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On the singular prescribed scalar curvature problem

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Abstract: Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. In this paper, we define and introduce the prescribed scalar curvature problem with singularities. Under some assumptions, we show that there exists a conformal metric $\overline{g}$ such that its scalar curvature $S_{\overline{g}}$ equals some given function. This problem is equivalent to studying the existence and regularity of the solution to what we will call the singular prescribed scalar curvature equation.

Key words: Yamabe problem, Second order elliptic equation, Variational method, Singular term.

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. In 1960, Yamabe [27] formulated the following problem : Does there exist a metric $\overline{g}$, conformal to g, such that the scalar curvature $S_{\overline{g}}$ is constant. He thought that he had solved this problem but unfortunately eight years later Trudinger [25] pointed out a serious difficulty in the Yamabe's article. The problem is now completely solved and is known as the Yamabe problem. The first step was given by Trudinger who had understood the gap of the Yamabe's proof when the scalar curvature $S_g \geq 0$. The second step was given by Aubin [4] in 1976, he solved the problem for any non locally conformally flat manifolds of dimension $n \geq 6$ and the last step was done by Schoen [23] in 1984. The reader can be refereed to [19] or [18] for more details on the subject. The method to solve the Yamabe problem is the following :

Let $u \in C^{\infty}(M)$, $u > 0$ be a function, the metric $\overline{g} = u^{N-2}g$ is a conformal metric to g where $N = \frac{2n}{n-2}$. Then, we can check out that the scalar curvatures S_g and $S_{\overline{g}}$ are related by the following equation see [18]:

$$\Delta_g u + C_n S_g u = C_n S_{\overline{g}} |u|^{N-2} u \tag{1}$$

where $\Delta_g = -div_g(\nabla_g)$ is the Laplacian-Beltrami operator and $C_n = \frac{n-2}{4(n-1)}$. Put

$$P_g = \Delta_g + C_n S_g.$$

The operator P_g is called the Yamabe operator and solving the Yamabe problem is equivalent to find a smooth positive function u solution of the following equation

$$P_g u = C|u|^{N-2}u \tag{2}$$

where C is a constant. In other words, we prescribe the scalar curvature by putting $C_n S_{\overline{g}} = C$ in (1) and we look for solution u to equation (2), again that is to say we are looking for the metric $\overline{g} = u^{N-2}g$. In order to

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obtain solutions of (2), Yamabe defined the quantity

$$\mu(M,g)=\inf_{u\in H_1^2(M),u\neq 0} Y(u) \tag{3}$$

where the Sobolev space $H_1^2(M)$ is the completion of the space $C^\infty(M)$ with respect to the norm

$$\|u\|_{H_1^2(M)}=(\int_M |\nabla_g u|^2+u^2 dv_g)^{\frac{1}{2}} \tag{4}$$

and

$$Y(u)=\frac{\int_M uP_g u dv_g}{(\int\limits_M u^N dv_g)^{\frac{2}{N}}}=\frac{\int_M |(\nabla_g u|^2+C_n S_g u^2)dv_g}{(\int\limits_M u^N dv_g)^{\frac{2}{N}}}.$$

The constant $\mu(M,g)$ is conformal invariant and it is known as the Yamabe invariant while Y is the Yamabe functional. If we write Euler-Lagrange equation associated to this functional, we will see that critical points of Y are solutions of equation (2). In particular, it follows that if u is positive, smooth and such that $Y(u)=\mu(M,g)$, then u is solution of (2) and $\overline{g}=u^{N-2}\ g$ is the desired metric of constant scalar curvature. To solve the problem, Aubin [4] showed in his works that it is sufficient to prove the following theorem (conjecture):

Theorem 1.1.
Let (M,g) be a compact Riemannian manifold of dimension $n\geq 3$. Assume that the Yamabe invariant

$$\mu(M,g)<K_0^{-2}(n,1),$$

then there exists a positive smooth function u such that $Y(u)=\mu(M,g)$.

Here the constant

$$K_0^2(n,1)=\frac{4}{n(n-2)\omega_n^{\frac{2}{n}}}$$

where ω_n stands for the volume of the unit n-sphere $\mathbb{S}^n\subset\mathbb{R}^{n+1}$. The following theorem is due to Aubin [4] and Schoen [23]:

Theorem 1.2. *Let (M,g) be a compact Riemannian manifold of dimension $n\geq 3$. Then we always have the following large inequality*

$$\mu(M,g)\leq K_0^{-2}(n,1) \tag{5}$$

and we only have equality in this inequality if and only if (M,g) is conformally diffeomorphic to the sphere $\mathbb{S}^n$.

In [21] Aubin and Madani assumed that the metric g satisfied the following assumption :

(H) : the metric $g\in H_2^p(M,T^*M\otimes T^*M)$ where $p>n$ and there exists a point P such that g is smooth in the ball $B(P,\delta)$,

where the space T^*M is the cotangent space of M and $B(P,\delta)$ is the geodesic ball of center P and of radius δ with $0<\delta<\frac{r_g(M)}{2}$ and $r_g(M)$ is the injectivity radius. The space $H_2^p(M,T^*M\otimes T^*M)$ is the space of all sections g $(2-$ covariant tensors) such that in normal coordinates the components g_{ij} of g are in $H_2^p(M)$ where $H_2^p(M)$ is the completion of the space $C^\infty(M)$ with respect to the norm

$$\|u\|_{H_2^p(M)} = \left(\int_M |\nabla_g^2 u|^p + |\nabla_g u|^p + |u|^p dv_g \right)^{\frac{1}{p}}. \tag{6}$$

By Sobolev's embedding, we get that for all $p > n$:

$$H_2^p(M, T^*M \otimes T^*M) \subset C^1(M, T^*M \otimes T^*M) \tag{7}$$

then the Christoffels symbols belong to $H_1^p(M) \subset C^0(M)$, the components of the Riemannian curvature tensor Rm_g, Ricci tensor Ric_g and the scalar curvature S_g are in $L^p(M)$. The assumption (H) allowed authors to introduce the singular Yamabe problem. Moreover, $\mu(M,g)$ is called the singular Yamabe invariant and P_g is the singular Yamabe operator.

From now, we assume that the metric g satisfies the assumption (H), the authors in [21] proved the following results :

Proposition 1.1.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. The operator P_g is weakly conformally invariant and if the singular Yamabe invariant $\mu(M,g) > 0$, P_g is coercive and invertible.

Solving the singular Yamabe problem is reduced to find a positive solution $u \in H_2^p(M)$ to the equation (2).

Theorem 1.3.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. If (M,g) is not conformal to the n-sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$, then there exists a metric $\overline{g} = u^{N-2}g$ conformal to g such that $u \in H_2^p(M)$, $u > 0$ and the scalar curvature $S_{\overline{g}}$ of $\overline{g}$ is constant.

In this paper, we would like to study a more general setting. We are very interested by solving the prescribed scalar curvature problem (under the assumption (H)), in other words we prescribe the scalar curvature by putting $S_{\overline{g}} = f$ where f is some positive $C^\infty(M)$ function on M and we look for the corresponding metric $\overline{g}$ and this is equivalent to study the existence and regularity of solutions to the following partial differential equation :

$$\Delta_g u + C_n S_g u = f|u|^{N-2}u. \tag{8}$$

The above equation (8) is elliptic, nonlinear with critical Sobolev growth and its second coefficient does not have the usual regularity, which allow us to talk about the singular prescribed scalar curvature equation. We also notice that there has been many results for second-order elliptic equations, see [1–10], [12], [14], [16–26] for more information on the subject. Many techniques have been used to solve second-order equations, and variational methods are the most suitable, for more detail about those methods, we refer the reader to [18, 19] and the references therein. [11, 13, 14] concern fourth order elliptic equation, they are cited here for some other methods used in this work.

2. Notations and preliminaries

In this section, we collect some basic facts and definitions which are used in the whole paper. Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. By Sobolev's embedding [18], one gets that

$$H_1^2(M) \subset L^q(M)$$

where $1 < q \leq N$, and this embedding is compact when $q < N$. The number $N = \frac{2n}{n-2}$ is known as the critical exponent of the Sobolev embedding.
The constant $K_0(n,1)$ introduced above is just the best constant in the following Sobolev inequality that asserts that there exists a constant $B > 0$ such that for any $u \in H_1^2(M)$,

$$(\int_M |u|^N dv_g)^{\frac{2}{N}} \leq K_0^2(n,1)\|\nabla_g u\|_2^2 + B\|u\|_2^2. \tag{9}$$

Under the assumption (H) on the metric g, the operator P_g is defined in the weak sense on $H_1^2(M)$, and it is easy to see that it is elliptic and self-adjoint. To obtain solutions of equation (8) we introduce the functional E on $H_1^2(M)$ as follows:

$$E(u) = \int_M |(\nabla_g u|^2 + C_n S_g u^2) dv_g$$

we will use classical variational methods by minimizing this functional. However, serious difficulties appear compared with the smooth case. In order, we define the quantity

$$\mu(M,g) = \inf_{\substack{u \in H \\ u \neq 0}} E(u) \tag{10}$$

where the set

$$H = \{u \in H_1^2(M) \quad \text{such that} \quad \int_M f|u|^N dv_g = 2^{\frac{N}{2}}\}$$

Clearly, the functional E is well defined in $H_1^2(M)$ and is of class C^1 and if we write Euler-Lagrange equation associated to this minimizing problem, we will get equation (8) up to a constant. Now, we state our main results :

Theorem 2.1.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $S_g \in L^p(M)$ where $p > n$, f a positive $C^\infty(M)$ function on M and $P \in M$ such that $f(P) = \sup_{x \in M} f(x)$. If

$$\mu(M,g) < 2(K_0^{-2}(n,1))(f(P))^{-\frac{2}{N}}$$

then, equation (8) has a nontrivial positive weak solution $u \in H_1^2(M)$ such that $E(u) = \mu(M,g)$ and $\int_M f|u|^N dv_g = 1$. Moreover, $u \in C^1(M)$ and $u > 0$, therefore there is a metric $\overline{g} = u^{N-2}g$ such that its scalar curvature $S_{\overline{g}} = f$.

This theorem is regarded as combined results between Theorem (3.1) and (3.2). Our paper is organized as follows : In section 1 and 2, we have just introduced some notations and preliminaries. In section 3, we establish the existence and regularity result to equation (8). Finally in section 4 we get an application.

3. Existence and regularity of the solution

In this section, we establish the existence and regularity result to equation (8).

Theorem 3.1.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $S_g \in L^p(M)$ where $p > n$, f a positive $C^\infty(M)$ function on M and $P \in M$ such that $f(P) = \sup_{x \in M} f(x)$. If

$$\mu(M,g) < 2(K_0^{-2}(n,1))(f(P))^{-\frac{2}{N}}$$

then, equation (8) has a nontrivial positive weak solution $u \in H_1^2(M)$ such that $E(u) = \mu(M,g)$ and $\int_M f|u|^N dv_g = 1$.

Proof.
Firstly, we show that $\mu(M,g)$ is finite. Since $p > \frac{n}{2}$, $S_g \in L^{\frac{n}{2}}(M)$, then for any $u \in H_1^2(M)$ and by Hölder's inequality one has,

$$\begin{aligned} \int_M |C_n S_g| u^2 dv_g &\leq (\int_M |C_n S_g|^{\frac{n}{2}} dv_g)^{\frac{2}{n}} (\int_M u^{\frac{2n}{n-2}} dv_g)^{\frac{n-2}{n}} \\ &\leq \|C_n S_g\|_{\frac{n}{2}} \|u\|_N^2. \end{aligned}$$

From the identity

$$\int_M f|u|^N dv_g = 2^{\frac{N}{2}}$$

and since $f > 0$, one can easily gets

$$\|u\|_N^2 \leq \frac{2}{(\inf_{x \in M} f(x))^{\frac{2}{N}}} \tag{11}$$

then,

$$\int_M |C_n S_g| u^2 dv_g \leq \frac{2\|C_n S_g\|_{\frac{n}{2}}}{(\inf_{x \in M} f(x))^{\frac{2}{N}}}. \tag{12}$$

Therefore, $\forall u \in H_1^2(M)$ one gets

$$\int_M (|\nabla_g u|^2 + C_n S_g u^2) dv_g \geq \int_M C_n S_g u^2 dv_g \geq \frac{-2\|C_n S_g\|_{\frac{n}{2}}}{(\inf_{x \in M} f(x))^{\frac{2}{N}}}.$$

Consequently, $\mu(M,g)$ is finite.
Secondly, we let $(u_m)_m \in H_1^2(M)$ be a minimizing sequence of $\mu(M,g)$, then the sequence u_m is such that

$$\mu(M,g) = \lim_{m \longrightarrow +\infty} \int_M (|\nabla_g u_m|^2 + C_n S_g u_m^2) dv_g, \tag{13}$$

and

$$\int_M f|u_m|^N dv_g = 2^{\frac{N}{2}}.$$

We note that it is easy to see that $(|u_m|)_m$ is also a minimizing sequence, hence, we can assume that $u_m \geq 0$. Thirdly, we are going to show that the sequence $(u_m)_m$ is bounded. For m large enough, we get

$$\int_M (|\nabla_g u_m|^2 + C_n S_g u_m^2) dv_g \leq \mu(M,g) + 1.$$

which implies that

$$\begin{aligned} \int_M |\nabla_g u_m|^2 dv_g &\leq \mu(M,g) + 1 - \int_M C_n S_g u_m^2 dv_g \\ &\leq \mu(M,g) + 1 + \int_M |C_n S_g| u_m^2 dv_g. \end{aligned}$$

By using (12), we get

$$\int_M |\nabla_g u_m|^2 dv_g \leq \mu(M,g) + 1 + \frac{2\|C_n S_g\|_{\frac{n}{2}}}{(\inf_{x\in M} f(x))^{\frac{N}{2}}}.$$

In the other hand, by the embedding $L^N(M) \subset L^2(M)$ and by (11), we get that there exists $c > 0$ such that,

$$\int_M u_m^2 dv_g \leq c\|u_m\|_N^2 \leq \frac{2c}{(\inf_{x\in M} f(x))^{\frac{N}{2}}}$$

this implies in turn that $(u_m)_m$ is bounded in $H_1^2(M)$, and after restriction to a subsequence still labeled $(u_m)_m$, we may assume that there exists $u \in H_1^2(M)$, $u \geq 0$ such that $u_m \longrightarrow u$ weakly in $H_1^2(M)$ and $u_m \longrightarrow u$ strongly in $L^q(M)$ for all $q < N$ and almost everywhere on M.
Putting $\varphi_m = u_m - u$, then, $\varphi_m \longrightarrow 0$ weakly in $H_1^2(M)$ and strongly in $L^q(M)$ for all $q < N$, then for all

m, one gets :

$$\begin{aligned}
\int_M u_m P_g u_m dv_g &= \int_M (\varphi_m + u) P_g u_m dv_g \\
&= \int_M (\varphi_m P_g u_m + v P_g u_m) dv_g \\
&= \int_M (u_m P_g \varphi_m + u_m P_g u) dv_g \\
&= \int_M ((u+\varphi_m) P_g u + (u+\varphi_m) P_g \varphi_m) dv_g \\
&= \int_M u P_g u dv_g + \int_M \varphi_m P_g \varphi_m dv_g + \int_M 2u P_g \varphi_m dv_g \\
&= \int_M u P_g u dv_g + \int_M |\nabla_g \varphi_m|^2 + C_n S_g \varphi_m^2 dv_g \\
&+ \ 2\int_M (\nabla_g u, \nabla_g \varphi_m) + C_n S_g u \varphi_m dv_g.
\end{aligned}$$

Since $(\varphi_m)_m$ goes to 0, weakly in $H_1^2(M)$ and as $S_g u \in L^2(M)$, one gets

$$\int_M (\nabla_g u, \nabla_g \varphi_m) + C_n S_g u \varphi_m dv_g \longrightarrow 0$$

in fact, since $p > n$, $S_g \in L^p(M) \subset L^n(M)$, then

$$\begin{aligned}
\int_M S_g^2 u^2 dv_g &\leq (\int_M S_g^{2\frac{N}{N-2}} dv_g)^{\frac{N-2}{N}} (\int_M u^{2\frac{N}{2}} dv_g)^{\frac{2}{N}} \\
&\leq (\int_M S_g^n dv_g)^{\frac{N-2}{N}} (\int_M u^N dv_g)^{\frac{2}{N}} < +\infty.
\end{aligned}$$

From the strong convergence of $(\varphi_m)_m$ to 0 in $H_1^2(M)$, we get

$$\int_M C_n S_g \varphi_m^2 dv_g \longrightarrow 0$$

therefore

$$\int_M u_m P_g u_m dv_g = \int_M u P_g u dv_g + \int_M |\nabla_g \varphi_m|^2 dv_g + o(1)$$

that is to say

$$\int_M u_m P_g u_m dv_g = \int_M u P_g u dv_g + \|\nabla_g \varphi_m\|_2^2 + o(1).$$

The definition of $\mu(M,g)$ implies that

$$\int_M uP_g u dv_g \geq \mu(M,g)(\int_M f|u^N|dv_g)^{\frac{2}{N}}$$

then

$$\int_M u_m P_g u_m dv_g \geq \mu(M,g)(\int_M f|u^N|dv_g)^{\frac{2}{N}} + \|\nabla_g \varphi_m\|_2^2 + o(1).$$

Again (13) means

$$\int_M u_m P_g u_m dv_g = \mu(M,g) + o(1)$$

which implies that

$$\mu(M,g) + o(1) \geq \mu(M,g)(\int_M f|u|^N dv_g)^{\frac{2}{N}} + \|\nabla_g \varphi_m\|_2^2 + o(1). \tag{14}$$

On the other hand, by Brezis-Lieb lemma applying to $(u_m)_m$, one has

$$2^{\frac{N}{2}} = \int_M f|u_m|^N dv_g = \int_M f|u|^N dv_g + \int_M f|\varphi_m|^N dv_g + o(1)$$

which implies,

$$2 \leq (\int_M f|u|^N dv_g)^{\frac{2}{N}} + (\int_M f|\varphi_m|^N dv_g)^{\frac{2}{N}} + o(1),$$

and since the inequality (14) can be written as

$$\|\nabla_g \varphi_m\|_2^2 \leq \mu(M,g)\left(1 - (\int_M f|u|^N dv_g)^{\frac{2}{N}}\right) + o(1),$$

one has,

$$2\|\nabla_g \varphi_m\|_2^2 \leq \mu(M,g)\left(2 - 2(\int_M f|u|^N dv_g)^{\frac{2}{N}}\right) + o(1),$$

then it follows that,

$$\begin{aligned} 2\|\nabla_g \varphi_m\|_2^2 &\leq \mu(M,g)\left((\int_M f|u|^N dv_g)^{\frac{2}{N}} + (\int_M f|\varphi_m|^N dv_g)^{\frac{2}{N}}\right) \\ &- 2\mu(M,g)\left((\int_M f|u|^N dv_g)^{\frac{2}{N}}\right) + o(1) \\ &\leq \mu(M,g)\left((\int_M f|\varphi_m|^N dv_g)^{\frac{2}{N}} - \int_M f|u|^N dv_g)^{\frac{2}{N}}\right) + o(1) \end{aligned}$$

we then get

$$2\|\nabla_g \varphi_m\|_2^2 \quad \leq \quad \mu(M,g)(\int_M f|\varphi_m|^N dv_g)^{\frac{2}{N}} + o(1).$$

By using Sobolev's inequality (9) in the right-hand side, the strong convergence of φ_m in $L^2(M)$ and since f is smooth on M, one has

$$2\|\nabla_g \varphi_m\|_2^2 \leq \mu(M,g)(\sup_{x\in M} f(x))^{\frac{2}{N}} K_0^2(n,1)\|\nabla_g \varphi_m\|_2^2 + o(1).$$

that is to say,

$$\left(2 - \mu(M,g)(f(P))^{\frac{2}{N}} K_0^2(n,1)\right)\|\nabla_g \varphi_m\|_2^2 \leq o(1).$$

Now, since

$$\mu(M,g) < 2(K_0^{-2}(n,1))(f(P))^{-\frac{2}{N}}$$

one finds,

$$\|\nabla_g \varphi_m\|_2^2 = o(1)$$

Consequently φ_m converges strongly to 0 in $H_1^2(M)$ which implies that u_m converges strongly to u in $H_1^2(M)$ and in $L^N(M)$. It follows that,

$$\lim \int_M f|u_m - u|^N dv_g = 0$$

which necessarily leads to

$$\int_M f|u|^N dv_g = 2^{\frac{N}{2}}$$

and since $f > 0$, u is a nontrivial. Finally, writing the Euler-Lagrange equation associated, one sees that u is a positive weak solution of (8). □

In [21], Madani proved a regularity result with f a constant function through an adaptation of Trudinger's regularity theorem [25]. In order, to study the regularity of our solution, we will follow the same procedure, and despite the presence of the non constant function f adds other difficulties, we can easily get the regularity of solution to equation (8). This result is formulated in the following theorem.

Theorem 3.2. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Let $S_g \in L^p(M)$ with $p > n$ and f be a positive $C^\infty(M)$ function on M. If $u \in H_1^2(M)$ is a nontrivial positive weak solution of*

$$\Delta_g u + C_n S_g u = f u^{N-1}, \tag{15}$$

then $u \in H_2^p(M) \subset C^1(M)$ and $u > 0$.

Proof.
To prove this theorem, it suffices to show that $u \in L^{N+\epsilon}(M)$ for some $\epsilon > 0$. Indeed, u satisfies the equation

$$\Delta_g u + (C_n S_g - f u^{N-2})u = 0,$$

and if $u \in L^{N+\epsilon}(M)$, the singular term $C_n S_g - f u^{N-2}$ must be in $L^r(M)$ where $r = \min(p, n+\epsilon) > n$, hence one has $\Delta_g u \in L^p(M)$ and by classical regularity theorem, we deduce that $u \in H_2^p(M)$.
Now we are going to show that $u \in L^{N+\epsilon}(M)$. As in Trudinger strategy, we let $l > 0$ be a real number and H, F be two continuous functions on $\mathbb{R}^+$ given by

$$H(x) = \begin{cases} t^\gamma & \text{if} \quad 0 \le t \le l \\ l^{q-1}(q l^{q-1} t - (q-1) l^q) & \text{if} \quad t > l \end{cases}$$

and

$$F(x) = \begin{cases} t^q & \text{if} \quad 0 \le t \le l \\ q l^{q-1} t - (q-1) l^q & \text{if} \quad t > l \end{cases}$$

where $\gamma = 2q - 1$ and $1 < q < \frac{n(p-1)}{n(p-2)}$. Since $u \ge 0$ and $u \in H_1^2(M)$, then it follows that $H \circ u$, $F \circ u$ are both in $H_1^2(M)$,

$$qH(t) = F(t)F'(t), \quad (F'(t))^2 \le qH'(t) \quad \text{and} \quad F^2(t) \ge tH(t). \tag{16}$$

Let u be a weak solution of (15), then for all $v \in H_1^2(M)$ one has,

$$\int_M \nabla_g u \nabla_g v dv_g + \int_M C_n S_g u v dv_g = \int_M f u^{N-1} v dv_g. \tag{17}$$

Now, as in the section 2, we define a cut-off function $\eta \in C^1(M)$ such that

$$\eta(x) = \begin{cases} 1 & \text{on} \quad B(P, \delta) \\ 0 & \text{on} \quad M - B(P, 2\delta) \end{cases}$$

Chosen, $v = \eta^2 H \circ u$, and plugging this function in (17), we get

$$\int_M \eta^2 H' \circ u |\nabla_g u|^2 dv_g + 2 \int_M \eta H \circ u \nabla_g u \nabla_g \eta dv_g = \int_M f u^{N-1} \eta^2 H \circ u dv_g - \int_M C_n S_g u \eta^2 H \circ u dv_g. \tag{18}$$

We put $h = F \circ u$ and let us evaluate each of the above integrals by using this function and the formulae (16). We have $\nabla_g h = F' \circ u \nabla_g u$ and by applying the second relationship of (16), one gets

$$|\nabla_g h|^2 = (F' \circ u)^2 |\nabla_g u|^2 \le q H' \circ u |\nabla_g u|^2,$$

we deduce that the first integral of (18) is bounded then it follows that,

$$\frac{1}{q} \|\eta \nabla_g h\|_2^2 \le \int_M \eta^2 H' \circ u |\nabla_g u|^2 dv_g.$$

The first relationship of (16) and the Cauchy-Schwartz inequality implies that the second integral of (18) is also bounded, therefore

$$2\int_M \eta H\circ u\nabla_g u\nabla_g\eta dv_g = \frac{2}{q}\int_M \eta h\nabla_g h\nabla_g\eta dv_g \geq \frac{-2}{q}\|h\nabla_g\eta\|_2\|\eta\nabla_g h\|_2.$$

By using the latter relationship of (16), one has $uH\circ u\leq h^2$ and along the same lines the two integrals of the right-hand side member in (18) will also be bounded, thus

$$|\int_M fu^{N-1}\eta^2 H\circ udv_g - \int_M C_nS_gu\eta^2H\circ udv_g| \leq (\sup_{x\in M} f(x))\|u\|_{N,2\delta}^{\frac{4}{n-2}}\|\eta h\|_N^2 + \|C_nS_g\|_p\|\eta h\|^2_{\frac{2p}{p-1}}.$$

where $\|u\|_{N,r}^N = \int_{B(P,r)} u^N dv_g$. After grouping these estimates together, the equality (18) becomes

$$\|\eta\nabla_g h\|_2^2 - 2\|h\nabla_g\eta\|_2\|\eta\nabla_g \overline{f}\|_2 \leq q[(\sup_{x\in M} f(x))\|u\|_{N,2\delta}^{\frac{4}{n-2}}\|\eta h\|_N^2 + \|C_nS_g\|_p\|\eta h\|^2_{\frac{2p}{p-1}}]. \tag{19}$$

Now, let a_1, b_1, c_1 and d_1 be real numbers such that ${a_1}^2 - 2a_1b \leq {c_1}^2 + {d_1}^2$, we easily obtain that $a_1 \leq 2b_1 + c_1 + d_1$, then (19) becomes

$$\|\eta\nabla_g h\|_2 \leq \sqrt{q\sup_{x\in M} f(x)}\|u\|_{N,2\delta}^{\frac{2}{n-2}}\|\eta h\|_N + \sqrt{q\|C_nS_g\|_p}\|\eta h\|_{\frac{2p}{p-1}} + 2\|h\nabla_g\eta\|_2. \tag{20}$$

By Sobolev's embedding, we then get that there exists a constant $c>0$ depending only on n such that

$$\|\eta h\|_N \leq c(\|\eta\nabla_g h\|_2 + \|h\nabla_g\eta\|_2 + \|h\eta\|_2.$$

Since $q<N$, and after using (20), we obtain

$$(1 - c\sqrt{N\sup_{x\in M} f(x)}\|u\|_{N,2\delta}^{\frac{2}{n-2}})\|\eta h\|_N \leq c(\sqrt{N\|C_nS_g\|_p}\|\eta h\|_{\frac{2p}{p-1}} + 3\|h\nabla_g\eta\|_2 + \|h\eta\|_2),$$

for δ sufficiently small, one has

$$\|u\|_{N,2\delta}^{\frac{2}{n-2}} \leq \frac{1}{2c\sqrt{N\sup_{x\in M} f(x)}}.$$

When l goes to $+\infty$, we then get that there exists a constant $C>0$ depending only on $n, \delta, \|\eta\|_\infty, \|\nabla_g\eta\|_\infty\|C_nS_g\|_p$ and f such that

$$\|u^q\|_{N,2\delta} \leq C(\|u^q\|_2 + \|u^q\|_{\frac{2p}{p-1}}).$$

The boundedness of u in $L^N(M)$ and as $\frac{2p}{p-1}q < N$ mean again that

$$\|u^q\|_{qN,2\delta} \leq C.$$

Since M is compact, it can be covered by a finite number of balls $\{B(P_i,\delta)\}_{i\in I}$ and let $(\eta_i)_{i\in I}$ be a partition of unity subordinated to the covering, then

$$\|u\|_{qN}^{qN} = \sum_{i\in I}\|\eta_i u\|_{qN,\delta_i}^{qN} \leq C.$$

it follows that, $u\in L^{qN}(M)$ with $qN > N$. □

4. Application

Corollary 4.1.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $S_g \in L^p(M)$ where $p > n$, f a positive $C^\infty(M)$ function on M and $P \in M$ such that $f(P) = \sup_{x \in M} f(x)$. If

$$\mu(M,g) < 2(K_0^{-2}(n,1))(f(P))^{-\frac{2}{N}}$$

Then there exists a metric $\overline{g} = u^{N-2}g$ conformal to g such that the scalar curvature $S_{\overline{g}} = f$.

Proof. As in the section 1, the singular Yamabe operator $P_g = \Delta_g + C_n S_g$ is weakly comformally invariant, and by Theorem (3.1) and (3.2), there exists $u \in C^1(M)$, $u > 0$ solution of the following equation

$$\Delta_g u + C_n S_g u = f|u|^{N-2}u.$$

On the other hand, by the weak conformal invariance of P_g and if $\overline{g} = u^{N-2}g$ is conformal to g, one has

$$\Delta_g u + C_n S_g u = C_n S_{\overline{g}}|u|^{N-2}u.$$

then, we deduce that the metric $\overline{g} = u^{N-2}g$ is such that its scalar curvature $S_{\overline{g}} = \frac{f}{C_n}$. □

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