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Strongly nonlinear unilateral anisotropic elliptic problem with L^1-data

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Abstract: This paper, deal with the existence of an entropy solution of quasilinear elliptic unilateral equation In the anisotropic Sobolev space, where the data belongs to $L^1(\Omega)$, and The nonlinear terms satisfy the sign and growth conditions.

Key words: Quasilinear elliptic unilateral problem, Anisotropic Sobolev spaces, Penalization methods, entropy solutions.

1. Introduction and Basic Assumptions

There are large numbers of papers which treated nonlinear unilateral elliptic problems due to their fundamental role in describing several phenomena, such as the study of fluid filtration in porous media, constrained heating, elastoplasticity, optimal control, see [1, 3] for more details.

In the Lebesgue Sobolev spaces , Bénilan in [4] was introduced the idea of entropy solutions adjusted to Boltzmann conditions. we cite also the works maked by the authors in [5, 6]. which were studied the existence solutions of this class ($B(t) = |t|^p$).

for the Sobolev space with variable exponent we refer the reader to the works [3, 10, 11].

In Orlicz spaces, among the difficulties that can be encountered in this study space is the non-homogeneity of the functions N (indirect definition of the norm). It is generally difficult to move results in L^p to Orlicz spaces. (see [12, 16]).

On the other hand, An elliptic systems of equations generalizing the following problem

$$-\sum_{i=1}^{N}\Big(|v_{x_i}(x)|^{p_i-2}v_{x_i}(x)\Big)_{x_i}+|v(x)|^{p_0-2}v(x)=g(x),$$

was treated by Domanska in [18] in the case when Ω is an unbounded domain.

In [17] the authors was solved the same problem under study with $N(x,v,\nabla v)=\operatorname{div}(N(v))$ and $N(v)$ a polynomial growth like u^q in L^p.

The major goal of this note is to demonstrate some results on the existence of entropy solution for

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boundary value problem as

$$\begin{cases} \sum_{i=1}^{N} D^i b_i(x,v,\nabla v) + N(x,v,\nabla v) = g - \operatorname{div}\varphi(v) \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega, \end{cases}$$

where $g \in L^1(\Omega)$, and $\varphi(\cdot) \in C^0\left(\mathbb{R}, \mathbb{R}^N\right)$, and several studies of certain elliptical and parabolic problems which are interested in the results of existence and uniqueness have been carried out by many researchers (see [7–9, 20–28]).

The main difficulty in this study cames from the fact that the term Av does not coercive in $W_0^{1,\vec{p}}(\Omega)$. For this reason, we use a penalization term $\frac{1}{n}|v|^{p_0-2}v$ in the problems (11) to show the main result of this note.

The great interest of studying problems of the anisotropic type comes from its application in mathematics, more precisely in a population dynamics model, in image restoration processes, and electro-rheological fluid flows (see [13, 14, 29]).

The plan of this note is organized as follows. The section 2 is devoted to definitions and auxiliary properties of anisotropic Sobolev spaces. The section3 includes the hypothesise on the function b_i for which the problem (8) admit at least one solution in the sense of the definition4.1 and in the final section we prove our main result.

2. Preliminary

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ with smooth boundary and, let $1 < p_1, \ldots, p_N < \infty$ be N real numbers,

$$p^+ = \max\{p_1, \ldots, p_N\}, p^- = \min\{p_1, \ldots, p_N\} \text{ and } \vec{p} = (p_1, \ldots, p_N).$$

We denote

$$\partial_i = \frac{\partial}{\partial x_i},$$

We set

$$\underline{\vec{p}} = \min\{p_0, p_1, p_2, \ldots, p_N\} \quad \text{then} \quad \underline{\vec{p}} > 1. \tag{1}$$

The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined by

$$W^{1,\vec{p}}(\Omega) = \left\{v \in L^{p_0}(\Omega), \quad D^i v \in L^{p_i}(\Omega), \quad i = 1, 2, \ldots, N\right\}$$

endowed with the norm

$$\|v\|_{1,\vec{p}} = \sum_{i=0}^{N} \left\|D^i v\right\|_{p_i} \tag{2}$$

$W_0^{1,\vec{p}}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$. The space $\left(W_0^{1,\vec{p}}(\Omega), \|v\|_{1,\vec{p}}\right)$ is a reflexive Banach space (for more details [30]).

Now, let us give the following key lemma of ,

Lemma 2.1.
Let Ω be a bounded domain in $\mathbb{R}^N$,

- *if* $\underline{p} < N$ *then* $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, \underline{p}^*[$ *with* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$,
- *if* $\underline{p} = N$ *then* $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, +\infty[$,
- *if* $\underline{p} > N$ *then* $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\bar{\Omega})$.

then the embedding bellow are compact

The proof of the preview lemma based on the compact embedding theorem for Sobolev spaces and the fact that the embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ *is continuous .*

Lemma 2.2.

The space $W^{-1,p'}(\Omega)$, *is the dual of the space* $W_0^{1,\vec{p}}(\Omega)$ *with* $p' = (p'_0, p'_1, \ldots, p'_N)$ *and* $\frac{1}{p'_i} + \frac{1}{p_i} = 1$. *For each* $F \in W^{-1,p'}(\Omega)$ *there exists* $F_i \in L^{p'_i}(\Omega)$ *for* $i = 0, 1, \ldots, N$, *such that* $F = F_0 - \sum_{i=1}^N D^i F_i$.

Then, for all $v \in W_0^{1,p}(\Omega)$ *we have*

$$\langle F, v\rangle = \sum_{i=0}^N \int_\Omega F_i D^i v dx.$$

We define a norm on the dual space as

$$\|F\|_{-1,p'} = \inf \sum_{i=0}^N \|F_i\|_{p'_i}.$$

Definition 2.1.
Let $k > 0$, the truncation function $T_k(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$, defined by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k\frac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and

$$T_0^{1,\vec{p}}(\Omega) := \left\{ v : \Omega \mapsto R \text{ measurable } / \ T_k(v) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0 \right\}.$$

Remark that, a function ($v$measurable) satisfying $T_k(v) \in W_0^{1,\vec{p}}(\Omega)$ for all $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. So, for any $v \in T_0^{1,\vec{p}}(\Omega)$ we can define the weak gradient of v, by ∇v.

Lemma 2.3.
Let $u \in T_0^{1,\vec{p}}(\Omega)$, *for* $i = 1, \ldots, N$, *there exists a unique measurable function* $v_i : \Omega \mapsto \mathbb{R}$ *such that*

$$D^i T_k(u) = v_i \cdot \chi_{\{|u|<k\}} \ \textit{a.e. in } \Omega, \ \textit{for any } k > 0,$$

with χ_A *named the characteristic function of a measurable set* A.

The idea of our proof based on the usual method used in [15] for the case of Sobolev spaces. we refer the reader to [14, 19] for more details.

3. Essential assumptions

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 2)$, we suppose that $p = (p_0, p_1, \ldots, p_N)$ verify the condition $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$.

Let us choosing $\varLambda : \Omega \longrightarrow \overline{\mathbb{R}}$ (measurable function), such that

$$\varLambda^+ \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

The subset $K_\varLambda$(**convex**) is given by

$$K_\varLambda = \left\{ v \in W_0^{1,\vec{p}}(\Omega) \text{ such that } v \geq \varLambda \text{ a.e. in } \Omega \right\}.$$

We consider $B: \ W_0^{1,\vec{p}}(\Omega) \longrightarrow W^{-1,\vec{p}'}(\Omega)$, defined by

$$Bv = -\sum_{i=1}^{N} \partial^i b_i(x, v, \nabla v),$$

where $b_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ are Carathéodory function verifying the conditions listed below

$$|b_i(x, s, \xi)| \leq \beta \left(R_i(x) + |s|^{\underline{p}-1} + |\xi_i|^{p_i - 1} \right) \text{ for any } i = 1, \ldots, N, \tag{3}$$

$$b_i(x, s, \xi)\xi_i \geq \alpha \, |\xi_i|^{p_i} \qquad \text{for any } i = 1, \ldots, N, \tag{4}$$

for all $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$, we have

$$[b_i(x, s, \xi) - b_i\,(x, s, \xi')]\,(\xi_i - \xi'_i) > 0 \quad \text{ for } \quad \xi_i \neq \xi'_i, \tag{5}$$

for a.e. $x \in \Omega$, and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $R_i(x) > 0$ belongs to $L^{p'_i}(\Omega)$ and $\alpha, \beta > 0$.

According to (4) and using the continuity of b_i with respect to ξ, we obtain

$$b_i(x, s, 0) = 0.$$

The term $N(x, s, \xi)$ be Carathéodory functions such that

$$N(x, s, \xi)s \geq 0, \tag{6}$$

$$|N(x, s, \xi)| \leq b(|s|) \left(c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i} \right), \tag{7}$$

with $b(\cdot)$ and $0 \leq c(\cdot) \in L^1(\Omega)$.

We consider the following problem with obstacle modeled by

$$\begin{cases} Bv + N(x, v, \nabla v) = g - \operatorname{div} \varphi(v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \\ v \geq \varLambda, & \end{cases} \tag{8}$$

where

$$g \in L^1(\Omega), \quad \text{ and } \quad \varphi(\cdot) \in C^0\left(\mathbb{R}, \mathbb{R}^N\right). \tag{9}$$

Remark 3.1. *The hypothesise* (3) *is necessary to ensure that* $|a_i(x,v,\nabla v)| \in L^{p_i'(\cdot)}(\Omega)$*. In the case of* $Bv = -\sum_{i=1}^{N} D^i b_i(x,\nabla v)$*, the condition* (3) *can be written as*

$$|b_i(x,\xi)| \leq \beta \left(R_i(x) + \sum_{i=1}^{N} |\xi_i|^{p_i-1} \right).$$

Lemma 3.1. *[13] Assuming that* (3)*-*(5) *hold, and let* $(v_n)_{n\in N}$ *be a sequence in* $W_0^{1,\vec{p}}(\Omega)$ *such that* $v_n \rightharpoonup v$ *in* $W_0^{1,\vec{p}}(\Omega)$ *and*

$$\begin{aligned}&\int_\Omega \left(|v_n|^{p-2} v_n - |v|^{p-2} v\right)(v_n - u)\,dx \\ &\quad + \sum_{i=1}^{N} \int_\Omega \left(b_i\left(x, v_n, \nabla v_n\right) - b_i\left(x, v_n, \nabla u\right)\right)\left(D^i v_n - D^i v\right) dx \to 0,\end{aligned}$$

then $v_n \to v$ *in* $W_0^{1,\vec{p}}(\Omega)$.

4. The Existence of an Entropy Solution

Firstly, let us define an entropy solutions for the problem under study (8) as follows.

Definition 4.1. A measurable function v is named entropy solution of the obstacle problem (8) if

$$\left\{\begin{array}{c} T_k(v) \in K_\Lambda, \\ \displaystyle\sum_{i=1}^{N} \int_\Omega b_i(x,v,\nabla v) D^i T_k(v-u)dx + \int_\Omega N(x,v,\nabla v) T_k(v-u)dx \\ \displaystyle\leq \int_\Omega g T_k(v-u)dx + \sum_{i=1}^{N} \int_\Omega \varphi_i(v) D^i T_k(v-u)dx \end{array}\right. \tag{10}$$

with $u \in K_\Lambda \cap L^\infty(\Omega)$.

the aim is to show the following result.

Theorem 4.1. *Suppose that* (3)*-*(7) *and* (9) *hold, then the problem* (8) *admit at least one solution in the sense of the definition 4.1.*

4.1. Proof of the main result

4.1.1. Approximate problem.

Let $(g_n)_{n\in N} \in W^{-1,p'}(\Omega) \cap L^1(\Omega)$ such that $g_n \to G$ in $L^1(\Omega)$ and $|g_n| \leq |g|$ ($g_n = T_n(g)$). We consider the penalized approximation.

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega b_i\left(x, T_n\left(v_n\right), \nabla v_n\right) D^i\left(v_n - u\right) dx + \frac{1}{n} \int_\Omega |v_n|^{p_0-2} v_n\left(v_n - u\right) dx \\ &+ \int_\Omega N_n\left(x, v_n, \nabla v_n\right)\left(v_n - u\right) dx \\ &\leq \int_\Omega g_n\left(v_n - u\right) dx + \sum_{i=1}^{N} \int_\Omega \varphi_{n,i}\left(v_n\right) D^i\left(v_n - u\right) dx,\end{aligned} \tag{11}$$

for any $v \in K_\psi$, where

$$\varphi_{n,i}(s) = \varphi_i\left(T_n(s)\right),$$

and

$$N_n(x,s,\xi) = T_n(N(x,s,\xi)),$$

We can show, similarly as in [2], that there exists at least one weak solution $v_n \in W_0^{1,\vec{p}(x)}(\Omega)$ of the problem (11).

4.1.2. A priori estimates

Suppose that $k \geq \max\left(1, \|\mathit{\Lambda}^+\|_\infty\right)$, we put $v = v_n - \eta T_k\left(v_n - \mathit{\Lambda}^+\right) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, for η very small we get $v \in K_{\mathit{\Lambda}}$, therefore v is an admissible test in (11), this implies

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega b_i\left(x, T_n\left(v_n\right), \nabla v_n\right) D^i T_k\left(v_n - \mathit{\Lambda}^+\right) dx + \frac{1}{n}\int_\Omega |v_n|^{p_0-2} v_n T_k\left(v_n - \mathit{\Lambda}^+\right) dx \\
&+ \int_\Omega N_n\left(x, v_n, \nabla v_n\right) T_k\left(v_n - \mathit{\Lambda}^+\right) dx \\
&\leq \int_\Omega g_n T_k\left(v_n - \mathit{\Lambda}^+\right) dx + \sum_{i=1}^N \int_\Omega \varphi_{n,i}\left(v_n\right) D^i T_k\left(v_n - \mathit{\Lambda}^+\right) dx.
\end{aligned} \tag{12}$$

For the term : $\sum_{i=1}^N \int_\Omega \varphi_{n,i}\left(v_n\right) D^i T_k\left(v_n - \mathit{\Lambda}^+\right) dx$, we have

$$\begin{aligned}
\sum_{i=1}^N \int_\Omega \varphi_{n,i}\left(v_n\right) D^i T_k\left(v_n - \mathit{\Lambda}^+\right) dx = &\sum_{i=1}^N \int_{\{|v_n - \mathit{\Lambda}^+| \leq k\}} \varphi_{n,i}\left(v_n\right) D^i v_n dx \\
&- \sum_{i=1}^N \int_{\{|v_n - \mathit{\Lambda}^+| \leq k\}} \varphi_{n,i}\left(v_n\right) D^i \mathit{\Lambda}^+ dx
\end{aligned} \tag{13}$$

Putting $\Phi_{n,i}(t) = \int_0^t \varphi_{n,i}(\tau) d\tau$, then $\Phi_{n,i}(0) = 0$ and $\Phi_{n,i}(\cdot) \in C^1(\mathbb{R})$, then the Green formula implies that

$$\begin{aligned}
\sum_{i=1}^N \int_{\{|v_n - \mathit{\Lambda}^+| \leq k\}} \varphi_{n,i}\left(v_n\right) D^i v_n dx &= \sum_{i=1}^N \int_\Omega \varphi_{n,i}\left(v_n\right) D^i v_n \cdot \chi_{\{|v_n - \mathit{\Lambda}^+| \leq k\}} dx \\
&= \sum_{i=1}^N \int_\Omega D^i \Phi_{n,i}\left(v_n \cdot \chi_{\{|v_n - \mathit{\Lambda}^+| \leq k\}}\right) dx \\
&= \sum_{i=1}^N \int_{\partial\Omega} \Phi_{n,i}\left(v_n \cdot \chi_{\{|v_n - \mathit{\Lambda}^+| \leq k\}}\right) \cdot n_i d\sigma = 0,
\end{aligned} \tag{14}$$

since $v_n = 0$ on $\partial\Omega$, with $n = (n_1, n_2, \ldots, n_N)$ is the normal vector on $\partial\Omega$.

therefore, we can have

$$
\begin{aligned}
\sum_{i=1}^{N}\left|\int_{\{|v_n-\Lambda^+|\le k\}}\varphi_{n,i}(v_n)\,D^i\Lambda^+dx\right| &\le \sum_{i=1}^{N}\int_{\left\{|v_n|\le k+\|\Lambda^+\|_\infty\right\}}|\varphi_{n,i}(v_n)|\left|D^i\Lambda^+\right|dx\\
&\le \sum_{i=1}^{N}\sup_{|s|\le k+\|\Lambda^+\|_\infty}|\varphi_i(s)|\int_\Omega\left|D^i\Lambda^+\right|dxf\le C_1
\end{aligned}
\tag{15}
$$

Using (6), we have

$$
\int_\Omega N_n(x,v_n,\nabla v_n)\,T_k\left(v_n-\Lambda^+\right)dx\ge 0,
\tag{16}
$$

Thanks to (13)–(16), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega b_i(x,T_n(v_n),\nabla v_n)\,D^iT_k\left(v_n-\Lambda^+\right)dx+\frac{1}{n}\int_\Omega|v_n|^{p_0-2}v_nT_k\left(v_n-\Lambda^+\right)dx\\
&\le k\int_\Omega|f(x)|dx+\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}b_i(x,T_n(v_n),\nabla v_n)\,D^i\Lambda^+dx+C_1
\end{aligned}
\tag{17}
$$

Since $k\ge\|\Lambda^+\|_\infty$, then $T_k(v_n-\Lambda^+)$ have the same sign as v_n on the set $\{|v_n-\Lambda^+|>k\}$, it follows that

$$
\begin{aligned}
\frac{1}{n}\int_\Omega|v_n|^{p_0-2}v_nT_k\left(v_n-\Lambda^+\right)dx&=\frac{1}{n}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{p_0-2}v_n\left(v_n-\Lambda^+\right)dx\\
&\quad+\frac{1}{n}\int_{\{|v_n-\Lambda^+|>k\}}|v_n|^{p_0-2}v_nT_k\left(v_n-\Lambda^+\right)dx\\
&\ge\frac{1}{n}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{p_0}dx-\frac{1}{n}\int_{\left\{|v_n-\Lambda^+|\le\ k\right\}}|v_n|^{p_0-1}\left|\Lambda^+\right|dx,
\end{aligned}
$$

taking into account (4), we get

$$
\begin{aligned}
&\alpha\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}\left|D^iv_n\right|^{p_i}dx+\frac{1}{n}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{p_0}dx\\
&\le\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}b_i(x,T_n(v_n),\nabla v_n)\,D^iv_ndx+\frac{1}{n}\int_\Omega|v_n|^{p_0-2}v_nT_k\left(v_n-\Lambda^+\right)dx\\
&\quad+\frac{1}{n}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{p_0-1}\left|\Lambda^+\right|dx\le k\int_\Omega\left(|f(x)|+|f_0(x)|\right)dx\\
&+\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}|b_i(x,T_n(v_n),\nabla v_n)|\left|D^i\Lambda^+\right|dx+\frac{1}{n}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{p_0-1}\left|\Lambda^+\right|dx.
\end{aligned}
\tag{18}
$$

By applying Young's inequality one has

$$
\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{p_0-1}\left|\Lambda^+\right|dx\le\frac{1}{2}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{p_0}dx+C_2,
\tag{19}
$$

Furthermore, by taking into account of (3) and by using Young's inequality, one has

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}|b_i(x,T_n(v_n),\nabla v_n)|\left|D^i\Lambda^+\right|dx\\
&\le \beta\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}R_i(x)\left|D^i\Lambda^+\right|dx+\beta\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{\underline{p}-1}\left|D^i\Lambda^+\right|dx\\
&+\beta\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}\left|D^iv_n\right|^{p_i-1}\left|D^i\Lambda^+\right|dx\\
&\le C_4+\varepsilon\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{\underline{p}}dx+\frac{1}{\varepsilon^{\underline{p}-1}}\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}\left|D^i\Lambda^+\right|^{\underline{p}}dx\\
&+C_5\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}\left|D^i\Lambda^+\right|^{p_i}dx
\end{aligned} \tag{20}$$

By thanking to (17)-(20), we conclude that

$$\begin{aligned}
&\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}\left|D^iv_n\right|^{p_i}dx+\frac{1}{2n}\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{p_0}dx\\
&\quad\le kC+N\varepsilon\int_{\{|v_n-\Lambda^+|\le k\}}|v_n|^{\underline{p}}dx+C_4(\varepsilon)
\end{aligned} \tag{21}$$

Therefore, there exists $C_5(k,\varepsilon)$ such that

$$\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|v_n-\Lambda^+|\le k\}}\left|D^iv_n\right|^{p_i}dx\le C_5(k,\varepsilon). \tag{22}$$

and since

$$\{x\in\Omega,|v_n|\le k\}\subset\left\{x\in\Omega,\left|v_n-\Lambda^+\right|\le k+\left\|\Lambda^+\right\|_\infty\right\}$$

Thus

$$\begin{aligned}
\sum_{i=0}^{N}\int_\Omega\left|D^iT_k(v_n)\right|^{p_i}dx&=\sum_{i=1}^{N}\int_{\{|v_n|\le k\}}\left|D^iv_n\right|^{p_i}dx+\int_\Omega|T_k(v_n)|^{p_0}dx\\
&\le\sum_{i=1}^{N}\int_{\left\{|v_n-\Lambda^+|\le k+\|\Lambda^+\|_\infty\right\}}\left|D^iv_n\right|^{p_i}dx+k^{p_0}|\Omega|\\
&\le C_6\left(k,\left\|\Lambda^+\right\|_\infty,\varepsilon\right)
\end{aligned}$$

and we have

$$\|T_k(v_n)\|_{1,\underline{p}}\le C_7\left(k,\left\|\Lambda^+\right\|_\infty,\varepsilon\right),$$

where $C_7>0$ does not depend on n. Thus, the sequence $(T_k(v_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega)$ uniformly in n, then there exists a subsequence still denoted $(T_k(v_n))_{n\in N}$ and a function $v_k\in W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} T_k(v_n)\rightharpoonup v_k & \text{weakly in } W_0^{1,\vec{p}}(\Omega)\\ T_k(v_n)\to v_k & \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e in } \Omega.\end{cases} \tag{23}$$

Now, using (21) and by applying Poincaré inequality, one has

$$\begin{aligned}
\|\nabla T_k(v_n)\|_{\underline{p}} &= \sum_{i=1}^{N} \int_\Omega \left|D^i T_k(v_n)\right|^{\underline{p}} dx \\
&\leq \sum_{i=1}^{N} \int_\Omega \left|D^i T_k(v_n)\right|^{p_i} dx + N|\Omega| \\
&\leq \frac{4k}{\alpha} C + \frac{4N\varepsilon}{\alpha} \|T_k(v_n)\|_{\underline{p}} + C_8(\varepsilon) \\
&\leq \frac{4k}{\alpha} C + C' \frac{4N\varepsilon}{\alpha} \|\nabla T_k(v_n)\|_{\underline{p}} + C_8(\varepsilon).
\end{aligned}$$

Therefore, by choosing ε very enough $\left(C' \frac{4N\varepsilon}{\alpha} \leq \frac{1}{2}\right)$, there exists a constant C_9 that does not depend on k and n, such that

$$\|\nabla T_k(v_n)\|_{\underline{p}} \leq C_9 k^{\frac{1}{\underline{p}}} \quad \text{for} \quad k \geq 1,$$

and we arrive at

$$\begin{aligned}
k \operatorname{meas}\{|v_n| > k\} &= \int_{\{|v_n|>k\}} |T_k(v_n)|\, dx \leq \int_\Omega |T_k(v_n)|\, dx \\
&\leq \|1\|_{p'} \|T_k(v_n)\|_{\underline{p}} \\
&\leq C \|\nabla T_k(v_n)\|_{\underline{p}} \\
&\leq C_{10} k^{\frac{1}{\underline{p}}},
\end{aligned} \tag{24}$$

which yields.

$$\operatorname{meas}\{|v_n| > k\} \leq C_{13} \frac{1}{k^{1-\frac{1}{\underline{p}}}} \longrightarrow 0 \text{ as } k \to \infty. \tag{25}$$

Now, we will prove that the sequence $(v_n)_n$ is a Cauchy sequence in measure. Indeed, we have for every $\delta > 0$

$$\begin{aligned}
&\operatorname{meas}\{|v_n - v_m| > \delta\} \leq \operatorname{meas}\{|v_n| > k\} + \operatorname{meas}\{|v_m| > k\} \\
&+ \operatorname{meas}\{|T_k(v_n) - T_k(v_m)| > \delta\}
\end{aligned}$$

suppose that $\varepsilon > 0$, in view of (25) we can take $k = k(\varepsilon)$ large enough such that

$$\operatorname{meas}\{|v_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \operatorname{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{26}$$

On the other hand, thanks to (23) we get

$$T_k(v_n) \longrightarrow \eta_k \text{ in } L^{\underline{p}}(\Omega) \text{ and a.e.in } \Omega.$$

Thus $(T_k(v_n))_{n\in N}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\operatorname{meas}\left\{\left|T_k\left(v_n\right)-T_k\left(v_m\right)\right|>\delta\right\} \leq \frac{\varepsilon}{3} \quad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{27}$$

By using (26) and (27), we conclude that : for all $\delta, \varepsilon>0$, there exists $n_0=n_0(\delta, \varepsilon)$

$$\operatorname{meas}\left\{\left|v_n-v_m\right|>\delta\right\} \leq \varepsilon \quad \text{for any } n, m \geq n_0.$$

Then $\left(v_n\right)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function v. By using (23) one has

$$\begin{cases} T_k\left(v_n\right) \rightharpoonup T_k(v) & \text{in } W_0^{1, \vec{p}}(\Omega) \\ T_k\left(v_n\right) \rightarrow T_k(v) & \text{in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{28}$$

4.1.3. Convergence of the gradient

From now on, we denote by $\varepsilon_i(n), i=1,2, \ldots$ various real-valued functions of real variables that converge to 0 as n tends to infinity. For $h>k>0$, we define

$$M:=4 k+h \quad \text{and} \quad \omega_n:=T_{2 k}\left(v_n-T_h\left(v_n\right)+T_k\left(v_n\right)-T_k(u)\right).$$

Choosing $v=v_n-\eta \omega_n$, we have $v \geq \Lambda$ for η small enough, thus v is an admissible test function in (11), and we obtain

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} b_i\left(x, T_n\left(v_n\right), \nabla v_n\right) D^i \omega_n d x+\frac{1}{n} \int_{\Omega}\left|v_n\right|^{p_0-2} v_n \omega_n d x \\ & +\int_{\Omega} N_n\left(x, v_n, \nabla v_n\right) \omega_n d x \leq \int_{\Omega} g_n \omega_n d x+\sum_{i=1}^N \int_{\Omega} \varphi_{n, i}\left(v_n\right) D^i \omega_n d x . \end{aligned}$$

It is easy to see that ω_n have the same sign as v_n on the set $\left\{\left|v_n\right|>k\right\}$ and $\omega_n=T_k\left(v_n\right)-T_k(v)$ on the set $\left\{\left|v_n\right| \leq k\right\}$, also we have $D^i \omega_n=0$ on $\left\{\left|v_n\right|>M\right\}$ then,

$$\begin{aligned} & \sum_{i=1}^N \int_{\left\{\left|v_n\right| \leq M\right\}} b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i \omega_n d x \\ & +\frac{1}{n} \int_{\left\{\left|v_n\right| \leq k\right\}}\left|v_n\right|^{p_0-2} v_n\left(T_k\left(v_n\right)-T_k(v)\right) d x+\int_{\left\{\left|v_n\right| \leq k\right\}} N_n\left(x, v_n, \nabla v_n\right) \omega_n d x \\ & \leq \int_{\Omega}\left|g_n(x)\right| d x+\int_{\left\{\left|v_n\right| \leq M\right\}} \varphi_{n, i}\left(T_M\left(v_n\right)\right) D^i \omega_n d x . \end{aligned} \tag{29}$$

The first term on the left-hand side of (29) is obtaining by

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|v_n|\leq M\}} b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i \omega_n dx \\
&\quad= \sum_{i=1}^{N} \int_{\{|v_n|\leq k\}} b_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right)\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right) dx \\
&\quad+ \sum_{i=1}^{N} \int_{\{k<|v_n|\leq M\}} b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i \omega_n dx \\
&= \sum_{i=1}^{N} \int_{\Omega}\left(b_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right)-b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right)\right) \\
&\qquad\qquad \times\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right) dx \\
&\quad+ \sum_{i=1}^{N} \int_{\Omega} b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right)\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right) dx \\
&\quad+ \sum_{i=1}^{N} \int_{\{|v_n|>k\}} b_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right) D^i T_k(v) dx \\
&\quad+ \sum_{i=1}^{N} \int_{\{k<|v_n|\leq M\}} b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i \omega_n dx .
\end{aligned} \tag{30}
$$

Lebesgue's dominated convergence theorem implies that $T_k\left(v_n\right) \rightarrow T_k(v)$ in $L^{p_i}(\Omega)$, therefore

$$
b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right) \rightarrow b_i\left(x, T_k(v), \nabla T_k(v)\right) \quad \text{ in } \quad L^{p_i^{\prime}}(\Omega),
$$

and since $D^i T_k\left(v_n\right)$ converges to $D^i T_k(v)$ weakly in $L^{p_i}(\Omega)$, we obtain

$$
\varepsilon_1(n)=\sum_{i=1}^{N} \int_{\Omega} b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right)\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right) dx \longrightarrow 0 \text{ as } n \rightarrow \infty. \tag{31}
$$

Now, by (4) and using the continuity of $a(x, s, \xi)$ we obtain $a(x, s, 0)=0$, therefore,

$$
\int_{\{|v_n|>k\}} a_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right) D^i T_k(v) dx=\int_{\{|v_n|>k\}} a_i\left(x, T_k\left(v_n\right), 0\right) D^i T_k(v) dx=0. \tag{32}
$$

Finally, taking $z_n=v_n-T_h\left(v_n\right)+T_k\left(v_n\right)-T_k(v)$, then

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|v_n|>k\}} b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i \omega_n dx \\
&= \sum_{i=1}^{N} \int_{\{|v_n|>k\} \cap\{|z_n| \leq 2k\}} b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i\left(v_n-T_h\left(v_n\right)+T_k\left(v_n\right)-T_k(v)\right) dx \\
&\geq -\sum_{i=1}^{N} \int_{\{|v_n|>k\}}\left|b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right)\right|\left|D^i T_k(v)\right| dx
\end{aligned}
$$

We have $\left(b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right)\right)_{n \in N}$ is bounded in $L^{p_i^{\prime}}(\Omega)$, then there exists $\vartheta_i \in L^{p_i^{\prime}}(\Omega)$ such that

$b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) \rightharpoonup \vartheta_i$ in $L^{p_i'}(\Omega)$. Therefore,

$$\begin{aligned}\varepsilon_2(n) &= \sum_{i=1}^N \int_{\{|v_n|>k\}} b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i T_k(v) dx \\ &\longrightarrow \sum_{i=1}^N \int_{\{|u|>k\}} \vartheta_i D^i T_k(v) dx = 0,\end{aligned}$$

thus

$$\sum_{i=1}^N \int_{\{|v_n|>k\}} b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i \omega_n dx \geq \varepsilon_2(n). \tag{33}$$

According to (30)-(33), we conclude that

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega \left(b_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right)-b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right)\right)\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right) dx \\ &\quad \leq \sum_{i=1}^N \int_\Omega b_i\left(x, T_M\left(v_n\right), \nabla T_M\left(v_n\right)\right) D^i \omega_n dx+\varepsilon_3(n).\end{aligned} \tag{34}$$

For the second term on the left-hand side of (29), using (50) we have

$$\varepsilon_4(n)=\frac{1}{n}\left|\int_{\{|v_n| \leq k\}}\left|v_n\right|^{p_0-2} v_n\left(T_k\left(v_n\right)-T_k(v)\right) dx\right| \leq \frac{2k}{n} \int_{\{|v_n| \leq k\}}\left|v_n\right|^{p_0-1} dx \longrightarrow 0 \text{ as } n \rightarrow \infty. \tag{35}$$

Using (29) and (34)-(35), we obtain

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega \left(b_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right)-b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right)\right)\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right) dx \\ &+\int_{\{|v_n| \leq k\}} N_n\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right) \omega_n dx \\ &\leq \int_\Omega \left(\left|g_n(x)\right|\right)\left|\omega_n\right| dx+\sum_{i=1}^N \int_{\{|v_n| \leq M\}} \varphi_{n,i}\left(T_M\left(v_n\right)\right) D^i \omega_n dx+\varepsilon_5(n).\end{aligned} \tag{36}$$

We have $g_n \rightarrow g$ in $L^1(\Omega)$ and $\omega_n \rightarrow T_{2k}\left(u-T_h(v)\right)$ weak-* in $L^{\infty}(\Omega)$, then

$$\int_\Omega g_n \omega_n dx=\int_\Omega g T_{2k}\left(u-T_h(v)\right) dx+\varepsilon_6(n). \tag{37}$$

Now, by choosing n very large, we obtain

$$\int_\Omega \varphi_{n,i}\left(T_M\left(v_n\right)\right) D^i \omega_n dx=\int_\Omega \varphi_i\left(T_M\left(v_n\right)\right) D^i \omega_n dx, \tag{38}$$

since $D^i \omega_n \rightarrow D^i T_{2k}\left(u-T_h(v)\right)$ in $L^{p_i}(\Omega)$, then

$$\int_\Omega \varphi_{n,i}\left(T_M\left(v_n\right)\right) D^i \omega_n dx=\int_\Omega \varphi_i\left(T_M(v)\right) D^i T_{2k}\left(u-T_h(v)\right) dx+\varepsilon_7(n). \tag{39}$$

Thanks to (7), we get

$$
\begin{aligned}
&\left|\int_{\{|v_n|\leq k\}} N_n\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right)\omega_n dx\right| \\
&\leq b_k \int_{\{|v_n|\leq k\}}\left(c(x)+\sum_{i=1}^N\left|D^i T_k\left(v_n\right)\right|^{p_i}\right)\left|\omega_n\right| dx \\
&\leq b_k \int_{\{|v_n|\leq k\}} c(x)\left|\omega_n\right| dx \\
&+\frac{b_k}{\alpha}\sum_{i=1}^N \int_\Omega\left(b_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right)-b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right)\right) \\
&\times\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right)\left|\omega_n\right| dx \\
&+\frac{b_k}{\alpha}\sum_{i=1}^N \int_\Omega b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right)\cdot\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right)\left|\omega_n\right| dx \\
&+\frac{b_k}{\alpha}\sum_{i=1}^N \int_\Omega a_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right)\cdot D^i T_k(v)\left|\omega_n\right| dx.
\end{aligned}
\tag{40}
$$

Since $T_k\left(v_n\right)-T_k(v) \rightharpoonup 0$ weak-$\star$ in $L^\infty(\Omega)$, then

$$
\int_{\{|v_n|\leq k\}} c(x)\left|\omega_n\right| dx=\int_{\{|v_n|\leq k\}} c(x)\left|T_k\left(v_n\right)-T_k(v)\right| dx \longrightarrow 0.
\tag{41}
$$

According to (36)-(41), one has

$$
\begin{aligned}
&\frac{b_k}{\alpha}\sum_{i=1}^N \int_\Omega\left(b_i\left(x, T_k\left(v_n\right), \nabla T_k\left(v_n\right)\right)-b_i\left(x, T_k\left(v_n\right), \nabla T_k(v)\right)\right) \\
&\times\left(D^i T_k\left(v_n\right)-D^i T_k(v)\right)\left|\omega_n\right| dx \\
&\geq\left|\int_{\{|v_n|\leq k\}} N_n\left(x, v_n, \nabla v_n\right)\omega_n dx\right|+\int_\Omega g_n T_{2k}\left(u-T_h(v)\right) dx \\
&+\sum_{i=1}^N \int_\Omega \varphi_i\left(T_M(v)\right) D^i T_{2k}\left(u-T_h(v)\right) dx+\varepsilon_8(n).
\end{aligned}
\tag{42}
$$

Moreover, we have

$$
\int_\Omega g_n T_{2k}\left(u-T_h(v)\right) dx \longrightarrow 0 \quad \text{ as } \quad h \rightarrow \infty.
\tag{43}
$$

For the second term on the right-hand side of (42), we choose $\Psi_i(t)=\int_0^t \varphi_i(\tau) d\tau$ then $\Psi_i(0)=0$ and $\Psi_i \in C^1(R)$. by analogy to (14), we show that

$$
\int_\Omega \varphi_i\left(T_M(v)\right) D^i T_{2k}\left(u-T_h(v)\right) dx=\int_\Omega \varphi_i\left(T_{2k+h}(v)\right) D^i T_{2k+h}(v) dx-\int_\Omega \varphi_i\left(T_h(v)\right) D^i T_h(v) dx=0.
\tag{44}
$$

Now, we take $v=v_n-\eta T_{2k}\left(v_n-T_h\left(v_n\right)\right)$ for η small enough, we can say that the function v is an admissible test in (8), we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} b_i\left(x,T_n\left(v_n\right),\nabla v_n\right)D^iT_{2k}\left(v_n-T_h\left(v_n\right)\right)dx+\frac{1}{n}\int_{\Omega}\left|v_n\right|^{p_0-2}v_nT_{2k}\left(v_n-T_h\left(v_n\right)\right)dx\\
&\qquad+\int_{\Omega}N(x,v,\nabla v)T_{2k}\left(v_n-T_h\left(v_n\right)\right)dx\\
&=\int_{\Omega}g_nT_{2k}\left(v_n-T_h\left(v_n\right)\right)dx+\sum_{i=1}^{N}\int_{\Omega}\varphi_{n,i}\left(v_n\right)D^iT_{2k}\left(v_n-T_h\left(v_n\right)\right)dx.
\end{aligned}\tag{45}$$

We have

$$\int_{\Omega}\varphi_{n,i}\left(v_n\right)D^iT_{2k}\left(v_n-T_h\left(v_n\right)\right)dx=0\quad\text{ for } i=1,\ldots,N$$

According to (4),(6),(8) and by applying Young's inequality, one has

$$\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{h<|v_n|\leq 2k+h\}}\left|D^iv_n\right|^{p_i}dx\leq\int_{\Omega}g_nT_{2k}\left(v_n-T_h\left(v_n\right)\right)dx.\tag{46}$$

Thus

$$\begin{aligned}
\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{h<|u|\leq 2k+h]}\left|D^iu\right|^{p_i}dx&\leq\frac{\alpha}{2}\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{[h<|v_n|\leq 2k+h|}\left|D^iv_n\right|^{p_i}dx\\
&\leq\lim_{n\to\infty}\int_{\Omega}g_nT_{2k}\left(v_n-T_h\left(v_n\right)\right)dx\\
&=\int_{\Omega}g_nT_{2k}\left(u-T_h(v)\right)dx.
\end{aligned}\tag{47}$$

By letting $h\longrightarrow+\infty$ in (47), we have

$$\limsup_{h\to\infty}\int_{\{h<|u|\leq 2k+h\}}\left|D^iu\right|^{p_i}dx=0.$$

Thanks to (43)-(43),and by letting $h,n\longrightarrow+\infty$ in (42), we have

$$\sum_{i=1}^{N}\int_{\Omega}\left(b_i\left(x,T_k\left(v_n\right),\nabla T_k\left(v_n\right)\right)-b_i\left(x,T_k\left(v_n\right),\nabla T_k(v)\right)\right)\left(D^iT_k\left(v_n\right)-D^iT_k(v)\right)dx\to 0,$$

and since $T_k\left(v_n\right)\to T_k(v)$ in $L^{p_0}(\Omega)$, therefore

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\left(b_i\left(x,T_k\left(v_n\right),\nabla T_k\left(v_n\right)\right)-b_i\left(x,T_k\left(v_n\right),\nabla T_k(v)\right)\right)\left(D^iT_k\left(v_n\right)-D^iT_k(v)\right)dx\\
&+\int_{\Omega}\left(\left|T_k\left(v_n\right)\right|^{p_0-2}T_k\left(v_n\right)-\left|T_k(v)\right|^{p_0-2}T_k(v)\right)\left(T_k\left(v_n\right)-T_k(v)\right)dx\longrightarrow 0\text{ as }n\to\infty
\end{aligned}\tag{48}$$

By using Lemma 3.1 and (19), we obtain

$$\begin{cases}T_k\left(v_n\right)\to T_k(v) & \text{strongly in } W_0^{1,\vec{p}}(\Omega),\\ D^iv_n\to D^iu & \text{a.e. in }\Omega\text{ for } i=1,\ldots,N.\end{cases}\tag{49}$$

4.1.4. The equi-integrability

i) The equi-integrability of $\left(\frac{1}{n}\left|v_n\right|^{p_0-2} v_n\right)_n$.

For passing to the limit in the approximate equation, we prove that

$$\frac{1}{n}\left|v_n\right|^{p_0-2} v_n \longrightarrow 0 \text{ strongly in } L^1(\Omega). \tag{50}$$

By using Vitali's Theorem, we show that $\left(\frac{1}{n}\left|v_n\right|^{p_0-2} v_n\right)_n$ is uniformly equiintegrable. Indeed, taking $v=v_n-\eta T_1\left(v_n-T_h\left(v_n\right)\right)$ as a test function in (11) for η small enough. Using (4), (6) and Young's inequality, we obtain

$$\begin{aligned}\alpha \sum_{i=1}^N \int_{\{h<|v_n| \leq h+1\}}\left|D^i v_n\right|^{p_i} d x+\frac{1}{n} \int_{\{h<|v_n|\}}&\left|v_n\right|^{p_0-2} v_n T_1\left(v_n-T_h\left(v_n\right)\right) d x \\ &\leq \int_{\{h<|v_n|\}}\left|g_n\right| d x,\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{n} \int_{\{h+1 \leq|v_n|\}}\left|v_n\right|^{p_0-1} d x &\leq \frac{1}{n} \int_{\{h<|v_n|\}}\left|v_n\right|^{p_0-2} v_n T_1\left(v_n-T_h\left(v_n\right)\right) d x \\ &\leq \int_{\{h<|v_n|\}}\left|g_n\right| d x,\end{aligned}$$

Let $\eta>0$ be fixed, then there exists $h(\eta) \geq 1$ such that

$$\frac{1}{n} \int_{\{h(\eta)<|v_n|\}}\left|v_n\right|^{p_0-1} d x \leq \frac{\eta}{2}, \tag{51}$$

then, for any measurable subset $E \subset \Omega$, one has

$$\frac{1}{n} \int_E\left|v_n\right|^{p_0-1} d x \leq \frac{1}{n} \int_E\left|T_{h(\eta)}\left(v_n\right)\right|^{p_0-1} d x+\frac{1}{n} \int_{\{h(\eta)<|v_n|\}}\left|v_n\right|^{p_0-1} d x, \tag{52}$$

We can easily remark that there exists $\beta(\eta)>0$ such that, for all $E \subseteq \Omega$ with meas $(E) \leq \beta(\eta)$, we have

$$\frac{1}{n} \int_E\left|T_{h(\eta)}\left(v_n\right)\right|^{p_0-1} d x \leq \frac{\eta}{2}. \tag{53}$$

Now, according to (51),(52) and (53), one has

$$\frac{1}{n} \int_E\left|v_n\right|^{p_0-1} d x \leq \eta \text{ for all } E \text{ such that } meas(E) \leq \beta(\eta). \tag{54}$$

We deduce that $\left(\frac{1}{n}\left|v_n\right|^{p_0-2} v_n\right)_n$ is uniformly equi-integrable, and since

$$\frac{1}{n}\left|v_n\right|^{p_0-2} v_n \longrightarrow 0 \text{ a.e in } \Omega.$$

By applying Vitali's Theorem, we obtain the convergence (50) .

ii) The equi-integrability of $\left(N_n\left(x, v_n, \nabla v_n\right)\right)_n$

We show that

$$N_n\left(x, v_n, \nabla v_n\right) \rightarrow N(x, v, \nabla v). \tag{55}$$

Thanks to Vitali's theorem, we prove that $\left(N_n\left(x, v_n, \nabla v_n\right)\right)_n$ are uniformly equi-integrable. We set

$$\bar{D}(s)=\frac{2}{\alpha} \int_0^s d(|\tau|) d\tau.$$

By choosing $T_1\left(v_n-T_h\left(v_n\right)\right) e^{\bar{D}\left(\left|v_n\right|\right)}$ as a test function in (11), we have

$$\begin{aligned}
& \sum_{i=1}^N \int_{\Omega} b_i\left(x, v_n, \nabla v_n\right) D^i T_1\left(v_n-T_h\left(v_n\right)\right) e^{\bar{D}\left(\left|v_n\right|\right)} dx \\
& +\frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} b_i\left(x, v_n, \nabla v_n\right) D^i v_n d\left(\left|v_n\right|\right)\left|T_1\left(v_n-T_h\left(v_n\right)\right)\right| e^{\bar{D}\left(\left|v_n\right|\right)} dx \\
& +\int_{\Omega} N_n\left(x, v_n, \nabla v_n\right) T_1\left(v_n-T_h\left(v_n\right)\right) e^{\bar{D}\left(\left|v_n\right|\right)} dx \\
& =\int_{\Omega} g_n T_1\left(v_n-T_h\left(v_n\right)\right) e^{\bar{D}\left(\left|v_n\right|\right)} dx+\sum_{i=1}^N \int_{\Omega} \varphi_{n, i}\left(v_n\right) D^i T_1\left(v_n-T_h\left(v_n\right)\right) e^{\bar{D}\left(\left|v_n\right|\right)} dx \\
& +\frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} \varphi_{n, i}\left(v_n\right) D^i v_n d\left(\left|v_n\right|\right)\left|T_1\left(v_n-T_h\left(v_n\right)\right)\right| e^{\bar{D}\left(\left|v_n\right|\right)} dx .
\end{aligned}$$

Thanking to (4), (8), and by applying the divergence lemma we obtain,

$$\begin{aligned}
& \alpha \sum_{i=1}^N \int_{\left\{h<\left|v_n\right| \leq h+1\right\}} b_i\left(x, v_n, \nabla v_n\right) D^i v_n e^{\bar{D}\left(\left|v_n\right|\right)} dx \\
& +2 \sum_{i=1}^N \int_{\left\{h<\left|v_n\right|\right\}}\left|D^i v_n\right|^{p_i} d\left(\left|v_n\right|\right)\left|T_1\left(v_n-T_h\left(v_n\right)\right)\right| e^{\bar{D}\left(\left|v_n\right|\right)} dx \\
& +\int_{\left\{h<\left|v_n\right|\right\}} N_n\left(x, v_n, \nabla v_n\right) T_1\left(v_n-T_h\left(v_n\right)\right) e^{\bar{D}\left(\left|v_n\right|\right)} dx \\
& \leq \int_{\left\{h<\left|v_n\right|\right\}}\left|g_n\right|+\sum_{i=1}^N \int_{\left\{h<\left|v_n\right|\right\}}\left|D^i v_n\right|^{p_i}\left|T_1\left(v_n-T_h\left(v_n\right)\right)\right| d\left(\left|v_n\right|\right) e^{\bar{D}\left(\left|v_n\right|\right)} dx .
\end{aligned} \tag{56}$$

Thus

$$\sum_{i=1}^N \int_{\left\{h+1<\left|v_n\right|\right\}} d\left(\left|v_n\right|\right)\left|D^i v_n\right|^{p_i} dx+\int_{\left\{h+1<\left|v_n\right|\right\}}\left|N_n\left(x, v_n, \nabla v_n\right)\right| dx \leq e^{\bar{D}(\infty)} \int_{\left\{h<\left|v_n\right|\right\}}|f| dx .$$

furthermore, for all $\eta>0$, there exists $h(\eta) \geq 1$ such that

$$\sum_{i=1}^N \int_{\left\{h(\eta)<\left|v_n\right|\right\}} d\left(\left|v_n\right|\right)\left|D^i v_n\right|^{p_i} dx+\int_{\left\{h(\eta)<\left|v_n\right|\right\}}\left|N_n\left(x, v_n, \nabla v_n\right)\right| dx \leq \frac{\eta}{2}. \tag{57}$$

Now, we put

$$b_{h(\eta)} := \max\{b(s) : |s| \leq h(\eta)\} \quad \text{and} \quad d_{h(\eta)} := \max\{d(s) : |s| \leq h(\eta)\},$$

for any measurable subset $E \subseteq \Omega$, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_E d(|v_n|) \left|D^i v_n\right|^{p_i} dx + \int_E |N_n(x, v_n, \nabla v_n)| \, dx \\ &\leq d_{h(\eta)} \sum_{i=1}^{N} \int_E \left|D^i T_{h(\eta)}(v_n)\right|^{p_i} dx + b_{h(\eta)} \int_E \left(c(x) + \sum_{i=1}^{N} \left|D^i T_{h(\eta)}(v_n)\right|^{p_i}\right) dx \\ &\quad + \sum_{i=1}^{N} \int_{\{h(\eta)<|v_n|\}} d(|v_n|) \left|D^i v_n\right|^{p_i} dx + \int_{\{h(\eta)<|v_n|\}} |N_n(x, v_n, \nabla v_n)| \, dx. \end{aligned} \tag{58}$$

From (49), there exists $\tau(\eta) > 0$ such that, for any meas $(E) \leq \tau(\eta)$ one has

$$d_{h(\eta)} \sum_{i=1}^{N} \int_E \left|D^i T_{h(\eta)}(v_n)\right|^{p_i} dx + b_{h(\eta)} \int_E \left(c(x) + \sum_{i=1}^{N} \left|D^i T_{h(\eta)}(v_n)\right|^{p_i}\right) dx \leq \frac{\eta}{2}. \tag{59}$$

Finally, by thanking to (57), (58) and (59), we obtain

$$\sum_{i=1}^{N} \int_E d(|v_n|) \left|D^i v_n\right|^{p_i} dx + \int_E |N_n(x, v_n, \nabla v_n)| \, dx \leq \eta \quad \text{for all} \quad \text{meas}(E) \leq \tau(\eta). \tag{60}$$

Thanks to (8), we can get and $(N_n(x, v_n, \nabla v_n))_n$ is uniformly equi-integrable, and since

$$N_n(x, v_n, \nabla v_n) \to N(x, v, \nabla v) \text{ a.e. in } \Omega.$$

By using Vitali's theorem, we deduce the convergence (55).

4.1.5. Passing to the limit

Let $v \in K_\psi \cap L^\infty(\Omega)$, by choosing $v_n - \eta T_k(v_n - u)$ as a test function in (11), with η smal enough, we get

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega b_i(x, T_n(v_n), \nabla v_n) D^i T_k(v_n - u) \, dx + \int_\Omega N_n(x, v_n, \nabla v_n) T_k(v_n - u) \, dx \\ &+ \frac{1}{n} \int_\Omega |v_n|^{p_0 - 2} v_n T_k(v_n - u) \, dx \leq \int_\Omega g_n T_k(v_n - u) \, dx + \sum_{i=1}^{N} \int_\Omega \varphi_{n,i}(v_n) D^i T_k(v_n - u) \, dx. \end{aligned} \tag{61}$$

Now, let $M = k + \|v\|_\infty$ then $\{|v_n - v| \leq k\} \subseteq \{|v_n| \leq M\}$, and we have

$$\begin{aligned} &\int_\Omega b_i(x, T_n(v_n), \nabla v_n) D^i T_k(v_n - u) \, dx \\ &= \int_\Omega b_i(x, T_M(v_n), \nabla T_M(v_n)) \left(D^i T_M(v_n) - D^i v\right) \chi_{\{|v_n - v| \leq k\}} dx \\ &= \int_\Omega \left(b_i(x, T_M(v_n), \nabla T_M(v_n)) - b_i(x, T_M(v_n), \nabla v)\right) \left(D^i T_M(v_n) - D^i v\right) \chi_{\{|v_n - v| \leq k\}} dx \\ &\quad + \int_\Omega b_i(x, T_M(v_n), \nabla v) \left(D^i T_M(v_n) - D^i v\right) \chi_{\{|v_n - v| \leq k\}} dx. \end{aligned}$$

It's clear that

$$\begin{aligned}\lim_{n\to\infty}&\int_\Omega b_i\left(x,T_M\left(v_n\right),\nabla v\right)\left(D^iT_M\left(v_n\right)-D^iv\right)\chi_{\{|v_n-v|\leq k\}}dx\\&=\int_\Omega b_i\left(x,T_M(v),\nabla v\right)\left(D^iT_M(v)-D^iv\right)\chi\{|u-v|\leq k\}dx.\end{aligned}$$

By using Fatou's Lemma, one has

$$\begin{aligned}&\liminf_{n\to\infty}\sum_{i=1}^N\int_\Omega b_i\left(x,T_n\left(v_n\right),\nabla v_n\right)D^iT_k\left(v_n-u\right)dx\\
\geq&\sum_{i=1}^N\int_\Omega\left(b_i\left(x,T_M(v),\nabla T_M(v)\right)-b_i\left(x,T_M(v),\nabla v\right)\right)\left(D^iT_M(v)-D^iv\right)\chi_{\{|u-v|\leq k\}}dx\\
&+\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega b_i\left(x,T_M\left(v_n\right),\nabla v\right)\left(D^iT_M\left(v_n\right)-D^iv\right)\chi_{\{|v_n-v|\leq k\}}dx\\
&=\sum_{i=1}^N\int_\Omega a_i\left(x,T_M(v),\nabla T_M(v)\right)\left(D^iT_M(v)-D^iv\right)\chi_{\{|u-v|\leq k\}}dx\\
&=\sum_{i=1}^N\int_\Omega a_i(x,v,\nabla v)D^iT_k(u-v)dx.\end{aligned}\tag{62}$$

Since $T_k\left(v_n-u\right)\to T_k(u-v)$ weak $-\star$ in $L^\infty(\Omega)$ and in view of (50) we conclude that

$$\frac{1}{n}\int_\Omega\left|v_n\right|^{p_0-2}v_nT_k\left(v_n-u\right)dx\longrightarrow 0,\tag{63}$$

and

$$\int_\Omega N_n\left(x,v_n,\nabla v_n\right)T_k\left(v_n-u\right)dx\longrightarrow\int_\Omega N(x,v,\nabla v)T_k\left(v_n-u\right)dx,\tag{64}$$

also, since g_n tends to f in $L^1(\Omega)$ then

$$\int_\Omega g_nT_k\left(v_n-u\right)dx\longrightarrow\int_\Omega gT_k(u-v)dx.\tag{65}$$

Also, we have $\varphi_{n,i}\left(v_n\right)=\varphi_i\left(T_M\left(v_n\right)\right)$ in $\{|v_n-v|\leq k\}$ for $n\geq M$, and since $T_k\left(v_n-u\right)\rightharpoonup T_k(u-v)$ in $W_0^{1,\vec{p}}(\Omega)$, then

$$\int_\Omega\varphi_{n,i}\left(v_n\right)D^iT_k\left(v_n-u\right)dx\longrightarrow\int_\Omega\varphi_i(v)D^iT_k(u-v)dx,\tag{66}$$

By combining (61)-(66), we deduce the proof of Theorem 4.1 .

Example 4.1.
Let $\varphi(\cdot)\equiv 0$, and $N(\cdot)\equiv 0$ we take,

$$b_i(x,v,\nabla v)=\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\quad\textit{for}\quad i=1,\ldots,N.$$

It is clear that the functions $b_i(x, v, \nabla v)$ verify (3)-(5).

Thanks to the Theorem 4.1, the anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\right) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

has at least an entropy solution, i.e. the function v verify $T_k(v) \in W^{1,\vec{p}}(\Omega)$ and

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{p_i-2} \frac{\partial v}{\partial x_i}\frac{\partial T_k(v-u)}{\partial x_i}dx \leq \int_\Omega gT_k(v-u)dx,$$

for any positive function $u \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

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