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Generalized optimal algebraic bounds for the exponential function

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Abstract: We present new generalized lower and upper bounds for the natural exponential function. These bounds are algebraic in nature and each involve a parameter a. Each bound is optimal as $a \to 0$.

Key words: Optimal bounds, exponential function, algebraic bounds.

1. Introduction

The well-known inequality

$$e^x \geq 1 + x \tag{1}$$

holds for all real numbers. If we restrict ourselves to the positive values of x then by change of variable it can be written as

$$e^x \leq \frac{1}{1-x}; \ 0 < x < 1. \tag{2}$$

For $1 \leq a \leq 2$, the inequalities

$$\left.\begin{aligned} e^x &\leq 1 - \frac{1}{a} + \frac{1}{a}\left[\frac{1 + \left(1 - \frac{1}{a}\right)x}{1 - \frac{x}{a}}\right]^a \\ &\leq \frac{1}{1-x}; \ 0 < x < 1 \end{aligned}\right\} \tag{3}$$

were established by Kim [7]. It is a generalization and refinement of (2). Other tighter bounds are:

$$(1+ax)^{\frac{1}{a}\sqrt{1+ax}} < e^x < (1+ax)^{\frac{1}{a}\sqrt{1+ax+\frac{1}{12}a^2x^2}}; \ a > 0, x > 0. \tag{4}$$

The double inequality (4) was appeared in [1]. It is observed that the smaller the value of a is, the sharper the bounds in (4) are. Recently, Bougoffa and Krasopoulos [5] proved that the inequalities

$$(1+x)^{\frac{a(a+1)x^2}{(1+x)^{a+1}+(1+x)^{-a}-(x+2)}} < e^x < (1+x)^{\frac{a}{a+1}\left[\frac{(1+x)^{a+1}-1}{(1+x)^a-1}\right]}; \ x > 0 \tag{5}$$

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are optimal as $a \to 0$ and sharper than the corresponding inequalities in (4). For some other bounds we refer the reader to [1–9] and references therein. The bounds of e^x in (4) and (5) are sharper but transcendental in nature. On the other hand the bounds of e^x in (1)–(3) are algebraic and simple, but they are not sharper. Algebraic bounds are always computationally efficient and can find applications in diverse areas. In this paper, our aim is to present generalized sharp bounds for e^x that are algebraic and optimal and also the alternatives to the bounds in (4) and (5).

2. Main results

In this section, we state and prove our main results. The first one states

Proposition 2.1. *Let $a > 0$ and $n = 0, 1, 2, 3, \cdots$. Then for $x \geq 0$, we have*

$$\phi(x) \leq e^x \tag{6}$$

where $\phi(x) = \frac{(1+ax)^{n+\frac{1}{a}}}{(1+a)(1+2a)\cdots(1+na)} + \frac{(1+a)(1+2a)\cdots(1+na)-1}{(1+a)(1+2a)\cdots(1+na)} + \phi_1(x)$

and

$$\phi_1(x) = \begin{cases} 0 & n = 0, 1 \\ \sum_{k=1}^{n-1} \frac{(1+a)(1+2a)\cdots(1+ka)-1}{(n-k)!(1+a)(1+2a)\cdots(1+ka)} x^{n-k} & n = 2, 3, \cdots. \end{cases}$$

Proof. For $x = 0$, the inequalities (6) clearly hold as equalities. For $x > 0$, we prove the proposition by principle of mathematical induction. In fact, for $n = 0$, we have

$$(1 + ax)^{\frac{1}{a}} < e^x \tag{7}$$

which is true and can be obtained from (1) after replacing x by ax. For $n = 1$, by integrating (7) as

$$\int_0^x (1 + at)^{\frac{1}{a}} dt < \int_0^x e^t dt,$$

we see that

$$\frac{(1 + ax)^{1+\frac{1}{a}}}{(1 + a)} + \frac{a}{1 + a} < e^x. \tag{8}$$

Similarly, for $n = 2$, integration of (8) gives

$$\int_0^x \frac{(1 + at)^{1+\frac{1}{a}}}{(1 + a)} dt + \frac{a}{1 + a} \int_0^x dt < \int_0^x e^t dt,$$

i.e.,

$$\frac{(1 + ax)^{2+\frac{1}{a}}}{(1 + a)(1 + 2a)} + \frac{(1 + a)(1 + 2a) - 1}{(1 + a)(1 + 2a)} + \frac{ax}{(1 + a)} < e^x. \tag{9}$$

We assume the statement to be true for $n = m > 2$, i.e., the following is true.

$$\begin{aligned} &\frac{(1 + ax)^{m+\frac{1}{a}}}{(1 + a)(1 + 2a) \cdots (1 + ma)} + \frac{(1 + a)(1 + 2a) \cdots (1 + ma) - 1}{(1 + a)(1 + 2a) \cdots (1 + ma)} \\ &\quad + \sum_{k=1}^{m-1} \frac{(1 + a)(1 + 2a) \cdots (1 + ka) - 1}{(m - k)!(1 + a)(1 + 2a) \cdots (1 + ka)} x^{m-k} < e^x. \end{aligned} \tag{10}$$

Now integrating (10) from 0 to x, we get

$$\frac{(1+ax)^{m+1+\frac{1}{a}}}{(1+a)(1+2a)\cdots(1+(m+1)a)}+\frac{(1+a)(1+2a)\cdots(1+(m+1)a)-1}{(1+a)(1+2a)\cdots(1+(m+1)a)}$$
$$+\sum_{k=1}^{m}\frac{(1+a)(1+2a)\cdots(1+ka)-1}{(m+1-k)!(1+a)(1+2a)\cdots(1+ka)}x^{m+1-k}<e^x.$$

This implies that the statement is true for $n=m+1$. By induction we infer that the statement is true for all $n=0,1,2,\cdots$. $\square$

In the proof of above theorem the inequalities (7)–(9) are particular cases and each succeeding inequality is sharper than the preceding one. Next we give generalized upper bounds for e^x in the following proposition.

Proposition 2.2. *Let $n=0,1,2,\cdots$. and $a>0$ be such that $a\neq 1, a\neq 1/2,\cdots,a\neq 1/n$. Then for $x\in[0,1/a)$ we have*

$$e^x\leq\psi(x) \tag{11}$$

where $\psi(x)=\frac{(1-a)(1-2a)\cdots(1-na)-1}{(1-a)(1-2a)\cdots(1-na)}+\frac{1}{(1-a)(1-2a)\cdots(1-na)}\left(\frac{1}{1-ax}\right)^{\frac{1}{a}-n}+\varphi_1(x)$

and

$$\psi_1(x)=\begin{cases}0 & n=0,1\\ \sum_{k=1}^{n-1}\frac{(1-a)(1-2a)\cdots(1-ka)-1}{(n-k)!(1-a)(1-2a)\cdots(1-ka)}x^{n-k} & n=2,3,\cdots.\end{cases}$$

Proof. Equalities hold for $x=0$. If $a>0$ and $x>0$ then for $\log(1-ax)$ to be defined $1-ax$ should be positive. i.e. $x<1/a$. Now, by series expansion we have

$$\log(1-ax)=-ax-\frac{a^2x^2}{2}-\frac{a^3x^3}{3}-\cdots.$$

Therefore, $\log(1-ax)<-ax$ or $1-ax<e^{-ax}$ which yields

$$e^x<\left(\frac{1}{1-ax}\right)^{\frac{1}{a}}. \tag{12}$$

The inequality (12) is the particular case of (11) for $n=0$. This inequality can also be obtained from (1) by making change of variable. Proceeding as in the case of proof of Proposition 2.1, the statement of Proposition 2.2 can be easily proved by induction. $\square$

The particular cases of Proposition 2.2 for $n=1,2$ are respectively given as

$$e^x<\frac{-a}{(1-a)}+\frac{1}{(1-a)}\left(\frac{1}{1-ax}\right)^{\frac{1}{a}-1} \tag{13}$$

and

$$e^x<\frac{(1-a)(1-2a)-1}{(1-a)(1-2a)}+\frac{1}{(1-a)(1-2a)}\left(\frac{1}{1-ax}\right)^{\frac{1}{a}-2}-\frac{ax}{(1-a)} \tag{14}$$

where a and x are as defined in the Proposition 2.2.

Lastly, it is worth noting that the bounds established in this paper are optimal as $a \to 0$, because of the limits

$$\lim_{a\to 0} \phi(x) = \lim_{a\to 0} (1+ax)^{n+\frac{1}{a}} = e^x$$

and

$$\lim_{a\to 0} \psi(x) = \lim_{a\to 0} \frac{1}{(1-a)(1-2a)\cdots(1-na)} \left(\frac{1}{1-ax}\right)^{\frac{1}{a}-n} = e^x.$$

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