Asia Mathematika
Volume: 7 Issue: 2 , (2023) Pages: 13 – 16
Available online at **www.asiamath.org**

Common fixed points on occasionally weakly compatible self- mappings in CMS

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Received: 17 Apr 2023 • **Accepted:** 11 Jul 2023 • **Published Online:** 31 Aug 2023

Abstract: In this paper, we prove a unique common fixed point theorem for occasionally weakly compatible self-mappings satisfying a generalized contractive type condition in CMS(Cone Metric Space). Our results are generalizing and improving some of the well known comparable results existing in the literature.

Key words: F ixed point, common fixed point, occasionally weakly compatible, cone metric space, normal cone.

1. Introduction and preliminaries

The fixed point theory is an important area of non-linear analysis. Recently Huang and Zhang [1] generalized the concept of a metric space into a cone metric space and they replaced the real numbers by an ordered Banach space and also proved some of the fixed point theorems in cone metric space with different types of contractive conditions.Later on Many authors extended these results in many ways and generalized in different ways (see for e.g., [3–11]). Recently Bhatt and Chandra [2] obtained some fixed point results in occasionally weakly compatible mappings in cone metric space. In this paper we obtained unique common fixed point result for occasionally weakly compatible condition in CMS.

We recall some definitions of cone metric spaces and some of their properties [1].

Definition 1.1. Let M be a real Banach space and Q be a subset of M.The set Q is called a cone if and only if

(a) Q is closed, nonempty and $Q \neq \{0\}$;

(b) $a, b \in R, a, b \geq 0, u, v \in Q \Longrightarrow au + bv \in Q$;

(c) $u \in Q$ and $-u \in Q \Longrightarrow u = 0$.

Definition 1.2. Let Q be a cone in a Banach space M define partial ordering $\leq$ with respect to Q by $u \leq v$ if and only if $u - v \in Q$.We shall write $u < v$ to indicate $u \leq v$ but $u \neq v$ while $u \ll v$ will stand for $u - v \in intQ$, where $intQ$ denotes the interior of the set Q. This cone Q is called an order cone.

Definition 1.3. Let M be a Banach Space and $Q \subset M$ be an order cone .The order cone Q is called normal if there exists $K > 0$ such that for all $u, v \in M$,

$0 \leq u \leq v$ implies $\| u \| \leq K \| v \|$.

The least positive number K satisfying the above inequality is called the normal constant of Q.

©Asia Mathematika, DOI: 10.5281/zenodo.8368963
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Definition 1.4. Let X be a nonempty set of M .Suppose that the map $d : X \times X \longrightarrow M$ satisfies :

$(d1)$ $0 \leq d(u, v)$ for all $u, v \in X$ and $d(u, v) = 0$ if and only if $u = v$;

$(d2)$ $d(u, v) = d(v, u)$ for all $u, v \in X$;

$(d3)$ $d(u, v) \leq d(u, z) + d(v, z)$ for all $u, v, z \in X$.

Then d is called a cone metric on X and (X, d) is called a CMS(Cone Metric Space).
It is obvious that the CMS(Cone Metric Spaces) generalize metric spaces.

Definition 1.5. Let (X, d) be a cone metric space.We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in M with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for any $0 \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$, for some fixed x in X. We denote this $x_n \longrightarrow x$ $(n \longrightarrow \infty)$.

A CMS(Cone Metric Space) X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 1.6 (9)**.** Let M and N be self-mappings of a set X. If $q = Mu = Nu$ for some u in X, then u is called a coincidence point of M and N, and q is called a point of coincidence of M and N.

Proposition 1.1. *Let M and N be occasionally weakly compatible self-mappings of a set X if and only if there is a point u in X which is coincidence point of M and N at which M and N are commute.*

Lemma 1.1. *Let X be a set, M, N are occasionally weakly compatible self-mappings of X. If M and N have a unique point of coincidence $q = Mu = Nu$, then q is the unique common fixed point of M and N.*

Definition 1.7. Let $\phi : R^+ \longleftrightarrow R^+$ be a function satisfying the condition $\phi(t) < t$ for each $t > 0$.

2. Main Results

Now we prove the main theorem

Theorem 2.1. *Let (X, d) be a cone metric space and M be a normal cone. Suppose that p and q are two self- mappings of X and satisfy the following conditions:*

$$d(pu, pv) \ \leq \ \phi(Max\{[\frac{d(qu, qv) + d(qu, pv)}{2}], d(qv, pu), d(qv, pv)\}) \qquad for \ \ all \ \ u, v \ \ \in \ \ X. \tag{1}$$

$$And \ \ \ p \ \ and \ \ \ q \ \ \ are \ \ occasionally \ weakly \ compatible. \tag{2}$$

Then p and q have a unique common fixed point.

Proof. Given (by (2)) p and q are occasionally weakly compatible, then there exists point $\alpha \in X$, $pq\alpha = qp\alpha$. We claim that, $p\alpha$ is the unique common fixed point of p and q. First we ascertain that $p\alpha$ is a fixed point of p. For if, $pp\alpha \neq p\alpha$, then by (1) we get that .

$$d(p\alpha, pp\alpha) \leq \phi(Max\{[\frac{d(q\alpha, qp\alpha) + d(q\alpha, pp\alpha)}{2}], d(qp\alpha, p\alpha), d(qp\alpha, pp\alpha)\}),$$

$$= \phi(Max\{[\frac{d(p\alpha, pq\alpha) + d(p\alpha, pp\alpha)}{2}], d(pq\alpha, p\alpha), d(pq\alpha, pp\alpha)\}),$$

$$= \phi(Max\{[\frac{d(p\alpha, pp\alpha) + d(p\alpha, pp\alpha)}{2}], d(pp\alpha, p\alpha), d(pp\alpha, pp\alpha)\}),$$

$$= \phi(Max\{d(p\alpha, pp\alpha), d(pp\alpha, p\alpha), 0\}),$$

$$= \phi(d(p\alpha, pp\alpha)),$$

$$\leq d(p\alpha, pp\alpha)),$$

, which is a contradiction.

Therefore, $pp\alpha = p\alpha$ and $pp\alpha = pq\alpha = qp\alpha = p\alpha$. Thus $p\alpha$ is a common fixed point of p and q.

Uniqueness: suppose that α, $\beta \in X$ such that $p\alpha = q\alpha = \alpha$ and $p\beta = q\beta = \beta$ and $\alpha \neq \beta$. Then by (1) we get that

$$d(\alpha, \beta) = d(p\alpha, p\beta) \leq \phi(Max\{[\frac{d(q\alpha, q\beta) + d(q\alpha, p\beta)}{2}], d(q\beta, p\alpha), d(q\beta, p\beta)\}),$$

$$= \phi(Max\{[\frac{d(\alpha, \beta) + d(\alpha, \beta)}{2}], d(\beta, \alpha), d(\beta, \beta)\}),$$

$$= \phi(Max\{d(\alpha, \beta), d(\beta, \alpha), 0\}),$$

$$= \phi(d(\alpha, \beta), d(\beta, \alpha), 0)),$$

$$\leq d(\alpha, \beta)$$

,, which is a contradiction.

Therefore, $\alpha = \beta$. Therefore, p and q have a unique common fixed point. This completes the proof of the theorem.

□

Remark 2.1. *Our results are more general then the results of [2].*

Acknowledgment

The author is grateful to the reviewers/editors for their great suggestions/corrections which helped to improve the quality of this article.

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