Asia Mathematika
Volume: 7 Issue: 2 , (2023) Pages: 1 – 12
Available online at **www.asiamath.org**

New common fixed point theorem for multi-valued mappings in b-metric spaces

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Received: 12 Apr 2023 • **Accepted:** 10 Jul 2023 • **Published Online:** 31 Aug 2023

Abstract: In this paper, we prove a common fixed point theorem for a pair of multi-valued mappings satisfying generalized contraction conditions in the setting of complete b-metric spaces. The proposed theorem expand and generalize several well-known comparable results in the literature.

Key words: Common fixed points, Multi-valued mappings, Generalized contraction, b-metric spaces.

1. Introduction

Fixed point theory plays an important role in applications of many branches of mathematics. It has been applied in computer sciences, game theory, physical sciences and models in economy. One such fixed points has started by Banach in 1922, ensures the existence and uniqueness of fixed points for a contraction mapping in complete metric spaces. After that a series of articles have been dedicated to the improvement of fixed point theory.

In 1969, Nadler [24] introduced the notion of a multi-valued (set-valued) and was the first author proved Banach fixed point theorem for a multi-valued mapping in a complete metric space by using the Hausdorff metric. The theory of multi-valued (set-valued) maps has applications in differential inclusions, economics, control theory, and fractional differential inclusions.

Afterward, in 1989, Backhtin [7] introduced the concept of b-metric space as a generalization of a metric space. In 1993, Czerwik [11] first presented a generalization of Banach fixed point theorem in b-metric spaces. Several researcher generalized and extended fixed point theorems for single and multi-valued contractions mappings on b-metric space (see[1–5, 8, 9, 15–17, 20, 21, 23, 24, 26] and many others).

In this paper we give fixed point theorems for multi-valued generalized contraction of two maps in b-metric space.

2. Preliminaries and Terminology

Definition 2.1. [7, 11] Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric with coefficient $s \geq 1$ if:

1. $d(x, y) = 0$ if and only if $x = y$;

©Asia Mathematika, DOI: 10.5281/zenodo.8368900
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2. $d(x,y) = d(y,x)$, for all $x, y \in X$;
3. $d(x,z) \leq s[d(x,y) + d(y,z)]$, for all $x, y, z \in X$

Then the pair (X,d) is called a b-metric space.

Remark 2.1. *It is clear that every metric space is b-metric space with $s = 1$, but in general, a b-metric space need not necessarily be a metric space [12].*

Example 2.1. *[9] Let $L_p([0,1]) = \{f : [0,1] \longrightarrow \mathbb{R} : \|f\|_{L_p([0,1])} < \infty\}$, $(0 < p < 1)$ and*

$$\|f\|_{L_p([0,1])} = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}.$$

Denote $X = L_p([0,1])$, define a mapping $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = \left(\int_0^1 |f(x) - g(x)|^p dx\right)^{\frac{1}{p}}. \tag{1}$$

for all $f, g \in X$. Then (X,d) is a b-metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

For more examples of b-metric space (see [13, 16, 17, 20]).

Definition 2.2. [13] Let (X,d) be a b-metric space and $\{x_n\}$ a sequence in X. We say that :

1. $\{x_n\}$ converges to x if $d(x_n, x) \to 0$, as $n \to +\infty$,
2. $\{x_n\}$ is Cauchy sequence if $d(x_n, x_m) \to 0$, as $n, m \to +\infty$,
3. (X,d) is complete if every Cauchy sequence in X is convergent.

Each convergent sequence in a b-metric space has a unique limit and it is also a Cauchy sequence. Moreover, in general, a b-metric is not necessarily continuous [10]. The following example illustrates this claim.

Example 2.2. *[13] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \longrightarrow \mathbb{R}^+$ as follows:*

$$d(m,n) = \begin{cases} 0 & \textit{if } m = n \\ |\frac{1}{m} - \frac{1}{n}| & \textit{if one of } m,n \textit{ is even and the other is even or } \infty \\ 5 & \textit{if one of } m,n \textit{ is odd and the other is odd or } \infty \\ 2 & \textit{otherwise } m = n. \end{cases}$$

Then (X,d) is a b-metric space with coefficient $s = \frac{5}{2}$. However, let $x_n = 2n$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(2n,\infty) = \lim_{n\to\infty} \frac{1}{2n} = 0$, that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \nrightarrow 5 = d(\infty, 1)$ as $n \to \infty$.

Let (X,d) be a complete b-metric space. In the sequel, we use the following notations:
$CB(X) = \{A : A$ is a nonempty closed and bounded subset of $X\}$,
$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$,
$\delta(A,B) = \sup\{d(a,B) : a \in A\}$,

$\delta(B,A) = \sup\{d(b,A) : b \in B\}$,
$H(A,B) = \max\{\delta(A,B), \delta(B,A)\} = \max\{\sup_{x\in B} D(x,A), \sup_{x\in A} D(x,B)\}$.
Notice that H is called the Hausdorff metric induced by the metric d.
Forward, we denote by $F(T)$ the set of all fixed points of a multi-valued mapping T, that is,

$$F(T) = \{p \in X : p \in Tp\}.$$

Definition 2.3. A point of $x_0 \in X$ is said to be a fixed point of the multi-valued mappings $T : X \longrightarrow CB(X)$ if $x_0 \in Tx_0$.

Lemma 2.1. *[12] Let (X,d) be a complete b-metric space. For any $A,B,C \in CB(X)$ and any $x,y \in X$, one has the following:*

1. *$d(x,B) \leq d(x,b)$, for any $b \in B$.*
2. *$\delta(A,B) \leq H(A,B)$.*
3. *$d(x,B) \leq H(A,B)$, for any $x \in A$.*
4. *$H(A,A) = 0$.*
5. *$H(A,B) = H(B,A)$.*
6. *$H(A,C) \leq s[H(A,B) + H(B,C)]$.*
7. *$d(x,A) \leq s[d(x,y) + d(y,A)]$.*

Lemma 2.2. *[13] Let (X,d) be a complete b-metric space and let $\{x_n\}$ be a sequence in X such that*

$$d(x_{n+1}, x_{n+2}) \leq \beta d(x_n, x_{n+1}), \text{ for all } n = 0,1,2,\ldots$$

where $0 \leq \beta < 1$. Then $\{x_n\}$ is a Cauchy sequence in X.

3. Mains results

Before proving our main results, we need the following Lemma:

Lemma 3.1. *Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$, $\alpha, \gamma_1, \gamma_2$ are nonnegative reals with $0 \leq \gamma_1 < \gamma_2$, and $S,T : X \to CB(X)$ be multi-valued maps satisfying, for all $x,y \in X$*

$$s^{\alpha}\delta(Sx,Ty) \leq N(x,y)M(x,y), \tag{2}$$

where

$$N(x,y) = \frac{\max\{d(x,y), D(x,Sx)+D(y,Ty), D(x,Ty)+D(y,Sx)+\gamma_1\}}{\delta(x,Sx)+\delta(y,Ty)+\gamma_2}, \tag{3}$$

and

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2s}\right\}. \tag{4}$$

Then every fixed point of S is a fixed point of T, and conversely.

Proof. Suppose that p is a fixed point of S. Using (2) and the definition of δ,

$$D(p,Tp) \leq \delta(p,Tp) \leq \delta(Sp,Tp) \leq \frac{1}{s^\alpha} N(p,p)M(p,p). \tag{5}$$

Where,

$$\begin{aligned} N(p,p) &= \frac{\max\{d(p,p), D(p,Sp)+D(p,Tp), D(p,Tp)+D(p,Sp)+\gamma_1\}}{\delta(p,Sp)+\delta(p,Tp)+\gamma_2} \\ &\leq \frac{D(p,Tp)+\gamma_1}{D(p,Tp)+\gamma_2} = \lambda < 1, \end{aligned}$$

and,

$$\begin{aligned} M(p,p) &= \max\left\{d(p,p), D(p,Sp), D(p,Tp), \frac{D(p,Tp)+D(p,Sp)}{2s}\right\} \\ &\leq D(p,Tp). \end{aligned}$$

From 5

$$D(p,Tp) \leq \frac{\lambda}{s^\alpha} D(p,Tp),$$

since $\frac{\lambda}{s^\alpha} < 1$, which implies that p is also a fixed point of T.

In a similar manner it can be shown that, if $p \in Tp$, then $p \in Sp$.

□

Now, we prove the main result in this section.

Theorem 3.1. *Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$, $\alpha, \gamma_1, \gamma_2$ are nonnegative reals with $0 \leq \gamma_1 < \gamma_2$, and $S,T : X \to CB(X)$ be multi-valued maps satisfying (2), (3) and (4). Then*
(a) S and T have at least one common fixed point $p \in X$.
(b) For n even, $\{(ST)^{n/2}x\}$ and $\{T(ST)^{n/2}x\}$ converge to a common fixed point for each $x \in X$.
(c) If p and q are distinct common fixed points of S and T, then

$$\frac{s^\alpha \gamma_2 - \gamma_1}{2} \leq d(p,q).$$

Proof. Part (a), let $x_0 \in X, x_1 \in Sx_0$ and define $\{x_n\}$ by

$$x_{2n+1} \in Sx_{2n},\ x_{2n+2} \in Tx_{2n+1},\ \text{for all } n \geq 0. \tag{6}$$

Without loss of generality, we assume that $x_n \neq x_{n+1}$ for each n. For, if there exist an n_0 such that $x_{n_0} \neq x_{n_0+1}$, then n_0 forms a common fixed point for S and T. More precisely, to see that x_{n_0} is the common fixed point of S and T, we consider n_0 in two cases. First, if $n_0 = 2n$. In this case, we have $x_{2n} = x_{2n+1} \in Sx_{2n}$, that is, x_{2n} is a fixed point of S, hence of T by Lemma 3.1. that is, $x_{2n} = x_{2n+1}$ is a common fixed point of S and T. Similarly, if $n_0 = 2n+1$.
Thus, throughout the proof, we suppose that $x_n \neq x_{n+1}$ for each n. For, if there exists an n_0 for which $x_{2n_0} \neq x_{2n_0+1}$, then, since $x_{2n_0+1} \in Sx_{2n_0}, x_{2n_0+1} \in F(S)$, and by Lemma 3.1, $x_{2n_0} \in F(T)$. Similarly,

$x_{2n_0+1} = x_{2n_0+2}$ for any n_0 implies that $x_{2n_0+1} \in F(T) \cap F(S)$.
First we to show that $\{x_n\}$ is a Cauchy sequence in X. For this, choose $x_{2n+1} \in Sx_{2n}$ such that

$$d(x_{2n}, x_{2n+1}) \leq \delta(Sx_{2n}, Tx_{2n-1}), \tag{7}$$

Similarly, choose $x_{2n+2} \in Tx_{2n+1}$ such that

$$d(x_{2n+1}, x_{2n+2}) \leq \delta(Sx_{2n}, Tx_{2n+1}). \tag{8}$$

Setting $d_{2n} = d(x_{2n+1}, x_{2n})$, the λ_n are defined by

$$\lambda_n = \frac{d_{n-1} + d_n + \dfrac{\gamma_1}{s}}{d_{n-1} + d_n + \gamma_2}. \tag{9}$$

It follows from (3) that

$$\begin{aligned}
N(x_{2n}, x_{2n-1}) &= \frac{\max\{d_{2n-1}, D(x_{2n}, Sx_{2n}) + D(x_{2n-1}, Tx_{2n-1}), D(x_{2n}, Tx_{2n-1}) + D(x_{2n-1}, Sx_{2n}) + \gamma_1\}}{\delta(x_{2n}, Sx_{2n}) + \delta(x_{2n-1}, Tx_{2n-1}) + \gamma_2} \\
&\leq \frac{\max\{d_{2n-1}, d_{2n} + d_{2n-1}, 0 + d(x_{2n-1}, x_{2n+1}) + \gamma_1\}}{d_{2n} + d_{2n-1} + \gamma_2} \\
&\leq \frac{\max\{d_{2n-1}, d_{2n} + d_{2n-1}, s\,[d_{2n-1} + d_{2n}] + \gamma_1\}}{d_{2n} + d_{2n-1} + \gamma_2} \\
&= s\frac{d_{2n-1} + d_{2n} + \dfrac{\gamma_1}{s}}{d_{2n-1} + d_{2n} + \gamma_2} = s\lambda_{2n},
\end{aligned} \tag{10}$$

where $\lambda_{2n} = \dfrac{d_{2n-1} + d_{2n} + \dfrac{\gamma_1}{s}}{d_{2n-1} + d_{2n} + \gamma_2} < 1$, for all $n > 0$ because $0 \leq \gamma_1 < \gamma_2$.
It follows from (4) that

$$\begin{aligned}
M(x_{2n}, x_{2n-1}) &= \max\left\{d(x_{2n}, x_{2n-1}), D(x_{2n}, Sx_{2n}), D(x_{2n-1}, Tx_{2n-1}), \frac{D(x_{2n}, Tx_{2n-1}) + D(x_{2n-1}, Sx_{2n})}{2s}\right\} \\
&\leq \max\left\{d_{2n-1}, d_{2n}, d_{2n-1}, \frac{0 + d(x_{2n-1}, x_{2n+1})}{2s}\right\} \\
&\leq \max\left\{d_{2n-1}, d_{2n}, \frac{d_{2n-1} + d_{2n}}{2}\right\} \\
&= \max\{d_{2n-1}, d_{2n}\}.
\end{aligned} \tag{11}$$

Using (2), (10) and (11) in (7) yields

$$d_{2n} \leq \delta(Sx_{2n}, Tx_{2n-1}) \leq \frac{\lambda_{2n}}{s^{\alpha-1}} \max\{d_{2n-1}, d_{2n}\}.$$

If $d_{2n} > d_{2n-1}$ for some n, then from the above inequality we have

$$d_{2n} \leq \frac{\lambda_{2n}}{s^{\alpha-1}} d_{2n},$$

a contradiction since each $x_n \neq x_{n+1}, d_{2n} > 0$ and $\frac{\lambda_{2n}}{s^{\alpha-1}} < 1$, then $\max\{d_{2n-1}, d_{2n}\} = d_{2n-1}$ for all $n > 0$. Also, by the above inequality we obtain

$$d_{2n} \leq \frac{\lambda_{2n}}{s^{\alpha-1}} d_{2n-1}. \tag{12}$$

Similarly, we can prove

$$d_{2n+1} \leq \frac{\lambda_{2n+1}}{s^{\alpha-1}} d_{2n}. \tag{13}$$

Combining (12) and (13), we can conclude that, for all $n > 0$,

$$d_n \leq \frac{\lambda_n}{s^{\alpha-1}} d_{n-1} < d_{n-1}. \tag{14}$$

Next, we show that $\lambda_n < \lambda_{n-1}$, for all $n > 0$.
Then, by inequality (14), we get $d_n < d_{n-1}$, and also $d_{n-1} < d_{n-2}$, which implies that $d_n < d_{n-2}$. Consequently

$$0 < d_n + d_{n-1} + \frac{\gamma_1}{s} < d_{n-1} + d_{n-2} + \frac{\gamma_1}{s},$$

and

$$0 < d_n + d_{n-1} + \gamma_2 < d_{n-1} + d_{n-2} + \gamma_2,$$

then by dividing, we get

$$\frac{d_n + d_{n-1} + \frac{\gamma_1}{s}}{d_n + d_{n-1} + \gamma_2} < \frac{d_{n-1} + d_{n-2} + \frac{\gamma_1}{s}}{d_{n-1} + d_{n-2} + \gamma_2}$$

is equivalent to $\lambda_n < \lambda_{n-1}$, continuing this process, we get

$$\lambda_n < \lambda_1. \tag{15}$$

Now, from (14) and (15), we have

$$d_n \leq \frac{\lambda_1}{s^{\alpha-1}} d_{n-1}. \tag{16}$$

Let $\beta = \frac{\lambda_1}{s^{\alpha-1}}$. Then, we have that $\beta \in [0, 1)$. Hence, by Lemma 2.2, we obtain that $\{x_n\}$ is a Cauchy sequence in (X, d). By completeness of (X, d), there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$.
Next, to show that p ai a fixed of T. For this, using triangular inequality, we have

$$\begin{aligned} D(p, Tp) &\leq s[d(p, x_{2n+1}) + D(x_{2n+1}, Tp)] \\ &\leq s[d(p, x_{2n+1}) + \delta(Sx_{2n}, Tp)]. \end{aligned} \tag{17}$$

It follows from (3) that

$$\begin{aligned} N(x_{2n}, p) &= \frac{\max\{d(x_{2n}, p), D(x_{2n}, Sx_{2n}) + D(p, Tp), D(x_{2n}, Tp) + D(p, Sx_{2n}) + \gamma_1\}}{\delta(x_{2n}, Sx_{2n}) + \delta(p, Tp) + \gamma_2} \\ &\leq \frac{\max\{d(x_{2n}, p), d(x_{2n}, x_{2n+1}) + d(p, Tp), d(x_{2n}, Tp) + d(p, x_{2n+1}) + \gamma_1\}}{d(x_{2n}, x_{2n+1}) + d(p, Tp) + \gamma_2}. \end{aligned} \tag{18}$$

It follows from (4) that

$$\begin{aligned} M(x_{2n},p) &= \max\left\{d(x_{2n},p), D(x_{2n},Sx_{2n}), D(p,Tp), \frac{D(x_{2n},Tp)+D(p,Sx_{2n})}{2s}\right\} \\ &\leq \max\left\{d(x_{2n},p), d(x_{2n},x_{2n+1}), D(p,Tp), \frac{d(x_{2n},Tp)+d(p,x_{2n+1})}{2s}\right\}. \end{aligned} \tag{19}$$

Substituting (18) and (19) into (17), using (2), and taking the limit of both sides as $n \longrightarrow \infty$, one obtains

$$D(p,Tp) \leq \frac{1}{s^{\alpha-1}} \frac{d(p,Tp)+\gamma_1}{d(p,Tp)+\gamma_2} D(p,Tp),$$

as $\dfrac{1}{s^{\alpha-1}} \dfrac{d(p,Tp)+\gamma_1}{d(p,Tp)+\gamma_2} < 1$, which implies $D(p,Tp) = 0$. Hence, we get that $p \in F(T)$. From Lemma 3.1, $p \in F(S)$. Accordingly, we conclude that S and T have a common fixed point p.
To prove (b), merely observe that, from (6) and the fact that x_0 is arbitrary, we may write.

$$x_{n+1} \in (ST)^{n/2}x \text{ and } x_{n+2} \in T(ST)^{n/2}x.$$

(c). Suppose that p and q are distinct common fixed points of S and T.
Then

$$d(p,q) \leq \delta(Sp,Tq). \tag{20}$$

It follows from (3) that

$$\begin{aligned} N(p,q) &= \frac{\max\{d(p,q), 0, D(p,Tq)+D(q,Sp)+\gamma_1\}}{\delta(p,Sp)+\delta(q,Tq)+\gamma_2} \\ &\leq \frac{\max\{d(p,q), d(p,q)+d(q,p)+\gamma_1\}}{d(p,Sp)+d(q,Tq)+\gamma_2} \\ &= \frac{2d(p,q)+\gamma_1}{\gamma_2}. \end{aligned}$$

It follows from (4) that

$$\begin{aligned} M(p,q) &= \max\left\{d(p,q), 0, 0, \frac{D(p,Tq)+D(q,Sp)}{2s}\right\} \\ &= d(p,q). \end{aligned}$$

Using (2) and substituting it into (20) gives

$$d(p,q) \leq \frac{2d(p,q)+\gamma_1}{s^{\alpha}\gamma_2} d(p,q).$$

which yields the result. This completes the proof. □

Corollary 3.1. *Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$, $\alpha, \gamma_1, \gamma_2$ are nonnegative reals with $0 \leq \gamma_1 < \gamma_2$, and $T : X \to CB(X)$ be a multivalued map satisfying for all $x,y \in X$*

$$s^{\alpha}\delta(Tx,Ty) \leq N(x,y)M(x,y), \tag{21}$$

where

$$N(x,y) = \frac{\max\{d(x,y), D(x,Tx)+D(y,Ty), D(x,Ty)+D(y,Tx)+\gamma_1\}}{\delta(x,Tx)+\delta(y,Ty)+\gamma_2}, \tag{22}$$

and

$$M(x,y) = \max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2s}\right\}. \tag{23}$$

Then
(a) T has at least one fixed point.
(b) $\{T^n x\}$ converge to a fixed point of T.
(c) If p and q are distinct fixed points of T, then

$$\frac{s^\alpha \gamma_2 - \gamma_1}{2} \le d(p,q).$$

Proof. Take $S = T$ in Theorem 3.1. $\square$

Corollary 3.2. *Let (X,d) be a complete b-metric space and $T : X \to CB(X)$ be a multivalued map satisfying for all $x, y \in X$*

$$s^\alpha d(Tx,Ty) \le \left(\frac{d(x,Ty)+d(y,Tx)+\gamma_1}{d(x,Tx)+d(y,Ty)+\gamma_2}\right) d(x,y), \tag{24}$$

Then
(a) T has at least one fixed point.
(b) $\{T^n x\}$ converge to a fixed point of T.
(c) If p and q are distinct fixed points of T, then

$$\frac{s^\alpha \gamma_2 - \gamma_1}{2} \le d(p,q).$$

Proof. Take $S = T$ in (2), $N(x,y) = \dfrac{d(x,Ty)+d(y,Tx)+\gamma_1}{d(x,Tx)+d(y,Ty)+\gamma_2}$ in (3) and $M(x,y) = d(x,y)$ in (4), from Theorem 3.1. $\square$

Example 3.1. *Let $X = \{-\frac{1}{2}, 0, \frac{1}{2}\}$ and let $d : X \longrightarrow \mathbb{R}^+$ defined by*

$$d(-\frac{1}{2},0) = 1, \quad d(-\frac{1}{2},-\frac{1}{2}) = 4, \quad d(0,\frac{1}{2}) = 2,$$

$$d(-\frac{1}{2},-\frac{1}{2}) = d(0,0) = d(\frac{1}{2},\frac{1}{2}) = 0,$$

$$d(x,y) = d(y,x), \quad \textit{for all} \quad x,y \in X.$$

(X,d) is a complete b-metric space with coefficient $s = \dfrac{4}{3}$, and $\alpha = 1, \gamma_1 = 1, \gamma_2 = 2$. Let $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} -\frac{1}{2}, & x = -\frac{1}{2}, \frac{1}{2}. \\ 0, & x = 0. \end{cases}$$

and we have

$$\begin{aligned} d(T(-\frac{1}{2}), T(0) &= d(-\frac{1}{2}, 0) = 1 \\ &\leq \frac{3}{4}\left(\frac{d(-\frac{1}{2}, T(0)) + d(0, T(-\frac{1}{2})) + 1}{d(-\frac{1}{2}, T(-\frac{1}{2})) + d(0, T(0)) + 2}\right) d(-\frac{1}{2}, 0) \\ &= \frac{9}{8}, \end{aligned}$$

and

$$\begin{aligned} d(T(\frac{1}{2}), T(0) &= d(-\frac{1}{2}, 0) = 1 \\ &\leq \frac{3}{4}\left(\frac{d(\frac{1}{2}, T(0)) + d(0, T(\frac{1}{2})) + 1}{d(\frac{1}{2}, T(\frac{1}{2})) + d(0, T(0)) + 2}\right) d(\frac{1}{2}, 0) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} d(T(0), T(-\frac{1}{2})) &= d(0, -\frac{1}{2}) = 1 \\ &\leq \frac{3}{4}\left(\frac{d(0, T(-\frac{1}{2})) + d((-\frac{1}{2}), T(0)) + 1}{d(0, T(0)) + d(-\frac{1}{2}, T(-\frac{1}{2})) + 2}\right) d(0, -\frac{1}{2}) \\ &= \frac{9}{8}, \end{aligned}$$

and also

$$\begin{aligned} d(T(0), T(\frac{1}{2})) &= d(0, -\frac{1}{2}) = 1 \\ &\leq \frac{3}{4}\left(\frac{d(0, T(\frac{1}{2}))) + d(\frac{1}{2}, T(0)) + 1}{d(\frac{1}{2}, T(\frac{1}{2})) + d(0, T(0)) + 2}\right) d(0, \frac{1}{2}) \\ &= 1. \end{aligned}$$

Therefore, T satisfies all the conditions of Corollary3.2. Then T has two distinct fixed points $\{-\frac{1}{2}, 0\}$ and $\frac{5}{6} \leq d(-\frac{1}{2}, 0) = 1$.

Corollary 3.3. *Let (X, d) be a complete b-metric space with a coefficient $s \geq 1$, $\alpha, \gamma_1, \gamma_2$ are nonnegative reals with $0 \leq \gamma_1 < \gamma_2$, and $S, T : X \to CB(X)$ be multi-valued maps satisfying, for all $x, y \in X$*

$$s^{\alpha} d(Sx, Ty) \leq N(x, y)M(x, y), \tag{25}$$

where

$$N(x, y) = \frac{\max\{d(x, y), d(x, Sx) + d(y, Ty), d(x, Ty) + d(y, Sx) + \gamma_1\}}{d(x, Sx) + dy, Ty) + \gamma_2}, \tag{26}$$

and

$$M(x, y) = \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2s}\right\}. \tag{27}$$

Then
(a) S and T have at least one common fixed point $p \in X$.
(b) For n even, $\{(ST)^{n/2}x\}$ and $\{T(ST)^{n/2}x\}$ converge to a common fixed point for each $x \in X$.
(c) If p and q are distinct common fixed points of S and T, then

$$\frac{s^{\alpha}\gamma_2 - \gamma_1}{2} \leq d(p,q).$$

Proof. Take $N(x,y) = \dfrac{\max\{d(x,y), d(x,Sx)+d(y,Ty), d(x,Ty)+d(y,Sx)+\gamma_1\}}{d(x,Sx)+dy,Ty)+\gamma_2}$ in (3) and $M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \dfrac{d(x,Ty)+d(y,Sx)}{2s}\right\}$ in (4), from Theorem 3.1. □

Example 3.2. *Let $X = \{0, \frac{1}{2}, \frac{5}{2}\}$, and let $d : X \times X \to [0,+\infty)$ be a mapping satisfies the following condition for all $x, y \in X$:*

1. *$d(x,y) = 0$, where $x = y$.*
2. *$d(0,\frac{1}{2}) = d(\frac{1}{2},0) = 1$, $\quad d(0,\frac{5}{2}) = d(\frac{5}{2},0) = 3$, $\quad d(\frac{1}{2},\frac{5}{2}) = d(\frac{5}{2},\frac{1}{2}) = 6$.*

Then, (X,d) is a complete b-metric space with coefficient $s = \frac{3}{2} > 1$. and let $CB(X) = \{0, \frac{1}{2}\}$. Consider mappings $T, S : X \to CB(X)$ define by

$$T(0) = 0, T\left(\frac{1}{2}\right) = \frac{1}{2}, T\left(\frac{5}{2}\right) = 0,$$

$$S(0) = 0, S\left(\frac{1}{2}\right) = \frac{1}{2}, S\left(\frac{5}{2}\right) = 0.$$

Let $\alpha = 1, \gamma_1 = 0, \gamma_2 = 1$. Now, we verify that the mappings S and T satisfy the condition (25) of Corollary 3.3. We have the following cases:
Case 1. *$d(Tx,Sy) = 0$, it is obvious.*
Case 2. *$d(Tx,Sy) \neq 0$, we have the following four cases to be considered.*
Case 2.1. *$x = 0, y = \frac{1}{2}$, we can get $s^{\alpha}d(Tx,Sy) = \frac{3}{2}$, then*

$$\begin{aligned}\frac{3}{2} &\leq 2 \\ &= 2 \times 1 \\ &= N(x,y)M(x,y),\end{aligned}$$

thus, the inequality (25) holds.
Case 2.2. *$x = \frac{1}{2}, y = \frac{5}{2}$, we can get $s^{\alpha}d(Tx,Sy) = \frac{3}{2}$, then*

$$\begin{aligned}\frac{3}{2} &\leq \frac{21}{2} \\ &= \frac{7}{4} \times 6 \\ &= N(x,y)M(x,y),\end{aligned}$$

thus, the inequality (25) holds.

Case 2.3. $x = \frac{1}{2}, y = 0,$ *we can get* $s^\alpha d(Tx, Sy) = \frac{3}{2},$ *then*

$$\begin{aligned} \frac{3}{2} &\leq 2 \\ &= 2 \times 1 \\ &= N(x,y)M(x,y), \end{aligned}$$

thus, the inequality (25) holds.

Case 2.4. $x = \frac{5}{2}, y = \frac{1}{2},$ *we can get* $s^\alpha d(Tx, Sy) = \frac{3}{2},$ *then*

$$\begin{aligned} \frac{3}{2} &\leq \frac{21}{2} \\ &= \frac{7}{4} \times 6 \\ &= N(x,y)M(x,y), \end{aligned}$$

thus, the inequality (25) holds.

Therefore, all the conditions of Corollary 3.3 are satisfied and, further, $\{0, \frac{1}{2}\}$ *is two common fixed point of the mappings* S *and* T*, and* $\frac{s^\alpha \gamma_2 - \gamma_1}{2} = \frac{3}{4} \leq d(0, \frac{1}{2}) = 1$*.*

Remark 3.1. *By choosing :*

$s = 1, \gamma_1 = 0$ *and* $\gamma_2 = 1$ *in Corollary 3.2, we get Theorem 1 of [21].*

$s = 1, \gamma_1 = 0$ *and* $\gamma_2 = 1$ *in Corollary 3.3, we get Theorem 2.1 of [25].*

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