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Characteristic equation of a matrix via Bell polynomials

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Abstract

We show that the coefficients of the characteristic equation of any matrix A_nxn can be written in terms of the complete Bell polynomials.

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Introduction

For an arbitrary matrix $\mathbf{A}_{nxn} = (\mathrm{A}^i{}_j)$ its characteristic equation [1-3]:

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0, \tag{1}$$

can be obtained, through several procedures [1, 4-7], directly from the condition det $(\mathrm{A}^i{}_j - \lambda\delta^i_j) = 0$. The approach of Leverrier-Takeno [4, 8-13] is a simple and interesting technique to construct (1) based in the traces of the powers $\mathbf{A}^r$, $r = 1, \ldots, n$. In fact, if we define the quantities:

$$\mathrm{a}_0 = 1, \qquad \mathrm{s}_\mathrm{k} = \mathrm{tr}\,\mathrm{A}^\mathrm{k}, \qquad \mathrm{k} = 1, 2, \ldots, \mathrm{n}, \tag{2}$$

then the process of Leverrier-Takeno implies (1) wherein the a_i are determined with the recurrence relation:

$$\mathrm{r}\,\mathrm{a}_\mathrm{r} + \mathrm{s}_1\,\mathrm{a}_{\mathrm{r}-1} + \mathrm{s}_2\,\mathrm{a}_{\mathrm{r}-2} + \ldots + \mathrm{s}_{\mathrm{r}-1}\,\mathrm{a}_1 + \mathrm{s}_\mathrm{r} = 0, \qquad \mathrm{r} = 1, 2, \ldots, \mathrm{n}, \tag{3}$$

therefore:

$$\begin{gathered} \mathrm{a}_1 = -\mathrm{s}_1\,, \qquad 2!\ \mathrm{a}_2 = (\mathrm{s}_1)^2 - \mathrm{s}_2\,, \qquad 3!\ \mathrm{a}_3 = -(\mathrm{s}_1)^3 + 3\,\mathrm{s}_1\,\mathrm{s}_2 - 2\,\mathrm{s}_3\,, \\ 4!\ \mathrm{a}_4 = (\mathrm{s}_1)^4 - 6\,(\mathrm{s}_1)^2\,\mathrm{s}_2 + 8\,\mathrm{s}_1\,\mathrm{s}_3 + 3\,(\mathrm{s}_2)^2 - 6\,\mathrm{s}_4\,, \\ 5!\ \mathrm{a}_5 = -(\mathrm{s}_1)^5 + 10\,(\mathrm{s}_1)^3\,\mathrm{s}_2 - 20\,(\mathrm{s}_1)^2\,\mathrm{s}_3 - 15\,\mathrm{s}_1(\mathrm{s}_2)^2 + 30\,\mathrm{s}_1\,\mathrm{s}_4 + 20\,\mathrm{s}_2\,\mathrm{s}_3 - 24\,\mathrm{s}_5, \ \ldots \end{gathered} \tag{4}$$

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in particular, det **A** = $(-1)^n\, a_n$, that is, the determinant of any matrix only depends on the traces s_r , which means that **A** and its transpose have the same determinant.

In [14-16] we find the general expression:

$$a_m = \frac{(-1)^m}{m!}\begin{vmatrix} s_1 & s_2 & s_3 & \cdots & s_{m-1} & s_m \\ m-1 & s_1 & s_2 & \cdots & s_{m-2} & s_{m-1} \\ 0 & m-2 & s_1 & \cdots & s_{m-3} & s_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & s_1 \end{vmatrix}, \qquad m = 1, \dots, n\,. \tag{5}$$

which allows reproduce the expressions (4). In Sec. 2 we show that the formula (5) permits relate the coefficients of the characteristic equation (1) with the complete Bell polynomials [17-21].

2 Bell polynomials

In [22, 23] we find the following expression for the Bell polynomials:

$$Y_m(x_1, x_2, \dots, x_m) = \begin{vmatrix} \binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & \cdots & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\ -1 & \binom{m-2}{0}x_1 & \cdots & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\ 0 & -1 & \cdots & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\ 0 & 0 & \cdots & -1 & \binom{0}{0}x_1 \end{vmatrix}, \tag{6}$$

therefore:

$$Y_0 = 1,\;\; Y_1 = x_1,\;\; Y_2 = x_1^2 + x_2,\;\; Y_3 = x_1^3 + 3x_1\, x_2 + x_3\,,\;\;\; Y_4 = x_1^4 + 6\, x_1^2\, x_2 + 4\, x_1\, x_3 + 3\, x_2^2 + x_4, \tag{7}$$

$$Y_5 = x_1^5 + 10\, x_1^3\, x_2 + 10\, x_1^2\, x_3 + 15\, x_1\, x_2^2 + 5\, x_1\, x_4 + 10\, x_2\, x_3 + x_5,\;\; \dots$$

We see that with (7) we can deduce (4) if we employ $x_1 = -s_1,\;\; x_2 = -s_2,\;\; x_3 = -2\, s_3,\;\; x_4 = -6\, s_4\,,\;\; x_5 = -24\, s_5, \dots$, that is:

$$a_m = \frac{1}{m!}\, Y_m(-0!\; s_1, -1!\; s_2, -2!\, s_3, -3!\; s_4, \dots, -(m-2)!\; s_{m-1}, -(m-1)!\; s_m). \tag{8}$$

In fact, it is simple to prove that (6) with $x_k = -(k-1)!\ s_k$ implies (5), thus the coefficients of the characteristic equation (1) are generated by the complete Bell polynomials [17-23].

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