limits_{t_1}^{t_2} (\mathrm{p}u_1 - \ddot{u}_1) d\gamma_\mathrm{p}$ and $B = \int\limits_{t_1}^{t_2} (\mathrm{p}u_2 - \ddot{u}_2) d\gamma_\mathrm{p}$. □

Now, we give a new version of the Holditch theorem given by [12, 14, 16] considering non linear points for homothetic planar motion in $\mathbb{C}_\mathrm{p}$. First of all, we assume that the non linear three fixed points X, Y and Z in $\mathbb{K}_\mathrm{p}$ for the homothetic planar motions draw the trajectories k_X, k_Y and k_Z with areas F_X, F_Y and F_Z, respectively. With a special choosing, we consider $X = (0, 0)$, $Y = (z + w, 0)$ and $Z = (z, u)$ $(z > u)$ in $\mathbb{K}_\mathrm{p}$. We know that the area F_X for the homothetic planar motion is calculated

$$F_X = F_0 + \frac{1}{2}h^2(t_0)\,\delta_p\left(\mathbf{x}\bar{\mathbf{x}} - \mathbf{x}\bar{\mathbf{s}} - \bar{\mathbf{x}}\mathbf{s}\right) + \zeta_1 x_1 + \zeta_2 x_2$$

where $\delta_p = \int\limits_{t_1}^{t_2} d\gamma_p$, $\zeta_1 = \frac{1}{2}\int\limits_{t_1}^{t_2} (-2hq_2 + u_2)\,dh$ and $\zeta_2 = \frac{1}{2}\int\limits_{t_1}^{t_2} (2hq_1 - u_1)\,dh$ considering [16]. In that case, for $X = (0, 0)$, $Y = (z + w, 0)$ and $Z = (z, u)$ the areas for homothetic planar motions are written by

$$F_X = F_0 \qquad for \qquad X = (0, 0),$$

$$F_Y = F_X + \frac{1}{2}h^2\delta_p\left((z + w)^2 - 2(z + w)s_1\right) + \zeta_1(z + w) \qquad for \qquad Y = (z + w, 0) \tag{9}$$

and

$$F_Z = F_X + \frac{1}{2}h^2\delta_p\left(z^2 - pu^2 - 2zs_1 + 2pus_2\right) + \zeta_1 z + \zeta_2 u \qquad for \qquad Z = (z,u)\,. \tag{10}$$

Therefore, using the equation (9) we get the first component of the Steiner point as

$$s_1 = \frac{F_X - F_Y}{h^2\,(z+w)\,\delta_p} + \frac{\zeta_1}{h^2\delta_p} + \frac{z+w}{2}. \tag{11}$$

Moreover, from the equations (10) and (11) we obtain

$$F_Z = \frac{zF_Y + wF_X}{z+w} - \frac{1}{2}h^2\delta_p\left(zw + pu^2\right) + ph^2us_2\delta_p + \zeta_2 u$$

where $\zeta_2 = \frac{1}{2}\int\limits_{t_1}^{t_2}(2hq_1 - u_1)\,dh$.

Theorem 3.2. *We assume that the one parameter homothetic motion in $\mathbb{C}_p$ with $S = S_G$ and the non linear points $X = (0,0)$, $Y = (z+w,0) \in \mathbb{K}_p$ move along the trajectories with the areas F_X and F_Y, respectively, then the point $Z = (z,u) \in \mathbb{K}_p$ draws the trajectory with the area*

$$F_Z = \frac{zF_Y + wF_X}{z+w} - \frac{1}{2}h^2\delta_p\left(zw + pu^2\right) - \sqrt{|p|}huL_{XY} + \zeta_2 u \tag{12}$$

where L_{XY} is the length of the enveloping curve (XY) obtained by substituting the specially selected points X and Y in equation (4).

Proof. Now, we suppose that the one-parameter homothetic planar motion $S = S_G$ in $\mathbb{C}_\mathrm{p}$. Therefore, we get $s_2 = \frac{B}{ph\delta_p}$. If equation (6) is adapted for $X = (0,0)$, $Y = (z+w,0)$ and $Z = (z,u)$ consequently, we have

$$F_Z = \frac{zF_Y + wF_X}{z+w} - \frac{1}{2}h^2\delta_p\left(zw + pu^2\right) - \sqrt{|p|}huL_{XY} + \zeta_2 u$$

where L_{XY} is the length of the enveloping curve (XY) obtained by substituting the specially selected points X and Y in equation (4). □

Therefore, the Holditch theorem for non linear points in homothetic motions of $\mathbb{C}_p$ can be given with following theorem.

Theorem 3.3. Main Theorem: *We suppose that the one parameter homothetic motion in $\mathbb{C}_p$ with $S = S_G$ and the non linear points $X = (0,0)$, $Y = (z+w,0)$ and $Z = (z,u) \in \mathbb{K}_p$ move along the trajectories with the areas F_X, F_Y and F_Z, respectively. Therefore, the area between F_X, F_Y and F_Z varies depending on the distance of Z to the line XY and the points X,Y to the projection point of Z, the rotation angle of the homothetic motion, the homothetic scale and the length of the envelope curve of (XY), while it is independent of the choosing of the curves.*

Corollary 3.1. *We assume that X, Y and Z are linear points for the homothetic planar motions in $\mathbb{C}_p$. Therefore, $u = 0$ and using the equation (12) we get*

$$F_Z = \frac{zF_Y + wF_X}{z+w} - \frac{1}{2}h^2\delta_p zw$$

In this case, this formula is the area formula in [16].

Corollary 3.2. *We consider that the homothetic scale is $h = 1$ in equation (12). Therefore, the area formula can be obtained as*

$$F_Z = \frac{zF_Y + wF_X}{z+w} - \frac{1}{2}\delta_p\left(zw + pu^2\right) - \sqrt{|p|}uL_{XY}.$$

This formula is area formula given in $\mathbb{C}_p$ in [14]

Corollary 3.3. *Now, we assume that both $u = 0$ and $h = 1$. Therefore, the equation (12) is obtained*

$$F_Z = \frac{zF_Y + wF_X}{z+w} - \frac{1}{2}\delta_p zw.$$

This equation is area formula in [12]

4. Conclusion

The Holditch theorem is a theorem that expresses the area of the trajectory drawn during the motion. To be more specific, the Holditch theorem in plane geometry emphasizes that if a fixed-length chord is allowed to rotate in a convex closed curve, the position of a point on the chord x units from one end and y units from the other end, the curve drawn by this point is less than the area of the original curve πxy. This theorem was first given in 1858 by the English mathematician Hamnet Holditch. Although not emphasized by Holditch, the proof of the theorem requires the chord to be short enough that the position of the point taken is a simple closed curve. The fact that the area of trajectories expressed in the Holditch theorem is independent of the curve (circle, ellipse, etc.) makes this theorem very interesting. Thus, the Holditch theorem has been included as one of Clifford A. Pickover's 250 milestones in the history of mathematics. It should be noted again that the most important feature of the theorem is that the formula that gives the area πxy is independent of both the shape and size of the original curve, and this formula gives the same formula as the area of an ellipse with axes x and y. Until now, Holditch's theorem has been generalized to many planes and spaces. But since the generalized complex plane mentioned in this study includes hyperbolic, dual, and complex planes, and planes in other possible choices of $p \in \mathbb{R}$, the study in this plane is a very extended study. In this study, we have generalized the studies of [12, 14, 16], which gives the Holditch theorem regarding areas for planar motions. Therefore, Holditch theorem in this study is the most general theorem including all the studies for planar motions so far.

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