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Theory of operators and Unitary operators on the power set and the way below relation on transitive binary relational sets

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Abstract: In this work, S. Abramsky and A. Jung introduced a method to construct a canonical partially ordered set from a pre-order set. The main goal of this monograph is to propose a new representation theory involving below relation ($\ll$(), lower (resp. upper) closure, upper (resp. a lower) cone, convex hull, co final set, directed set, domain set and Scott-open (resp Scott-closed) set. We also study the family of all isolated points above (resp. below), isolated point in transitive relational sets. Order preserving monotone and idempotent function between two posets are introduced and discussed, where the latter become representable in all cases, and still rich enough to allow geometric, topological and combinatorial applications. Throughout the text, we shall give evidence of the geometric potential of these new ideas. These concepts extend in many aspects of the known geometric theory of poset, but they also raise new perspectives in the topological world. In particular, we believe that our results and techniques may be of interest in connection with several of the famous conjectures and constructions for algebra and topology. The readers consider them as a source of new concepts, techniques and problems for algebraic theory.

Key words: way below relation, pre-oreder binary relational sets, poset

1. Introduction

The present paper contains a brief description of partial orders (poset), in particular in posets and lattices, way below relation and more types of functions in transitive relations. A poset consists of a set together with a binary relation that indicates that, for certain pairs of elements in the set, one of the elements precedes the other. Such a relation is called a partial order to reflect the fact that not every pair of elements needs to be related: for some pairs, it may be that neither element precedes the other in the poset. Thus, posets generalize the more familiar total orders. The main aim is to generalize the concepts of poset, by replacing transitive binary relation. In the context of the poset, we propose several new methods of constructing below relation ($\ll$), lower (resp. upper) closure, upper (resp. a lower) cone, convex hull , cofinal set, directed set, domain set and Scott-open (resp Scott-closed) set. We also study the family of all isolated points above (resp. below), isolated point in transitive relational sets. Order preserving monotone and idempotent function between two posets are introduced and discused. Several formalisms can be used to express variability in a poset. We take a unified approach to study poset and related concepts. The concept of chain complete poset [15] and the concept of domain [5] are identical although they have different names. In this paper, we stress the importance of the two kinds of operators preserving monotonicity and idempotent property The work of this paper is organized as follows. We shall first briefly introduce poset and related concepts. In Section 2, two kinds of operators are combined , and used to study the relation between Unitary operators on the power set and transitive relations. In Section 3, two kinds of operators above are combined to construct algebraic

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lattices, and used to study the relation between The way below relation and some types of functions on transitive relations We first list some necessary notions and relevant properties from the classical order theory in the sequel

Definition 1.1. Let $\leq$ be a binary relation set on $X \neq \phi$. Then;

(1) $\leq$ is called reflexive (RR for short) iff $\forall\, x \in X,\ x \leq x$ [9];

(2) $\leq$ is called antisymetric ((AR for short) iff $\forall\, x, y \in X,\ x \leq y$ and $y \leq x \Rightarrow x = y$ [9];

(3) $\leq$ is called transitive (TR for short)) iff $\forall\, x, y, z \in X,\ x \leq y$ and $y \leq z \Rightarrow x = z$[9];

(4) $\leq$ is called symetric ((SR for short) iff $\forall\, x, y \in X,\ x \leq y \Rightarrow y \leq x$ [9];

(5) $\leq$ is called interpolative (denoted by, IR) iff $\forall\, x, z \in X$, with $x \leq z,\ \exists\ y \in X\ \ s.t.\ x \leq y \leq z$ [3].

(6) If $\leq$ satisfies the conditions (1), (2) and (3), then $(X, \leq)$ is called partialy order set (Poset) [13];

(7) If $\leq$ satisfies the conditions (1), and (3), then $(X, \leq)$ is called pre-orderd set (Quasi set) (PRS for short) [9];

(8) If $\leq$ satisfies the conditions (1), (2), (3) and (4), then $(X, \leq)$ is called an equvalence set,

(9) If $\leq$ satisfies the conditions (3) and (5), then $(X, \leq)$ is a Continuous information system [7].

(10) If $\leq$ satisfies the conditions (3), and $\forall\, x \in X$, and for every finite subset A of X the following axiom holds: if $\forall\, y \in A,\ y \leq x$ then $\exists\ z \in X\ \ s.t. \forall\, y \in A,\ y \leq z$ and $z \leq x$, then $(X, \leq)$ is abstract basis [12].

Definition 1.2. For any poset X, let $A, B, \lambda \subseteq X$. Then:

(1) The lower (resp. upper) bounded subset in X of λ is denoted by $lb(\lambda)$ (resp. $ub(\lambda)$) and defined as follows: $lb(\lambda) = \{x \in X:\ \forall\, y \in \lambda,\ \ x \leq y)$ (resp. $ub(\lambda) = \{x \in X:\ \forall\, y \in \lambda,\ \ y \leq x)$.

(2) The subset of least (resp. largest) elements of a subset λ is denoted by $le(\lambda)$ (resp. $la(\lambda)$) and defined as follows:

$le(\lambda) = \{x \in \lambda:\ \forall\, y \in \lambda,\ \ x \leq y)$ (resp. $la(\lambda) = \{x \in \lambda:\ \forall\, y \in \lambda,\ \ y \leq x)$. Each element in $le(\lambda)$ (resp. $la(\lambda)$) is aclled a least (resp. largest) element of λ [9].

(3) The infimum (resp. supremum) subset in X is denoted by $\bigwedge(\lambda)$ (resp. $\bigvee(\lambda)$) and defined as follows:

$\bigwedge(\lambda) = la(lb(\lambda))$ (resp. $\bigvee(\lambda) = le(ub(\lambda))$. Each element in $\bigwedge(\lambda)$ (resp. $\bigvee(\lambda)$) is aclled a infimum (resp. supremum) element of λ [9].

(4) The lower (resp. upper) closure in X of λ is denoted by $\downarrow \lambda$ (resp. $\uparrow \lambda$) and defined as follows: $\downarrow \lambda = \{x \in X : \exists\ \ y \in \lambda\ \ s.t.\ x \leq y\ \}$(resp. $\uparrow \lambda = \{x \in X : \exists\ \ y \in \lambda\ \ s.t.\ y \leq x\ \}$)[3];

(5) An upper (resp. a lower) cone of X iff $\exists\, x \in \lambda\ \ s.t.\ \lambda = \uparrow x$ (resp. $\lambda = \downarrow x$)[3].

(6) The convex hull A is denoted by $\updownarrow A$ and defined as follows: $\updownarrow A = \downarrow A \cap \uparrow A$ [3];

(7) Let $A, B \subseteq X$. B is called cofinal in A iff $B \subseteq A \subseteq \downarrow (B)$[3].

(8) λ is called directed subset of X iff $\lambda \neq \phi$ and $\forall\, x, y \in \lambda,\ \exists\, z \in \lambda\ \ s.t.\ x \leq z$ and $y \leq z$ [3];

(9) $\leq$ is called a domain iff for every directed subset λ of X, $\bigvee(\lambda)$ exists.

(10) A subset λ of the domain (resp. Poset) X is called directed closed (d-closed for short) iff $\forall$ directed subset D of λ, $\bigvee(D) \in \lambda$; [3]

(11) A subset λ of the Poset X is called Scott-closed iff λ is d-closed lower subset of X [11] ;

(12) λ is called d-(resp. Scott-) open iff λ^c d-(resp. Scott-) closed [3, 11];

Proposition 1.1.[5]Let $(X, \leq)$ be a poset, let $\lambda \subseteq X$ such that $\bigvee(\lambda)$ exists. Then $\bigcap\limits_{x \in \lambda} \uparrow x = \uparrow \bigvee(\lambda)$.

Proposition 1.2.[5] Let $(X, \leq)$ be a poset, let $\lambda, \mu \subseteq X$ and let $\lambda \subseteq \mu$. If $\bigvee(\lambda) \neq \phi$ and $\bigvee(\mu) \neq \phi$. Then $\bigvee(\lambda) \leq \bigvee(\mu)$.

Proposition 1.3.[5] ILet $(X, \leq)$ be a poset. Let $A, B \subseteq X$, Then:

(1) $\updownarrow A$ is convex;

(2) A is convex iff $\updownarrow A = A$;

(3) If $A \subseteq B$. and B is convex then, $\updownarrow A \subseteq B$.

Proposition 1.4.[5] Let $(X, \leq)$ be a poset. If λ be an upper subset of X, then it is d-open.

Proposition 1.5.[5] Let $(X, \leq)$ be a poset, and let λ be a convex subset of X. and $(\downarrow \lambda)$ is d-closed, then λ so is.

Proposition 1.6.[5] Let $(X, \leq)$ be a poset, and $\forall\, j \in J$, λ_j be a family of lower (resp. upper) subset of X, then $(\bigcap_{j \in J} \lambda_j)$ and $(\bigcup_{j \in J} \lambda_j)$ are lower (resp. upper) subsets of X

Proposition 1.7.(Corollary page 45.[5]).Let $(X, \leq)$ be a poset. If λ is directed subset iff $(\downarrow \lambda)$ so is.

Proposition 1.8.[5] Let $(X, \leq)$ be a poset, and let $\lambda, \mu \subseteq X$ such that $(\downarrow \lambda) = (\downarrow \mu)$. If λ is directed subset iff μ so is.

Proposition 1.9.[5] Let $(X, \leq)$ be a poset, and let $x, y \in X$. Then:

(1) $x \leq y \Leftrightarrow x \in \downarrow y \Leftrightarrow \downarrow x \subseteq \downarrow y \Leftrightarrow y \in \uparrow x \Leftrightarrow \uparrow x \subseteq \uparrow y$;

(2) $x \in \downarrow x \Leftrightarrow x \in \uparrow x$;

(3) $\downarrow x \cap \uparrow x = \{x\}$.

Proposition 1.10.[5]. Let $(X, \leq)$ be a poset, and let If λ_j is a an upper (resp.lower) subsets of (X_j) $\forall j \in J$, then $\prod_{j \in J} (\lambda_j)$ is a an upper (resp. lower) subsets of $\prod_{j \in J} X_j$.

Proposition 1.11.[5] Let (X,$\leq$) be a poset $\{(X_j, \leq_j) : j \in J\}$ be a family of a poset, and let $\lambda_j \subseteq X_j$ $\forall j \in J$. Then

(1) $\uparrow (\prod_{j \in J} \lambda_j) = (\prod_{j \in J} \uparrow (\lambda_j))$

(2) $\downarrow (\prod_{j \in J} \lambda_j) = (\prod_{j \in J} \downarrow (\lambda_j))$

Remark 1.1. Each point of X is an upper bound of the empty set. Then $\bigvee(\phi)$ exists iff X has a least element $\perp$. In this case $\bigvee(\phi) = \perp$.

Proposition 1.12.[5] Let $(X, \leq)$ be a poset, and let $\lambda, \mu \subseteq X$. If μ is directed subset and cofinal in λ, then $\lambda$is directed subset and $\bigvee(\lambda) = \bigvee(\mu)$.

Definition 1.3. (1) For any poset X, :Let $x, y \in X$. We say x below (resp. y is way above) y (resp. x), denoted by $x \ll y$ iff $\forall\, D \subseteq X$, $s.t.$ D is directed subset of X with $\bigvee D$ exists and $y \leq \bigvee D$, $\exists\, d \in D$ $s.t.$ $x \leq d$. The family of all elements in X each of which way above (resp. way below) x is denoted and defined as follows: $\Uparrow x = \{y \in X : x \ll y\}$ (resp. $\Downarrow x = \{y \in X : y \ll x\}$ [11];

(2) Let $x \in X$. If $x \ll x$, then x is said to be isolated. The family of all isolated points above (resp. below) $x \in X$ is denoted and defined by: $\uparrow^{\circ} x = \{y \in X : y \ll y$ and $x \leq y\}$ (resp. $\downarrow_{\circ} x = \{y \in X : y \ll y$ and $y \leq x\}$ [11].

Proposition 1.13.[5] Let $(X, \leq)$ be a domain,.If X has a least element $\perp$, then $\perp \ll x$ holds for all $x \in X$.

Proposition 1.14.[5]. Let $(X, \leq)$ be a domain, and let D is directed subset of X. such that $x = \bigvee(D)$ and x is isolated. Then $x \in D$.

Definition 1.7. [9] Let $(X, \leq_1)$ and $(X, \leq_2)$ be a poset, and let $f : (X, \leq_1) \to (Y, \leq_2)$ be a function. Then

(1) f is monotone iff $f(\lambda) \leq_2 f(\mu)$, whenever $\lambda \leq_1 \mu$,

(2) f is called order preserving (monotone) iff $f(x_1) \leq_2 f(x_2)$, whenever $x_1 \leq_1 x_2$.

Lemma 1.1. [9] Let X be a set and let $f : X \to X$ be a function. Then f is idempotent iff $f(X) = fix(f)$, where $fix(f) = \{x \in X : f(x) = x\}$.

2- Operators and Unitary operators on the power set of TR

Definition 2.1. A TR is a pair $(X, \leq)$ where $X \neq \phi$ and $\leq$ is a transitive binary relations set.

Example 2.1. Poset, equivalence set, PRS, Continuous information system and abstract basis are TR.

The following diagram illustrates some relations between some types of TR

$$\begin{array}{ccc}
\textit{Abstract base} & \Rightarrow & \textit{Continuous information system} \\
 & & \Downarrow \\
 & & TR \\
 & & \Uparrow \\
\textit{Partially ordered set} & \Rightarrow & PRS \\
 & & \Uparrow \\
 & & \textit{Equvalence set}
\end{array}$$

Remark 2.1. Abstract basis $\Rightarrow$ Continuous information system**,** the converse is not true, so give the following example

Example 2.2. Let $X = \{a, b, c\}$ and let $\leq = \{(a, a), (b, b), (a, c), (b, c)\}$, because

$$\begin{array}{l}
a \leq a \Rightarrow \exists\, a \in X \ s.t.,\ a \leq a \leq a; \\
b \leq b \Rightarrow \exists\, b \in X \ s.t.,\ b \leq b \leq b,\ ; \\
a \leq c \Rightarrow \exists\, a \in X \ s.t.,\ a \leq a \leq c, \text{ and} \\
b \leq c \Rightarrow \exists\, b \in X \ s.t., b \leq b \leq c.
\end{array}$$

Then $\leq$ transitive and interpolative. Now, Let $\lambda = \{a, b\}$. Then $a \leq c$ and $b \leq c$, but $\forall\, x \in X,\ x \nleq c$. Hence $(X, \leq)$ is not abstract basis.

Definition 2.2. Let $\{(X_j, \leq_j) : j \in J\}$ be a family of TR. The product relation $\leq$ on $\prod_{j \in J} X_j$ of $\{\leq_j : j \in J\}$ is defined as follows: $(x_j)_{j \in J} \leq (y_j)_{j \in J}$ iff $x_j \leq y_j \quad \forall\, j \in J$.

Definition 2.3. Let $(X, \leq)$ be a TR. the dual of $\leq$ is denoted and defined as follows: $\leq^d = \{(x, y) : (y, x) \in \leq\}$.

Theorem 2.1. $\{(X_j, \leq_j) : j \in J\}$ be a family of TR, $(\prod_{j \in J} X_j, \leq)$ is TR, where $\leq$ is the product of $\{\leq_j : j \in J\}$.

Proof Let $(x_j)_{j \in J} \leq (y_j)_{j \in J}$, $(y_j)_{j \in J} \leq (z_j)_{j \in J}$. Then $\forall\, j \in J$, $x_j \leq_j y_j$ and $y_j \leq_j z_j$ so that $\forall\, j \in J$, $x_j \leq_j z_j$. Thus $(x_j)_{j \in J} \leq (z_j)_{j \in J}$. Then $\leq$ is TR. Hence $(\prod_{j \in J} X_j, \leq)$ is TR.

Definition 2.4. Let $(X, \leq)$ be a TR. Then:

(1) The lower (resp. upper) bounded subset in X of λ is denoted by $lb(\lambda)$ (resp. $ub(\lambda)$) and defined as follows: $lb(\lambda) = \{x \in X: \forall\, y \in \lambda,\ \ x \leq y)$ (resp. $ub(\lambda) = \{x \in X: \forall\, y \in \lambda,\ \ y \leq x)$.

(2) The minimal (resp. maximal) subset in X of λ is denoted by $\min(\lambda)$ (resp. $, \max(\lambda)$) and defined as follows: $\min(\lambda) = \{x \in \lambda : \text{if } \ y \in \lambda, \text{ and } \ y \leq x, \text{ then } x = y)$ (resp. $\max(\lambda) = \{x \in \lambda : if \ \ y \in \lambda, \text{ and } \ x \leq y, \text{then } x = y)$.

(3) The subset of least (resp. largest) elements of a subset λ is denoted by $le(\lambda)$ (resp. $la(\lambda)$) and defined as follows:

$le(\lambda) = \{x \in \lambda: \forall\, y \in \lambda,\ \ x \leq y)$ (resp. $la(\lambda) = \{x \in \lambda: \forall\, y \in \lambda,\ \ y \leq x)$. Each element in $le(\lambda)$ (resp. $la(\lambda)$) is aclled a least (resp. largest) element of λ

(4) The infimum (resp. supremum) subset in X of λ is denoted by $\bigwedge(\lambda)$ (resp. $\bigvee(\lambda)$) and defined as follows:

$\bigwedge(\lambda) = la(lb(\lambda))$ (resp. $\bigvee(\lambda) = le(ub(\lambda))$. Each element in $\bigwedge(\lambda)$ (resp. $\bigvee(\lambda)$) is aclled an infimum (resp. supremum) element of λ

(5) The lower (resp. upper) closure in X of λ is denoted by $\downarrow \lambda$ (resp. $\uparrow \lambda$) and defined as follows: $\downarrow \lambda = \{x \in X : \exists \ \ y \in \lambda \ \ s.t.\ x \leq y \ \}$ (resp. $\uparrow \lambda = \{x \in X : \exists \ \ y \in \lambda \ \ s.t.\ y \leq x \ \}$)

(6) An upper (resp. a lower) cone of X iff $\exists\, x \in \lambda \ \ s.t.\ \lambda = \uparrow x$ (resp. $\lambda = \downarrow x$)

(7) The convex hull A is denoted by $\updownarrow A$ and defined as follows: $\updownarrow A = \downarrow A \cap \uparrow A$

(8) Let $A, B \subseteq X$. B is called cofinal in A iff $B \subseteq A \subseteq \downarrow (B)$.

(9) Let $(X, \leq)$ be a TR, then $\leq$ is called a domain iff for every directed subset λ of X, $\bigvee(\lambda)$ exists.

Remark 2.2 In [13], If $(X, \leq)$ is a posets**,** then concepts .$le(\lambda)$, $la(\lambda)$, $\bigwedge(\lambda)$, $\bigvee(\lambda)$ is singlton. if it exists.but in TR need not be singlton.so give the following example,

Example 2.3. Let $X = \{a, b, c, d, e\}$ and let $\leq = \{(a, a),\ (b, b),\ (a, b),\ (b, a)\}$. Then $(X, \leq)$ be a TR, and $\bigvee(\{a, b\}) = \bigwedge(\{a, b\}) = le(\{a, b\}) = la(\{a, b\}) = \{a, b\}$.

Theorem 2.2 Let $(X, \leq)$ be a TR. Then the following statments are satisfied;

(1) x is an upper (resp. lower) bounded of a subset λ of a TR $(X, \leq)$ iff x is an lower (resp. upper) bounded of $(X, \leq^d)$.

(2) x is least (resp. largest) elements of a subset λ of a TR $(X, \leq)$ iff x is largest (resp. least) elements of a subset λ of a TR $(X, \leq^d)$

(3) $x \in \bigvee(\lambda)$ (resp., $\bigwedge(\lambda)$) of a TR $(X, \leq)$ iff x $x \in \bigwedge(\lambda)$(resp., $\bigvee(\lambda)$) of a TR $(X, \leq^d)$

(4) $\downarrow \lambda$ (resp. $\uparrow \lambda$) of a TR $(X, \leq)$ equals $\uparrow \lambda$ (resp. $\downarrow \lambda$) of a TR $(X, \leq^d)$

(5) minimal (λ) (resp. maximal(λ)) of a TR $(X, \leq)$ equals maximal (λ) (resp. minimal (λ)) of a TR $(X, \leq^d)$.

Remark 2.3 Let $(X, \leq)$ be a PRS, then $\forall\, \lambda \subseteq X$, $\lambda \subseteq \downarrow \lambda$(resp., $\lambda \subseteq \uparrow \lambda$), but if $(X, \leq)$ be a TR, then λ need not be a subset of $\downarrow \lambda$(resp., $\uparrow \lambda$), so give the following example.

Example 2.4. Let $X = \{x, y, z\}$ and let $\leq = \{(x, x),\ (x, y)\}$. Take $\lambda = X$. Then $X \subsetneq \{x\} = \downarrow X$ (resp., $X \subsetneq \{x, y\} = \uparrow X$).

Remark 2.4 Let $(X, \leq)$ be a PRS, then $\forall\, \lambda \subseteq X$, $\lambda \subseteq \downarrow \lambda \subseteq \downarrow (\downarrow \lambda)$ (resp., $\uparrow \lambda \subseteq \uparrow (\uparrow \lambda)$), but if $(X, \leq)$ be a TR, then λ need not be a subset of $\downarrow (\downarrow \lambda)$(resp., $\uparrow (\uparrow \lambda)$), so give the following example.

Example 2.5. Let $X = \{x, y, z\}$ and let $\leq = \{(x, x), (x, y)\}$. Take $\lambda = X$. Then $\uparrow (\uparrow \lambda) = \{x\} \supsetneqq \uparrow \lambda$. (resp., $\downarrow (\downarrow \lambda) = \{x\} \supsetneqq \downarrow \lambda = \{x, y\}$.).

Remark 2.5 Let $(X, \leq)$ be a PRS, then $\forall \lambda \subseteq X$, $\lambda \subseteq \updownarrow \lambda$, but if $(X, \leq)$ be a TR, then λ need not be a subset of $\updownarrow \lambda$, so give the following example.

Example 2.6. Let $X = \{x, y, z\}$ and let $\leq = \{(x, x), (x, y)\}$. Take $\lambda = X$. Then $\updownarrow \lambda = (\uparrow \lambda) \cap (\downarrow \lambda) = \{x\} \cap \{x, y\} = \{x\} \supsetneqq \lambda$.

Remark 2.6 Let $(X, \leq)$ be a PRS, then $\forall \lambda \subseteq X$, $\updownarrow \lambda \subseteq \updownarrow (\updownarrow \lambda)$, but if $(X, \leq)$ be a TR, then $\updownarrow \lambda$ need not be a subset of $\updownarrow (\updownarrow \lambda)$, so give the following example.

Example 2.7. Let $X = \{x, y, z, l\}$ and let $\leq = \{(x, y), (y, z), (x, z)\}$. Take $\lambda = \{x, y, z\}$ of X. Then $\downarrow \lambda = \{x, y\}$ and $\uparrow \lambda = \{y, z\}$, then $\updownarrow \lambda = (\uparrow \lambda) \cap (\downarrow \lambda) = \{y\}$. Thus $\updownarrow (\updownarrow \lambda) = \updownarrow (\{y\}) = (\uparrow \{y\}) \cap (\downarrow \{y\}) = \{x\} \cap \{z\} = \phi$. Hence $\updownarrow \lambda \subsetneqq \updownarrow (\updownarrow \lambda)$.

Theorem 2.3 Let $(X, \leq)$ be a TR, and let $\{\lambda_j : j \in J\}$ be a family of subset of X. Then:

$$\begin{array}{llcl}
(1) & \downarrow (\bigcup\limits_{j \in J} \lambda_j) & = & \bigcup\limits_{j \in J} \downarrow (\lambda_j) \\
(2) & \uparrow (\bigcup\limits_{j \in J} \lambda_j) & = & \bigcup\limits_{j \in J} \uparrow (\lambda_j) \\
(3) & \downarrow (\bigcap\limits_{j \in J} \lambda_j) & \subseteq & \bigcap\limits_{j \in J} \downarrow (\lambda_j) \\
(4) & \uparrow (\bigcap\limits_{j \in J} \lambda_j) & \subseteq & \bigcap\limits_{j \in J} \uparrow (\lambda_j)
\end{array}$$

Proof (1) Let $x \in \downarrow (\bigcup\limits_{j \in J} \lambda_j) \Leftrightarrow \exists\ y \in (\bigcup\limits_{j \in J} \lambda_j)$ such that $x \leq y \Leftrightarrow \exists\ \lambda_j$ such that $j \in J$, $y \in \lambda_j$, $x \leq y \Leftrightarrow x \in \downarrow (\lambda_j)$ for some $j \in J \Leftrightarrow x \in \bigcup\limits_{b \in F_2} \downarrow (\lambda_j)$.

(2) $x \in \uparrow (\bigcup\limits_{j \in J} \lambda_j) \Leftrightarrow \exists\ y \in (\bigcup\limits_{j \in J} \lambda_j)$ such that $y \leq x \Leftrightarrow \exists\ \lambda_j$ such that $j \in J$, $y \in \lambda_j$, $y \leq x \Leftrightarrow x \in \uparrow (\lambda_j)$ for some $j \in J \Leftrightarrow x \in \bigcup\limits_{j \in J} \uparrow (\lambda_j)$.

(3) $x \in \downarrow (\bigcap\limits_{j \in J} \lambda_j) \Rightarrow \exists\ y \in (\bigcap\limits_{j \in J} \lambda_j)$ such that $x \leq y \Rightarrow \forall\ j \in J$, $\exists\, y \in \lambda_j$, such that $x \leq y \Rightarrow x \in \downarrow (\lambda_j)\ \forall\ j \in J \Longrightarrow x \in \bigcap\limits_{j \in J} \downarrow (\lambda_j)$; and

(4) $x \in \uparrow (\bigcap\limits_{j \in J} \lambda_j) \Rightarrow \exists\ y \in (\bigcap\limits_{j \in J} \lambda_j)$ such that $y \leq x \Rightarrow \forall\ j \in J$, $\exists\, y \in \lambda_j$, such that $y \leq x \Rightarrow x \in \uparrow (\lambda_j)\ \forall\ j \in J \Longrightarrow x \in \bigcap\limits_{j \in J} \uparrow (\lambda_j)$.

The following theorem is a generalization of the corresponding result in Proposition 1.1.

Theorem 2.4 Let $(X, \leq)$ be a TR, and let $\lambda \subseteq X$ such that $\bigvee(\lambda)$ exists. Then $\bigcap\limits_{x \in \lambda} \uparrow x = \uparrow \bigvee(\lambda)$.

Proof Let $y \in \bigcap\limits_{x \in \lambda} \uparrow x \Leftrightarrow \forall\ x \in \lambda$, $y \in \uparrow x \Leftrightarrow y \geq z\ \forall\ z \in \bigvee(\lambda) \Leftrightarrow y \in \uparrow \bigvee(\lambda)$.

The following theorem is a generalization of the corresponding result in Proposition 1.2.

Theorem 2.5 Let $(X, \leq)$ be a TR, and let $\lambda, \mu \subseteq X$ and let $\lambda \subseteq \mu$. If $\bigvee(\lambda) \neq \phi$ and $\bigvee(\mu) \neq \phi$. Then $\forall\alpha \bigvee(\lambda),\ \forall\beta \bigvee(\mu),\ \alpha \leq \beta$.

Proof If $\forall\ x \in ub(\lambda),\ \forall\ y \in ub(\mu),\ y \geq x$. If $\alpha \in \bigvee(\lambda) = le(ub(\lambda))$ and $\beta \in \bigvee(\mu) = le(ub(\mu)),\ \alpha \leq x\ \ \forall\ x \in ub(\lambda)$, so that $\alpha \leq y\ \ \forall\ y \in ub(\mu)$. so that $\alpha \leq \beta$.

Definition 2.4. Let $(X, \leq)$ be a TR, and let $\lambda, \mu \subseteq X$. λ is called:

(1) An lower (resp. upper) subset of X iff $\downarrow \lambda \subseteq \lambda$ (resp. $\uparrow \lambda \subseteq \lambda$)). The family of all lower (resp. upper) subset in λ will denoted by $LS(\lambda)$ (resp., $.US(\lambda)$),

(2) A directed (resp. filtered) subset of X iff $\lambda \neq \phi$ and every $x \neq y$ in $\lambda,\ \exists\ z \in \lambda \cap lb(\{x, y\})$;

(3) The ideal (resp. filter) subset of X iff λ is directed lower subset (resp. filtered upper subset) in X;

(4) A principal ideal (resp. principal filter) of X iff $\max(\lambda) \neq \phi$ (resp. $\min(\lambda) \neq \phi$);

(5) A bounded subset from above (resp. below) of X iff iff $ub(\lambda) \neq \phi$ (resp. $lb(\lambda) \neq \phi$);

(6) A cofinal in μ iff $\lambda \subseteq \mu \subseteq \downarrow (\lambda)$;

(7) A convex subset of X iff $\forall\ x, y \in \lambda$ such that $x \leq z$ and $z \leq y,\ z \in \lambda$;

(8) An upper (resp. lower) cone of X iff $\exists\ x \in \lambda$ such that $\lambda = \uparrow x$ (resp. $\lambda = \downarrow x$);

(9) A d-closed subset iff $\forall$ directed subset D of X such that $D \subseteq \lambda$, we have $\bigvee(D) \subseteq \lambda$;

(10) A d-open subset of X iff λ^c is a d-closed subset;

(11) A Scott-closed subset of X iff λ is d-closed lower subset ;

(12) A Scott-open subset of X iff λ is d-closed upper subset of ;

The following theorem is a generalization of the corresponding result in Proposition 1.3.

Theorem 2.6 Let $(X, \leq)$ be a TR, and let λ be a convex subset of X, $\updownarrow \lambda \subseteq \lambda$.

Proof $y \in \updownarrow \lambda = (\uparrow \lambda) \cap (\downarrow \lambda)$ so that $y \in \uparrow \lambda$ and $y \in \downarrow \lambda$ so that $\exists\ x, z \in \lambda$ with $x \leq y$ and $y \leq z$. Since λ convex $y \in \lambda$

Remark 2.7 Let $(X, \leq)$ be a PRS, then $\forall\ \lambda \subseteq X,\ \lambda \subseteq \uparrow \lambda$ and $\lambda \subseteq \downarrow \lambda$ so that $\lambda \subseteq (\updownarrow \lambda)$, but if $(X, \leq)$ be a TR, $\lambda \subseteq \uparrow \lambda$ and $\lambda \subseteq \downarrow \lambda$ so that $\lambda \subseteq (\updownarrow \lambda)$ need not be true, so give the following example.

Example 2.8. Let $X = \{x, y, z, l, m\}$ Now $\leq = \{(x, y),\ (y, l), (x, l).(l, l)\}$. is TR. Take $\lambda = \{x, y, z\}$ of X. Then $\downarrow \lambda = \{x\}$ so that $\lambda \subsetneq \downarrow \lambda$ and $\uparrow \lambda = \{y, l\}$, so that $\lambda \subsetneq \uparrow \lambda$

and $\updownarrow \lambda = (\uparrow \lambda) \cap (\downarrow \lambda) = \phi$ so that $\lambda \subsetneq \updownarrow \lambda$.

The following theorem is a generalization of the corresponding result in Proposition 1.3(1)

.

Theorem 2.7 Let $(X, \leq)$ be a TR, Then $\updownarrow \lambda$ is convex.

Proof Let $x,\ y \in \updownarrow \lambda = (\uparrow \lambda) \cap (\downarrow \lambda)$ and let $x \leq z \leq y$. Then $\exists\ x_1 \in \lambda$ such that $y \leq y_1$ and $\exists\ x_1 \in \lambda$ such that $x_1 \leq x$ so $x_1 \leq x \leq z \leq y \leq y_1$. Because $(X, \leq)$ is a TR, Then $z \in \updownarrow \lambda$ so that $\updownarrow \lambda$ is convex.

Theorem 2.8 Let $(X, \leq)$ be a TR, and let $\lambda \subseteq X$, Then:

λ is a d-open subset of X iff $\forall$ directed subset D of X such that $\bigvee(D) \cap \lambda \neq \phi$, we have $D \cap \lambda \neq \phi$

Proof Lgically: for $\lambda \subseteq X$, Let E is a family of all directed subset of of X, and $P \equiv' D \in E$, $Q \equiv' D \subseteq \lambda'$, and

$R \equiv' \bigvee(D) \subseteq \lambda'$. Now λ is a d-closed $\equiv P \wedge (Q \Rightarrow R) \equiv P \wedge (\neg R \Rightarrow \neg Q)$. So λ^c is a d-open iff $\forall D \in E$ such that $\bigvee(D) \cap \lambda^c \neq \phi$, we have $D \cap \lambda^c \neq \phi$, where $\neg R$ and $\neg Q$ are the negations of R and Q respectively. We can simply say that, μ is a d-open iff $\forall D \in E$ such that $\bigvee(D) \cap \mu \neq \phi$, and $D \cap \mu \neq \phi$.

The following theorem is a generalization of the corresponding result in Proposition 1.4.

Theorem 2.9 Let $(X, \leq)$ be a TR, and let λ be an upper subset of X, then it is d-open.

Proof Let λ be an upper subset of X and let D be a directed subset of λ. Now, $m \in \bigvee(D)$ implies that $\exists$ $k \in D \subseteq \lambda$ such that $k \leq m$. Then $m \in (\uparrow \lambda) \subseteq \lambda$, so that $m \in \lambda$. Thus $\bigvee(D) \subseteq \lambda$.

The following theorem is a generalization of the corresponding result in Proposition 1.5.

Theorem 2.10 Let $(X, \leq)$ be a PRS, and let λ be a convex subset of X. If $(\downarrow \lambda)$ is d-closed, then λ so is.

Proof Let D be a directed subset of λ. Now. Because $\leq$ is reflexive, then $D \in \lambda \subseteq (\downarrow \lambda)$. Since $(\downarrow \lambda)$ is d-closed, then $\bigvee(D) \subseteq (\downarrow \lambda)$. So $\forall\, m \in \bigvee(D)$ $\exists$ $x \in \lambda$ such that $m \leq x$. Since $D \neq \phi$, then $\exists$ $d \in D$ such that $d \leq m$. Now, $d \leq m \leq x$. Since λ is convex , then $m \in \lambda$ so that $\bigvee(D) \subseteq \lambda$. Hence λ is d-closed.

Theorem 2.11 Let $(X, \leq)$ be a TR, and let λ be a subset of X, then

(1) $(\downarrow \lambda) \subseteq \lambda$ iff $(\uparrow \lambda^c) \subseteq \lambda^c$;

(2) $(\uparrow \lambda) \subseteq \lambda$ iff $(\downarrow \lambda^c) \subseteq \lambda^c$.

Proof (1) $\Rightarrow$: Let $x \in (\uparrow \lambda^c)$ implies that $\exists\, y \in \lambda^c$ such that $y \leq x$ implies that $\exists\, y \notin \lambda$ such that $y \leq x$ implies that $\exists\, y \notin (\downarrow \lambda)$ such that $y \leq x$ implies that $\forall\, z \in \lambda$, $y \nleq z$ and $y \leq x$ implies that $x \notin \lambda$ implies that $x \in \lambda^c$;

$\Leftarrow$: Let $x \in (\downarrow \lambda)$ implies that $\exists\, y \in \lambda$ such that $x \leq y$ implies that $\exists\, y \notin \lambda^c$ such that $x \leq y$ implies that $\exists\, y \notin (\uparrow \lambda^c)$ such that $x \leq y$ implies that $\forall\, z \in \lambda^c$, $z \nleq y$ and $x \leq y$ implies that $x \notin \lambda^c$ implies that $x \in \lambda$.

(2) Not need to prove this case,because this statement is equivalent to (1).

Corollary 2.1. Let $(X, \leq)$ be a TR, and let λ be a subset of X, then λ is lower (resp. an upper) subset of X iff λ^c is upper (resp. an lower).

The following theorem is a generalization of the corresponding result in Proposition 1.6.

Theorem 2.12 Let $(X, \leq)$ be a TR, and let λ_j be a family of lower (resp. upper) subset of X, then $(\bigcap_{j \in J} \lambda_j)$ and $(\bigcup_{j \in J} \lambda_j)$ are lower (resp. upper) subsets of X

Proof. (1) From Theorem 2.3 (3) we have $\downarrow (\bigcap_{j \in J} \lambda_j) \subseteq \bigcap_{j \in J} \downarrow (\lambda_j)$ so that $\downarrow (\bigcap_{j \in J} \lambda_j) \subseteq \bigcap_{j \in J} (\lambda_j)$. Hence $\bigcap_{j \in J} (\lambda_j)$ is lower subset of X; And

From Theorem 2.3 (4) we have $\uparrow (\bigcap_{j \in J} \lambda_j) \subseteq \bigcap_{j \in J} \uparrow (\lambda_j)$ so that $\uparrow (\bigcap_{j \in J} \lambda_j) \subseteq \bigcap_{j \in J} (\lambda_j)$. Hence $\bigcap_{j \in J} (\lambda_j)$ is upper subset of X.

(2) Let $x \in \downarrow (\bigcup_{j \in J} \lambda_j)$ implies that $\exists\, j \in J$ such that $y \leq \lambda_j$ and $x \leq y$ implies that $x \in \downarrow (\lambda_j)$ implies that $x \in (\lambda_j)$ implies that $x \in (\bigcup_{j \in J} \lambda_j)$. Hence $(\bigcup_{j \in J} \lambda_j)$ is lower subset of X; And

Let $x \in \uparrow (\bigcup_{j \in J} \lambda_j)$ implies that $\exists\, j \in J$ such that $y \leq \lambda_j$ and $y \leq x$ implies that $x \in \uparrow (\lambda_j)$ implies that $x \in (\lambda_j)$ implies that $x \in (\bigcup_{j \in J} \lambda_j)$. Hence $(\bigcup_{j \in J} \lambda_j)$ is upper subset of X.

Lemma 2.1 Let $(X, \leq)$ be a PRS. If λ is directed subset, then $(\downarrow \lambda)$ so is.

Proof. Let $x_1,\ x_2 \in (\downarrow \lambda)$ such that $x_1 \neq x_2$. Then $\exists\ y_1, y_2 \in \lambda$ such that $x_1 \leq y_1,\ x_2 \leq y_2$. So $\exists\ x \in ub(\{x_1,\ x_2\})$. Since $\lambda \subseteq (\downarrow \lambda)$, then $x \in (\downarrow \lambda)$. Thus $x \in (\downarrow \lambda) \cap ub(\{x_1,\ x_2\})$. Hence $(\downarrow \lambda)$ is directed subset.

The following Corollary is a generalization of the corresponding result in Proposition 1.7.

Corollary 2.2. Let $(X, \leq)$ be a PRS. If λ is directed subset iff $(\downarrow \lambda)$ so is.

The following theorem is a generalization of the corresponding result in Proposition 1.8.

Theorem 2.13 Let $(X, \leq)$ be a PRS, and let $\lambda, \mu \subseteq X$ such that $(\downarrow \lambda) = (\downarrow \mu)$. If λ is directed subset, then μ so is

Proof. From Lemma 2.1, since λ is directed subset, then $(\downarrow \lambda)$ is directed subset, implies that $(\downarrow \mu)$ is directed subset. Now prove that μ is directed subset, let $x_1,\ x_2 \in \mu$ such that $x_1 \neq x_2$, then $x_1,\ x_2 \in (\downarrow \mu)$. So $\exists\ x \in (\downarrow \mu)$ such that $x \in (\downarrow \mu) \cap ub(\{x_1,\ x_2\})$. Then $\exists\ y \in \mu$ such that $x \leq y$. Now $x_1 \leq x \leq y$ and $x_2 \leq x \leq y$ so that $y \in \mu \cap ub(\{x_1,\ x_2\})$. Hence μ is directed subset.

Corollary 2.3. Let $(X, \leq)$ be a PRS, and let $\lambda, \mu \subseteq X$ such that $(\downarrow \lambda) = (\downarrow \mu)$. If λ is directed subset iff μ so is

The following theorem is a generalization of the corresponding result in Proposition 1.9.

Theorem 2.14 Let $(X, \leq)$ be a PRS, and let $x, y \in X$. Then:

(1) $x \leq y$ iff $x \in \downarrow y$ iff $\downarrow x \subseteq \downarrow y$ iff $y \in \uparrow x$ iff $\uparrow x \subseteq \uparrow y$.

(2) $x \in \downarrow x \Leftrightarrow x \in \uparrow x$;

(3) $\downarrow x \cap \uparrow x = \{x\}$.

Proof. Since $x \leq y \Leftrightarrow x \in \downarrow y \Leftrightarrow \downarrow x \subseteq \downarrow y$; and since $x \leq y \Leftrightarrow y \in \uparrow x \Leftrightarrow \uparrow x \subseteq \uparrow y$. The results obtained.

The following theorem is a generalization of the corresponding result in Proposition 1.10 and Proposition 1.11.

Theorem 2.15. Let $(X, \leq)$ be a TR, and let $\{(X_j, \leq_j) : j \in J\}$ be a family of TR, and let $\lambda_j \subseteq X_j$ $\forall j \in J$. Then:

(1) If $\lambda_j \subseteq US(X_j)$ (resp. $LS(X_j)$) $\forall j \in J$, then $\prod_{j \in J} (\lambda_j)$ is a an upper (resp. lower) subsets of $(\prod_{j \in J} X_j, \leq)$.

(2) $\uparrow (\prod_{j \in J} \lambda_j) = (\prod_{j \in J} \uparrow (\lambda_j))$

(3) $\downarrow (\prod_{j \in J} \lambda_j) = (\prod_{j \in J} \downarrow (\lambda_j))$

(3) If $\lambda_j \subseteq US(X_j)$ (resp. $LS(X_j)$) $\forall j \in J$, then $\prod_{j \in J} (\lambda_j)$ is a an upper (resp. lower) subsets of $(\prod_{j \in J} X_j, \leq)$.

Proof (1) Let $\lambda_j \subseteq US(X_j)$(resp. $LS(X_j)$) $\forall j \in J$. Hence $\uparrow (\lambda_j) \subseteq (\lambda_j)$(resp.$\downarrow (\lambda_j) \subseteq (\lambda_j)$), then $\uparrow(\prod_{j\in J}\lambda_j)\subseteq(\prod_{j\in J}(\lambda_j))$(resp.$\downarrow(\prod_{j\in J}\lambda_j)\subseteq(\prod_{j\in J}(\lambda_j))$). Thus $\prod_{j\in J}(\lambda_j)$ is a an upper (resp. lower) subsets of $\prod_{j\in J}X_j$.

(2) and (3) Obvious;

The following theorem is a generalization of the corresponding result in Proposition 1.3. and Remark 1.1.

Theorem 2.16 Let $(X,\leq)$ be a TR, and let $\lambda,\mu\subseteq X$. If $\lambda\subseteq\mu$ and μ is convex. Then
(1) $\updownarrow(\lambda)\subseteq\mu$;
(2) $\bigvee(\phi)=le(X)$.

Proof (1) Let $x\in\updownarrow(\lambda)$. Since $x\in(\downarrow\lambda)$, So $\exists\, y\in\lambda$ such that $x\leq y$. And since $x\in(\uparrow\lambda)$, So $\exists\, z\in\lambda$ such that $z\leq x$. since $\lambda\subseteq\mu$ implies that $z,y\in\mu$. And since μ is convex, then $x\in\mu$. Hence $\updownarrow(\lambda)\subseteq\mu$.
(2) Obvious.

The following theorem is a generalization of the corresponding result in Proposition 1.12.

Theorem 2.17 Let $(X,\leq)$ be a TR. and let $\lambda,\mu\subseteq X$. If μ is directed subset and cofinal in λ, then $\lambda$is directed subset and $\bigvee(\lambda)=\bigvee(\mu)$.

Proof Since $\mu\subseteq\lambda$, then $\lambda\neq\phi$. Let $x,y\in\lambda$ such that $x\neq y$. Then $\exists\, b_1,b_2\in\mu$ such that x such that $x\leq b_1$, $y\leq b_2$ and $z\in\lambda\cap ub(\{b_1,b_2\})$. Hence μ is directed subset. Also, since $\mu\subseteq\lambda$, then $ub(\{\lambda\}\subseteq ub(\{\mu\}$ (1) and let $y\notin ub(\{\lambda\}$ implies that $\exists\, a\in\lambda$ such that $a\nleq y$ implies that $\exists\, a\in(\downarrow\mu)$ such that $a\nleq y$ implies that $\exists\, b\in\mu$ such that $a\leq b$ and $a\nleq y$ implies that $b\in\mu$ such that $b\nleq y$ implies that $y\notin ub(\{\mu\}$ (2). From (1) and (2) implies that $ub(\{\lambda\}=ub(\{\mu\}$. Hence $\bigvee(\lambda)=\bigvee(\mu)$.

Theorem 2.18 Let $(X,\leq)$ be a TR. and let $\lambda,\mu\subseteq X$.
(1) If $x\in ub(\lambda)\cap\lambda$ and If $x\leq x$. Then $x\in\bigvee(\lambda)$
(2) If $ub(\lambda)=ub(\lambda)$. Then $\bigvee(\lambda)=\bigvee(\mu)$.

Proof Obvious.

3- The way below relation and some types of functions on TR

Definition 3.1 Let $(X,\leq)$ be a TR,

(1) Let $x,y\in X$. We say x way below (resp. y is way above) y (resp. x), denoted by $x\ll y$ iff $\forall\, D\subseteq X$, $s.t.$ D is directed subset of X, if $y\leq\downarrow(\bigvee D)$, $\exists\, d\in D$ $s.t.$ $x\leq d$.

The family of all elements in X each of which way above (resp. way below) x is denoted and defined as follows: $\Uparrow x=\{y\in X: x\ll y\}$ (resp. $\Downarrow x=\{y\in X: y\ll x\}$.

(2) Let $x\in X$. If $x\ll x$, then x is said to be isolated. The family of all isolated points above (resp. below) $x\in X$ is denoted $K(X)$ and defined by: $\uparrow^{\circ} x=\{y\in X: y\ll y$ and $x\leq y\}$ (resp. $\downarrow_{\circ} x=\{y\in X: y\ll y$ and $y\leq x\}$.

Theorem 3.1. Let $(X,\leq)$ be a TR. let $x,y,z\in X$. Then :

(1) If $x\leq y$ and $y\ll z$, then $x\ll z$;
(2) If $x\ll y$ and $y\leq z$, then $x\ll z$;

(3) If $\bigvee(\{y\}) \neq \phi$. and $x \ll y$, then $x \leq y$;

(4) If $\bigvee(\{y\}) \neq \phi$. or $\bigvee(\{z\}) \neq \phi$, $x \ll y$ and $y \ll z$, then $x \ll z$.

Proof (1) Let D be a directed subset of X s.t. $z \in \downarrow \bigvee(D)$. Then $\exists\, d \in D$ $s.t.\ y \leq d$.Then $x \leq d$ and hence $x \ll z$.

(2) Let D be a directed subset of X s.t. $z \in \downarrow \bigvee(D)$. Then $\exists\, k \in \bigvee(D)$ $s.t.\ z \leq k$.Thus $y \leq k$ and so $y \in \downarrow \bigvee(D)$. Therefore $\exists\, l \in D$ $s.t.\ x \leq l$. hence $x \ll z$.

(3) Let $D = \{y\}$ and assume that $x \ll y$. Then $\exists\, d \in D$ $s.t.\ x \leq d$ but $y = d$. Thus $x \leq y$;

(4) The proof follow directly From (1) and (3) above.

Definition 3.2 A TR $(X, \leq)$ is called a domain iff for every directed subset λ of X, $\bigvee(\lambda) \neq \phi$.

Lemma 3.1 Let $(X, \leq)$ be a TR, let $\Downarrow x$ be a directed subset of $\forall\, x \in X$. Then $\forall\, z \in X$, $D = \bigcup\{\Downarrow a : a \in \Downarrow z\}$ is called directed subset.

Proof Let $\lambda, \mu \in D$ s.t., $\lambda \neq \mu$. Then $a_1,\ a_2 \in \Downarrow z \in ub(\{\lambda, \mu\}) \cap \Downarrow a \subseteq ub(\{\lambda, \mu\}) \cap D$. Thus D is called directed subset.

Lemma 2.2. Let $(X, \leq)$ be a TR, let $\forall\, x \in X$. Then $\forall\, x \in X$, $ub(\bigcup\{\Downarrow a : a \in \Downarrow x) = ub(\bigcup\{\bigvee(\Downarrow a) : a \in \Downarrow x)$. Thus $\bigvee(\bigcup\{(\Downarrow a) : a \in \Downarrow x) = \bigvee(\bigcup\{\bigvee(\Downarrow a) : a \in \Downarrow x)$.

s.t., $\lambda \in \Downarrow a_1$ and $\mu \in \Downarrow a_2$. If $a_1 = a_2$, the result hlolds. the otherwise let $a \in ub(\{a_1,\ a_2\}) \cap \Downarrow z$ so that Theorem 3.1 (2). $\lambda, \mu \in \Downarrow a$. Since $\Downarrow a$ is called directed subset, then $\exists \rho$

Proof.

$$\begin{array}{ccc}
\lambda \in ub(\bigcup\{(\Downarrow a) : a \in \Downarrow x) & \Leftrightarrow & \forall \mu \in \bigcup\{(\Downarrow a) : a \in \Downarrow x\}, \\
\lambda \geq \mu & \Leftrightarrow & \forall \mu \in (\Downarrow a),\ a \in (\Downarrow x), \\
\lambda \geq \mu & \Leftrightarrow & \forall \rho \in \bigvee(\Downarrow a),\ a \in (\Downarrow x), \\
\lambda \geq \rho & \Leftrightarrow & \lambda \in ub(\bigcup \bigvee(\Downarrow a) : a \in \Downarrow x).
\end{array}$$

The following theorem is a generalization of the corresponding result in Proposition 1.13.

Theorem 3.2. Let $(X, \leq)$ be a TR. let $le(X) \neq \phi$. Then $\forall \perp \in le(X)$, $\forall\, x \in X$, $\perp \ll x$.

Proof Let $x \in \downarrow \bigvee(D)$, where D is directed subset of X. $\exists\, d \in D$ $s.t.\ \perp \leq d$ for some $\perp \in le(X)$. Hence $\perp \ll x$.

The following theorem is a generalization of the corresponding result in Proposition 1.14.

Theorem 3.3. Let $(X, \leq)$ be a TR. such that $\leq$ is antisymmetric relation. If D is directed subset of X and $x \in \bigvee(D) \cap K(X)$, then $x \in D$.

Proof Let $x \in \downarrow \bigvee(D)$, where D is directed subset of X. $\exists\, d \in D$ such that. $x \leq d \leq x$. $\leq\leq$ Hence$x \in D$.

Theorem 3.4. Let $(X, \leq)$ be a TR, and let $\{(X_j, \leq_j) : j \in J\}$ be a family of TR, If $\forall\, j \in J.$, X_j is a domain, then $\prod\limits_{j \in J}(X_j)$ is a domain.

Proof Let λ be directed subset of $\prod\limits_{j \in J}(X_j) = \{f : J \to \prod\limits_{j \in J}(X_j)$ such that $f(j) \in X_j\ \forall\, j \in J\ \}$. Then one can deduce that $\forall\, j \in J$, $\lambda_j = \prod\limits_{j \in J}(\lambda) = \{f(j) : f \in \lambda\}$ is a directed subset of X_j. So, $\bigvee(\lambda_j) \neq \phi$. Let $k_j \in \bigvee(\lambda_j)$ $\forall\, j \in J$ so that $(k_j)_{\ j \in J} \in \bigvee(\lambda)$, that mean $\forall$ λ be directed subset, $\bigvee(\lambda_j) \neq \phi$. Hence $\prod\limits_{j \in J}(X_j)$ is a domain.

Definition 3.3. Let $(X, \leq_1)$ and $(X, \leq_2)$ be TR, , $x_1,\ x_2 . \in X$ and let $f:(X, \leq_1) \rightarrow (Y, \leq_2)$ be a function. Then

(1) f is $t-$monotone iff $f(x_1) \leq_2 f(x_2)$, whenever $x_1 \leq_1 x_2$, and $x_2 \leq_1 x_2$,

(2) f is called order preserving (monotone) iff $f(x_1) \leq_2 f(x_2)$, whenever $x_1 \leq_1 x_2$.

Theorem 3.5. Let $(X, \leq_1)$ and $(X, \leq_2)$ be TR, and let $f:(X, \leq_1) \rightarrow (Y, \leq_2)$ be a function. If f is $t-$monotone, then $\downarrow(f(\lambda)) = \downarrow(f(\downarrow(\lambda)) \ \forall\ \lambda \subseteq X$ such that. $\forall\ x \in \lambda,\ x \leq_1 x$

Proof Let $\lambda \subseteq X$ such that. $\forall\ x \in \lambda,\ x \leq_1 x$. So, $\lambda \subseteq \downarrow(\lambda)$. Then $\downarrow(f(\lambda)) \subseteq \downarrow(f(\downarrow(\lambda))$. We need to prove $\downarrow(f(\downarrow(\lambda)) \subseteq \downarrow(f(\lambda))$. So, let $y \in \downarrow(f(\downarrow(\lambda))$ implies that $\exists\ n \in f(\downarrow(\lambda)$ such that $y \leq_2 n$ implies that $\exists\ l \in \downarrow(\lambda)$ such that $f(l) = n$ and $y \leq_2 n$ implies that $\exists\ k \in (\lambda)$ such that $l \leq_1 k,\ n = f(l) \leq_2 f(k)$ and $y \leq_2 n$ implies that $\exists\ h = f(k) \in f((\lambda)$ such that $y \leq_2 h$ implies that $y \in \downarrow(f(\lambda))$.

Theorem 3.6. Let $(X, \leq_1)$ and $(X, \leq_2)$ be PRS, then a function $f:(X, \leq_1) \rightarrow (Y, \leq_2)$ is $t-$monotone, iff it is monotone.

Proof Obvious

Theorem 3.7. Let $(X, \leq_1)$ and $(X, \leq_2)$ be PRS, and let $f:(X, \leq_1) \rightarrow (Y, \leq_2)$ be a function. Then f is monotone, iff $\downarrow(f(\lambda)) = \downarrow(f(\downarrow(\lambda)) \ \forall\ \lambda \subseteq X$.

Proof From Theorem 3.5 and Theorem 3.6 and the other direction, Let $x_1,\ x_2 . \in X$ such that $x_1 \leq_1 x_2$. Now, $x_1 \in \downarrow x_2$ implies that $f(x_1) \in f(\downarrow(x_2)) \subseteq \downarrow(f(\downarrow(x_2)) = \downarrow(f(x_2))$ implies that $f(x_1) \leq_2 f(x_2)$.

Definition 3.4.. Let $(X, \leq_1)$ and $(X, \leq_2)$ be TR, and let a function $f:(X, \leq_1) \rightarrow (Y, \leq_2)$ is called:

(1) idempotent iff $f \circ f = f$;

(2) below the identity iff $f \leq id_X$, that mean $\forall\ x \in X.,\ f(x) \leq x$;

(3) Projection iff f is idempotent and monotone;

(4) Kernal operator iff f is idempotent, below the identity and monotone.

The following theorem is a generalization of the corresponding result in Lemma 1.1

Theorem 3.8. Let X be a set and let $f : X \rightarrow X$ be a function. Then f is idempotent iff $f(X) = fix(f)$, where $fix(f) = \{x \in X : f(x) = x\}$.

Proof $\Rightarrow$:) First let $y \in f(X)$. Then $\exists\ x \in X$ such that $f(x) = y$. Now, $f(x) = y = f(f(x)) = f(y)$ so that $y = f(y)$ so that $y \in fix(f)$. Hence $f(X) \subseteq fix(f)$. Second, let $y \in fix(f)$. Then $y = f(y)$ so that $y \in f(f)$. Thus $fix(f) \subseteq f(X)$.

$\Leftarrow$:) let $z \in X$. Then $f \circ f(z) = f(f(z)) = f(z)$. Hence f is idempotent

Definition 3.5. Let $(X, \leq)$ be TR, and let $\lambda \subseteq X$. Then the relation $\leq_\lambda$ on λ is defined by $\leq_\lambda = (\leq) \cap (\lambda \times \lambda)$.

Theorem 3.9. Let $(X, \leq)$ be TR, and let $\lambda \subseteq X$. Then $(\lambda,\ \leq_\lambda)$ is a TR.

Proof Let $(x, y),\ (y, z) \in (\leq) \cap (\lambda \times \lambda) = \leq_\lambda$. Then $(x, z) \in \leq$ so that $(x, z) \in \leq_\lambda$. Hence $(\lambda,\ \leq_\lambda)$ is a TR.

Theorem 3.10. Let $(X, \leq)$ be PRS (resp. poset), and let $\lambda \subseteq X$. Then $(\lambda, \leq_\lambda)$ is PRS (resp. poset).

Proof (1) Assume that $\leq$ is antisymetric Let $(x, y), (y, x) \in (\leq_\lambda)$. Then $(x, y), (y, x) \in \leq$ so that $x = y$. Hence if $\leq$ is antisymetric, then $\leq_\lambda$ so is.

(2) Assume that $\leq$ is reflexive Let $x \in \lambda$. Then $(x, x) \in \lambda \times \lambda$. so that $\cap \leq = \leq_\lambda$. Hence if $\leq$ is reflexive, then $\leq_\lambda$ so is. From (1) , (2) and Theorem 3.9, the result hold.

Remark 3.1. Let X be a set, $\lambda, \mu \subseteq X$, let $(\lambda, \leq_\lambda)$ is sub-poset of a poset $(X, \leq)$ such that $x \in \bigvee_\lambda(\mu)$ for some $\mu \subseteq \lambda$, but $x \notin \bigvee_X(\mu)$. So give the following example.

Example 3.1 Let $X = \{x, y, z, l\}, \lambda = \{x, y, z\}, \mu = \{x, y\}$ Now $\leq = \{(x, x), (y, y), (z, z).(l, l), (x, l), (y, l), (l, z).(x, z), (y, z)\}$. Then $\lambda \times \lambda = \{(x, x), (x, y), (x, z).(y, x), (y, y), (y, z), (z, x).(z, y), (z, z)\}$. Hence $\leq_\lambda = \{(x, x), (x, z), (y, y).(y, z), (z, z)\}$. Thus $\bigvee_\lambda(\mu) = z$ but $\bigvee_X(\mu) = l$.

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