Asia Mathematika
Volume: 10 Issue: 1 (2026) Pages: 1 – 11
Available online at **www.asiamath.org**

On the possibility of IF structures

T. Witczak *
Institute of Mathematics, University of Silesia
Bankowa 14, 40-007, Katowice, Poland.

Received: 10 Jan 2026 • **Accepted:** 04 May 2026 • **Published Online:** 30 May 2026

Abstract: In this paper we investigate the idea of IF pseudo-intuitionistic sets. We combine two possible approaches to the problem of division of the initial universe of discourse into three mutually disjoint parts.

First, we use ideals to model "small" (possibly empty) intersection between the ranges of accepted and rejected objects. In this way we generalize Çoker's intuitionistic sets (implementing the same algebraic operations of union, intersection and complement).

Second, we use filters to describe the fact that the union of both components is "big" (but not necessarily identical with the whole universe). Thus, our structures consist of a non-empty universe X together with some fixed ideal and fixed filter (on X). In this setting, we introduce our IF pseudo-intuitionistic sets and we analyze their algebraic properties. We implement the notion of point and we propose several possible directions of future research.

Key words: Possibly paraconsistent sets, intuitionistic sets, neutrosophic crisp sets, ideals, filters, IF pseudo-intuitionistic sets

1. Introduction

We introduced *possibly paraconsistent sets* in [17]. We showed that they are isomorphic with so-called intuitionistic sets (in the sense of Çoker) and weak rough sets (known as double or flou sets too). Hence, they form an appropriate description of the situation in which our initial universe of discourse is divided into three mutually disjoint parts.

In case of *weak rough* sets (see [19]) these parts are: A_1, consisting of those elements that are necessary; A_2 that is a superset of A_1 and gathers those objects that are possible; and A_2^c that describes those elements that remain without evaluation. Note that weak rough sets are known as interval-valued sets too (see [11]).

In the setting of *intuitionistic* sets we have: A_T (where we collect those elements that are "true" or accepted); A_F that is separate from A_T and covers "false" (rejected) objects; and $(A_T \cup A_F)^c$ consisting of neutral elements. As for the comprehensive study of intuitionistic topological spaces, the reader should check [10]. Moreover, Çoker's sets can be associated with the idea of intuitionistic fuzzy sets (first defined and studied by Atanassov in [2]).

Finally, in the framework of *possibly paraconsistent* sets we have: A (these are strictly accepted elements), $A^{\sim}$ which is a superset of A^c and stores the rest of elements. In particular, in $A \cap A^{\sim}$ we have questionable objects.

Obviously, in the last setting $A \cup A^{\sim} = X$. But this condition can be relaxed without any change in the definition of union, intersection and complement. Additionally, we can impose some simple condition on the

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©Asia Mathematika, DOI: 10.5281/zenodo.20982168
*Correspondence: tm.witczak@gmail.com

intersection of A and $A^{\sim}$ (and this condition will not spoil the operators in question). The crucial thing is to assume that the union of both components belongs to some fixed filter on X, while the intersection belongs to a fixed ideal.

This environment is a modification of the previous one that was presented in [18]. We added filters and we eliminated the third component A_3 (such that $(A_1 \cup A_2 \subseteq A_3)$). The second change is motivated by the fact that we wanted to reduce the level of complexity of the whole project and to concentrate on ideals and filters.

In general, all the concepts mentioned above can be treated as "crisp" approaches to the problem of uncertainty. There are other solutions of this kind. For example, in [7] and [8] we presented *picture sets*. They are a crisp version of picture fuzzy sets (that were studied by Cuong in [6]). Each picture set on a non-empty universe X can be defined as an ordered triple of the form (A_1, A_2, A_3) where the only requirement is that $A_1 \cap A_3 = \emptyset$.

Picture fuzzy approach evolved in various directions: for example, some authors study picture fuzzy ideals in the context of rings (see [15]).

In [16] we can find *triple sets*: they form a bridge between weak rough sets and Çoker's sets. Each object of this kind is an ordered triple of the form (A_1, A_2, A_3), where $A_1 \subseteq A_2$ and $A_2 \cap A_3 = \emptyset$. These sets were recognized independently in [1] under the name of NCT sets.

However, picture and triple sets are just subclasses of *neutrosophic crisp sets* that were introduced and analyzed e.g. in [12] and [13].

Even more complex structures are analyzed: in [3] we have interval-valued intuitionistic sets (together with their topology). In [9] the authors defined interval-valued neutrosophic crisp sets.

2. Basic notions

In this section we define several basic notions that will be widely used throughout the whole paper. Clearly, the first two definitions belong to the mathematical folklore.

Definition 2.1. Let X be a non-empty universe. Suppose that $\mathcal{F}$ is a non-empty family of subsets of X, i.e. $\mathcal{F} \subseteq P(X)$. Assume that the following conditions are satisfied:

1. If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$ (closure under supersets).
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ (closure under finite intersections).

Then we say that $\mathcal{F}$ is a *filter* on X. It is *proper* if $\emptyset \notin \mathcal{F}$ (which means that $\mathcal{F} \neq P(X)$).

A filter is *principal* if it contains only the supersets of some fixed $A \subseteq X$. In such a case, we can say that $\mathcal{F}$ is *generated* by A.

Definition 2.2. Let X be a non-empty universe. Suppose that $\mathcal{I}$ is a non-empty family of subsets of X, i.e. $\mathcal{I} \subseteq P(X)$. Assume that the following conditions are satisfied:

1. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (closure under subsets).
2. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (closure under finite unions).

Then we say that $\mathcal{I}$ is an *ideal* on X. It is *proper* if $X \notin \mathcal{I}$ (which means that $\mathcal{I} \neq P(X)$).

Definition 2.3. Let X be a non-empty universe. Assume that $\mathcal{I}$ is an ideal on X and $\mathcal{F}$ is a filter on X. Then the ordered triple of the form $(X, \mathcal{I}, \mathcal{F})$ will be called an *IF structure* (on X, that is, with X as its *support*).

Definition 2.4. Suppose that $(X, \mathcal{I}, \mathcal{F})$ is an IF structure on X. Assume that A and $A^{\sim}$ are two subsets of X such that $A \cup A^{\sim} \in \mathcal{F}$ and $A \cap A^{\sim} \in \mathcal{I}$. Then the ordered pair $\mathcal{A} = (A, A^{\sim})$ will be called an *IF pseudo-intuitionistic set* on X (that is, an *IFpi* set on X).

Example 2.1. *Suppose that $X = \{a, b, c, d, e, f, g, h\}$ and $\mathcal{F}$ is a principal filter generated by $\{a, b, c, d\}$. Assume that $\mathcal{I} = \{\emptyset, \{b\}\}$. Clearly, $\mathcal{I}$ is an ideal on X.*

Take $A = \{a, b\}$ and $A^{\sim} = \{b, c, d, f\}$. Then $\mathcal{A} = (A, A^{\sim})$ is an example of IFpi set on X because $A \cup A^{\sim} = \{a, b, c, d, f\} \in \mathcal{F}$ and $A \cap A^{\sim} = \{b\} \in \mathcal{I}$. Another example is $\mathcal{B} = (B, B^{\sim})$, where $B = \{a, c\}$ and $B^{\sim} = \{b, d, e, f\}$. Now consider $\mathcal{C} = (\{a, b\}, \{f\})$. This object is not an IFpi set on X with respect to $\mathcal{I}$ and $\mathcal{F}$ defined above. The intersection of both components is empty (hence it belongs to $\mathcal{I}$) but their union does not belong to $\mathcal{F}$.

Example 2.2. *Suppose that $X = \mathbb{N} = \{0, 1, 2, 3, ...\}$. Let $\mathcal{F}$ be the set of all cofinite subsets of X, i.e. those sets whose complement is finite. Let $\mathcal{I} = \{\emptyset, \{7\}, \{8\}, \{7, 8\}\}$.*

Suppose that $A = \{5, 6, 7, 10, 12, 14, 16, 18, ...\}$ and $A^{\sim} = \{7, 8, 9, 11, 13, 15, 17, ...\}$. Then the complement of $A \cup A^{\sim}$ is $\{0, 1, 2, 3, 4\}$ and it is finite. Hence, $A \cup A^{\sim} \in \mathcal{F}$. On the other hand, $A \cap A^{\sim} = \{7\} \in \mathcal{I}$. Hence, $\mathcal{A} = (A, A^{\sim})$ is an IFpi set on X.

Example 2.3. *Let X be an arbitrary set. Suppose that $\mathcal{F} = \{X\}$ and $\mathcal{I} = \{\emptyset\}$. Now the only properly defined IFpi sets are those that satisfy the following two conditions: that $A \cup A^{\sim} = X$ and $A \cap A^{\sim} = \emptyset$. Hence, they are ordered pairs of the form (A, A^{c}).*

For example, let $X = \{p, q, r, s, t, u, v, w\}$, $\mathcal{F} = \{X\}$ and $\mathcal{I} = \{\emptyset\}$. Then we can define $\mathcal{A} = (\{p, q, r\}, \{s, t, u, v, w\})$ or $\mathcal{B} = (\{p, r, t, v\}, \{q, s, u, w\})$.

Example 2.4. *Let $X = \{a, b, c, d\}$. Suppose that $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, while $\mathcal{F} = \{X, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}$. We can say that $\mathcal{F}$ is a* dual *of $\mathcal{I}$ (it consists of complements of the elements of $\mathcal{I}$). For example, $\mathcal{A} = (\{a, b\}, \{b, c, d\})$ is an IFpi set on X. The same can be said about $\mathcal{B} = (\{a\}, \{c, d\})$.*

Remark 2.1. *Let $(X, \mathcal{I}, \mathcal{F})$ be an IF structure and suppose that $\mathcal{A}$ is an IFpi set on X. Let $M = A \cap A^{\sim}$. Now assume that $P_1 = A \setminus M$, $P_2 = M$ and $P_3 = A^{\sim} \setminus M$. Then the ordered triple (P_1, P_2, P_3) is a picture set in the sense of [7].*

2.1. Explanation and justification

One could say that in IFpi sets we combine and (at the same time) generalize certain properties of both Çoker's intuitionistic sets and possibly paraconsistent sets. Basically, the idea of the former is that both components are strictly separate. However, we can imagine that in some cases it is enough to assume that the overlap of the two components is "small" (but not necessarily empty). Then the collection of these objects that are uncertain, paraconsistent (or temporarily of unclear status) is somewhat "negligible".

Of course the fact that ideals gather "small" sets is best visible in the context of infinite universes. This aspect is not so clear when our universe is finite (and thus, all its subsets and families of subsets are finite).

However, even then we can notice that the intersection of "accepted" and "rejected" is bounded by some specific set. This is because in finite universe X every ideal can be interpreted as the powerset of some $B \subseteq X$.

In a similar manner, we say that filters gather those sets that are somewhat "big" (but not necessarily equal to the whole universe). These elements of our universe of discourse that are "accepted" (true) or "rejected" (false) are *evaluated*. Hence, the set of evaluated objects is "big", while its complement is "small" and thus negligible.

The conclusion is that our system describes the following situation. First, we divided our universe of discourse into two parts: evaluated and unevaluated objects. The amount of unevaluated objects is "small" (at least theoretically), while the amount of evaluated objects is "big". Second, the evaluated objects can be accepted, rejected or paraconsistent. However, the amount of paraconsistent objects is "small".

As we know, *negligible sets* in mathematics are negligible from some point of view. This context can vary, depending on the framework and applications. For example, "small" can mean finite sets (in contrast to the infinite ones). It can also mean "measure zero" (in contrast to the positive measure). Hence, our framework is very general. In practice, Çoker's intuitionistic sets and similar structures are often considered in the context of finite universes. Obviously, it is not obligatory. Thus, we open certain more theoretical ways of thought. Perhaps the most fruitful applications of our approach will be those utilizing infinite (in particular, uncountable) universes.

3. Algebraic operations

Now let us define several algebraic operations involving IFpi sets.

3.1. Union and intersection

The most natural thing is to reproduce the general concept of logical disjunction and conjunction in our new setting.

Definition 3.1. Suppose that $X \neq \emptyset$ and $\mathcal{F}$ is a filter on X, while $\mathcal{I}$ is an ideal on X. Assume that $\mathcal{A}$, $\mathcal{B}$ are two IFpi sets on X. Then we define their:

1. *(Standard) union*: $\mathcal{A} \cup \mathcal{B} = (A \cup B, A^{\sim} \cap B^{\sim})$.
2. *(Standard) intersection*: $\mathcal{A} \cap \mathcal{B} = (A \cap B, A^{\sim} \cup B^{\sim})$.

Lemma 3.1. *All the operations defined in Def. 3.1 return IFpi sets.*

Proof. : Let us concentrate on the first case. We shall use classical distributivity laws.

1. First, we have $(A \cup B) \cup (A^{\sim} \cap B^{\sim}) = (A \cup B \cup A^{\sim}) \cap (A \cup B \cup B^{\sim})$. Clearly, $A \cup A^{\sim} \in \mathcal{F}$ and $A \cup A^{\sim} \subseteq A \cup A^{\sim} \cup B$. Hence $A \cup A^{\sim} \cup B \in \mathcal{F}$ (due to the fact that filters are closed under supersets). In the same way, we prove that $A \cup B \cup B^{\sim} \in \mathcal{F}$. But then the intersection of both these components belongs to $\mathcal{F}$ too.
   Second, we have $(A \cup B) \cap (A^{\sim} \cap B^{\sim}) = (A \cap A^{\sim} \cap B^{\sim}) \cup (B \cap A^{\sim} \cap B^{\sim})$.
   By the assumption, $A \cap A^{\sim} \in \mathcal{I}$ and $B \cap B^{\sim} \in \mathcal{I}$. Clearly, $A \cap A^{\sim} \cap B^{\sim} \subseteq A \cap A^{\sim}$, so $A \cap A^{\sim} \cap B^{\sim} \in \mathcal{I}$ (because ideals are closed under subsets). In the same way we prove that $B \cap A^{\sim} \cap B^{\sim} \in \mathcal{I}$. But then the union of both these components belongs to $\mathcal{I}$.

2. Similar to the first case. For example, think about $(A \cap B) \cap (A^{\sim} \cup B^{\sim})$. It is $(A \cap B \cap A^{\sim}) \cup (A \cap B \cap B^{\sim})$. Now it is clear that both components belong to $\mathcal{I}$, hence the same can be said about their union.

□

Remark 3.1. *Consider the following two hypothetical operations:*

1. Strong union*:* $\mathcal{A} \vee \mathcal{B} = (A \cup B, A^{\sim} \cup B^{\sim})$.
2. Strong intersection*:* $\mathcal{A} \wedge \mathcal{B} = (A \cap B, A^{\sim} \cap B^{\sim})$.

The problem is that they do not necessarily return IFpi sets. For example, take $X = \{a, b, c, d, e, f\}$ *with* $\mathcal{I} = \{\emptyset\}$ *and* $\mathcal{F} = \{X\}$. *Now both* $\mathcal{A} = (\{a, b\}, \{c, d, e, f\})$ *and* $\mathcal{B} = (\{a, c\}, \{b, d, e, f\})$ *are IFpi sets. However,* $\mathcal{A} \vee \mathcal{B} = (\{a, b, c\}, \{b, c, d, e, f\})$. *Clearly,* $\{a, b, c\} \cap \{b, c, d, e, f\} = \{b, c\} \notin \{\emptyset\} = \mathcal{I}$.

In a similar manner, we can prove that "strong intersection" does not guarantee that the resulting object will satisfy the conditions of IPpi set.

However, if our $\mathcal{I}$ *is improper (namely, equal to* $P(X)$*), then we can use strong union. Check this:* $(A \cup B) \cup (A^{\sim} \cup B^{\sim}) \in \mathcal{F}$ *(no matter what is the exact structure of* $\mathcal{F}$*) just because* $A \cup A^{\sim} \in \mathcal{F}$, $B \cup B^{\sim} \in \mathcal{F}$ *and filters are upwards closed. On the other hand, it will be always true that* $(A \cup B) \cap (A^{\sim} \cup B^{\sim}) \in P(X) = \mathcal{I}$.

Analogously, if $\mathcal{F} = P(X)$*, then we can use strong intersection.*

3.2. About inclusion

Now we should define the notion of inclusion.

Definition 3.2. Let X be a non-empty universe. Assume that $\mathcal{I}$ is an ideal on X, while $\mathcal{F}$ is a filter on X. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two IFpi sets on X. We say that $\mathcal{A}$ is *contained* in $\mathcal{B}$ (and we write $\mathcal{A} \subseteq \mathcal{B}$) if and only if $A \subseteq B$ and $B^{\sim} \subseteq A^{\sim}$.

Lemma 3.2. *Suppose that* X *is a non-empty universe together with some ideal* $\mathcal{I}$ *and some filter* $\mathcal{F}$. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two IFpi sets on* X. *Then the following conditions are equivalent:*

1. $\mathcal{A} \subseteq \mathcal{B}$.
2. $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$.
3. $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$.

Proof. (sketch):

For example, assume that $\mathcal{A} \subseteq \mathcal{B}$. Then consider $\mathcal{A} \cup \mathcal{B} = (A \cup B, A^{\sim} \cap B^{\sim})$. We know that $A \subseteq B$, so $A \cup B = B$ and we know that $B^{\sim} \subseteq A^{\sim}$, so $A^{\sim} \cap B^{\sim} = B^{\sim}$. But then the resulting set is exactly $(B, B^{\sim}) = \mathcal{B}$.

Now suppose that $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$. This means that $A \cup B = B$ and $A^{\sim} \cap B^{\sim} = B^{\sim}$. But then $A \subseteq B$ and $B^{\sim} \subseteq A^{\sim}$.

The rest of the proof is similar. □

3.3. The notion of complement

The following approach to the problem of negation is rather typical.

Definition 3.3. Let $\mathcal{A}$ be an IFpi set on X (where we have some ideal $\mathcal{I}$ and some filter $\mathcal{F}$). Then the *complement* of $\mathcal{A}$ is defined as $\mathcal{A}^c = (A^{\sim}, A)$.

As for the properties of this complement, they will be studied later in the next section.

3.4. Distinguished sets

First of all, we need to define IFpi empty set and IFpi universal set.

Definition 3.4. Assume that $(X, \mathcal{I}, \mathcal{F})$ is an IF space. Then $\tilde{\emptyset} = (\emptyset, X)$ is called the *empty* IFpi set and $\tilde{\mathcal{X}} = (X, \emptyset)$ is called the *universal* IFpi set.

Note that $\tilde{0} = (\emptyset, \emptyset)$ makes sense (as an IFpi set) only if $\emptyset \cup \emptyset = \emptyset \in \mathcal{F}$. But this means that $\mathcal{F} = P(X)$. Hence, it applies only to some IF structures (but not to all of them).

Something similar can be said about $\overline{X} = (X, X)$. If $\mathcal{I} = P(X)$, then $X \cap X = X \in \mathcal{I}$. Clearly, for every filter $\mathcal{F}$ it will be true that $X \cup X = X \in \mathcal{F}$.

Moreover, in the setting of Çoker's intuitionistic sets we have exactly one object that is identical with its complement: it is $(\emptyset, \emptyset)$ (no matter what is the universe). As for the possibly paraconsistent sets, this object will be (X, X).

Now think about IF structures. Assume that $\mathcal{A} = (A, A^{\sim}) = (A^{\sim}, A)$ for some $\mathcal{A}$. Clearly, this means that $A = A^{\sim}$. But this happens when A belongs both to $\mathcal{I}$ and $\mathcal{F}$. In general, this is possible.

4. Algebraic properties and identities

The following theorems are devoted to proving basic algebraic identities involving the operators defined above.

Theorem 4.1. *Assume that $(X, \mathcal{I}, \mathcal{F})$ is an IFpi structure. Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are three IFpi sets on X. Then the following properties are true:*

1. $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$ *and* $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$ *(commutativity).*
2. $\mathcal{A} \cap \mathcal{A} = \mathcal{A}$ *and* $\mathcal{A} \cup \mathcal{A} = \mathcal{A}$ *(idempotence).*
3. $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$ *and* $(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$ *(de Morgan laws).*
4. $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$ *and* $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$ *(distributivity laws).*
5. $\mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{A}$ *and* $\mathcal{A} \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A}$ *(absorption laws).*

Proof. The proof is rather standard and similar to the analogous one that holds e.g. for Çoker's intuitionistic sets. □

Note that the theorem above tells us that every IF structure satisfies all axioms of de Morgan algebra.

Remark 4.1. *In general, IFpi sets on IF structures do not satisfy the law of the excluded middle nor the law of non-contradiction.*

For example, take $X = \{a, b, c, d, e, f, g, h\}$. *Let* $\mathcal{I} = \{\emptyset, \{c\}\}$ *and* $\mathcal{F} = \{\{a, b, c, d, e, f, g\}, X\}$. *Take* $\mathcal{A} = (\{a, b, c, e\}, \{c, d, f, g\})$. *Clearly, it is an IFpi set because* $A \cap A^{\sim} = \{c\} \in \mathcal{I}$ *and* $A \cup A^{\sim} = \{a, b, c, d, e, f, g\} \in \mathcal{F}$. *Now,* $\mathcal{A}^c = (\{c, d, f, g\}, \{a, b, c, e\})$. *Let us calculate* $\mathcal{A} \cup \mathcal{A}^c = (\{a, b, c, d, e, f, g\}, \{c\}) \neq \tilde{\mathcal{X}}$. *Moreover,* $\mathcal{A} \cap \mathcal{A}^c = (\{c\}, \{a, b, c, d, e, f, g\}) \neq \tilde{\emptyset}$.

The reader can prepare even less complicated counter-examples.

The next lemma describes basic properties of $\tilde{\emptyset}$ and $\tilde{\mathcal{X}}$.

Lemma 4.1. *Suppose that* $(X, \mathcal{I}, \mathcal{F})$ *is an IF structure on* X. *Let* $\mathcal{A}$ *be an IFpi set on* X. *Then the following identities and relationships hold:*

1. $\tilde{\emptyset} \subseteq \mathcal{A} \subseteq \tilde{X}$.
2. $\tilde{\emptyset}^c = \tilde{\mathcal{X}}$ *and* $\tilde{\mathcal{X}}^c = \tilde{\emptyset}^c$.
3. $\mathcal{A} \cap \tilde{\emptyset} = \tilde{\emptyset}$ *and* $\mathcal{A} \cup \tilde{\emptyset} = \mathcal{A}$.
4. $\mathcal{A} \cap \tilde{\mathcal{X}} = \mathcal{A}$ *and* $\mathcal{A} \cup \tilde{\mathcal{X}} = \tilde{\mathcal{X}}$.

Hence, even if the algebra of IFpi sets is not Boolean, it still reasonable to identify $\tilde{\emptyset}$ with the notion of "empty" set and $\tilde{\mathcal{X}}$ with the notion of "universal set".

4.1. Arbitrary union and intersection

Our basic operators of union and intersection are binary. However, they can be easily generalized.

Definition 4.1. Suppose that $(X, \mathcal{I}, \mathcal{F})$ is an IF structure and let $\{\mathcal{A}_i : i \in J\}$ be an indexed family of IFpi sets on X (where $J \neq \emptyset$). Then we define:

1. *Arbitrary standard union*: $\bigcup_{i \in J} \mathcal{A}_i = (\bigcup A_i, \bigcap A_i^{\sim})$.
2. *Arbitrary standard intersection*: $\bigcap_{i \in J} \mathcal{A}_i = (\bigcap A_i, \bigcup A_i^{\sim})$.

One can prove that elementary properties of these operators are satisfied. In particular, we mean de Morgan laws.

5. Some special subclasses of IFpi sets

We can distinguish several special subclasses of IFpi sets in our environment.

Definition 5.1. Suppose that $(X, \mathcal{I}, \mathcal{F})$ is an IF structure. Assume that $\mathcal{A} = (A, A^{\sim})$ is an IFpi set on X. Then we say that $\mathcal{A}$ is :

1. *IFpi consistent* if $A \cap A^{\sim} = \emptyset$. The family of all such sets (on a given X) will be denoted as IFpi**C**.
2. *IFpi nested* if $A \subseteq A^{\sim}$ or $A^{\sim} \subseteq A$. The family of all such sets will be denoted as IFpi**N**. Additionally, we can distinguish between *left-side nested* IFpi sets (satisfying at least $A \subseteq A^{\sim}$) and *right-side nested* IFpi sets (satisfying at least $A^{\sim} \subseteq A$). These subclasses will be denoted as IFpi**LN** and IFpi**RN**.
3. *IFpi complement invariant* if $A = A^{\sim}$. The family of all such sets will be denoted as IFpi**ComI**.
4. *IFpi quasi-boolean* if $A \cup A^{\sim} = X$. The family of all such sets will be denoted as IFpi**B**.

In general, all these classes of objects are reasonable:

Lemma 5.1. *Suppose that* $(X, \mathcal{I}, \mathcal{F})$ *is an IF structure. Then IFpi***C** *and IFpi***B** *are non-empty.*

Proof. It is enough to consider $\tilde{\emptyset}$ and $\tilde{\mathcal{X}}$: they are both IFpi consistent and IFpi quasi-boolean. Besides, we see that IFpi$\mathbf{C} \cap$ IFpi$\mathbf{B} \neq \emptyset$. In general, it is sufficient to take any $\mathcal{A}$ for which $A^{\sim} = A^c$. For example, it can be $A = \{a\}$, where $a \in X$.

□

Lemma 5.2. *Suppose that* $(X, \mathcal{I}, \mathcal{F})$ *is an IF structure. Then IFpi*$\mathbf{N}$ *is non-empty.*

Proof. Again, it is obvious that $\tilde{\emptyset}$ and $\tilde{\mathcal{X}}$ are correct here. But let us assume that $A = \emptyset$ and $A^{\sim} \in \mathcal{F}$. Then $\mathcal{A} = (A, A^{\sim}) = (\emptyset, A^{\sim})$ satisfies the required condition. Moreover, $\mathcal{A}$ belongs to IFpi$\mathbf{C}$ but not necessarily to IFpi$\mathbf{B}$. ☐

Remark 5.1. *We said that "in general" it is possible that* $A = A^{\sim}$. *We meant the fact that we can easily build an IF structure in which such sets exist. However, it does not mean that IFpi*$\mathbf{ComI}$ *is non-empty in* every *IF structure. Assume that* $X \neq \emptyset$ *and* $\mathcal{I} \cap \mathcal{F} = \emptyset$. *Now assume that there is some IFpi set* $\mathcal{A} = (A, A^{\sim})$ *on* X *satisfying the condition* $A = A^{\sim}$. *But this would imply that* $A \cap A^{\sim} = A \in \mathcal{I}$ *and* $A \cup A^{\sim} = A \in \mathcal{F}$. *Clearly, this is not possible.*

For example, we can assume that $\mathcal{I} = \{\emptyset\}$ *and* $\mathcal{F} = \{X\}$.

One can prove the following lemma:

Lemma 5.3. *Suppose that* $(X, \mathcal{I}, \mathcal{F})$ *is an IF structure. Assume that* $\mathcal{A}$ *and* $\mathcal{B}$ *are IFpi quasi-boolean. Then* $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}$ *are IFpi quasi-boolean. If* $\mathcal{I} = P(X)$, *then* $\mathcal{A} \vee \mathcal{B}$ *is IFpi quasi-boolean.*

Proof. First, think about $\cup$. We can repeat the reasoning from Lemma 3.1. The only important thing is to check the intersection of resulting components. As we know, $(A \cup B) \cap (A^{\sim} \cap B^{\sim}) = (A \cap A^{\sim} \cap B^{\sim}) \cup (B \cap A^{\sim} \cap B^{\sim})$. Now, $A \cap A^{\sim} = \emptyset$ and $B \cap B^{\sim} = \emptyset$. Hence we obtain $(\emptyset \cap B^{\sim}) \cup (\emptyset \cap A^{\sim}) = \emptyset \cup \emptyset = \emptyset$. Thus, the resulting IFpi set is IF quasi-boolean.

In case of $\cap$ our approach will be similar.

Now suppose that $\mathcal{I} = P(X)$. According to Remark 3.1, we have: $\mathcal{A} \vee \mathcal{B} = (A \cup B, A^{\sim} \cup B^{\sim})$.

As for the intersection of both components: as we already know, it will be always true that $(A \cup B) \cap (A^{\sim} \cup B^{\sim}) \in P(X) = \mathcal{I}$.

As for the union of these components, we have: $(A \cup B) \cup (A^{\sim} \cup B^{\sim}) = (A \cup A^{\sim}) \cup (B \cup B^{\sim}) = X \cup X = X$. ☐

Lemma 5.4. *Suppose that* $(X, \mathcal{I}, \mathcal{F})$ *is an IF structure. Assume that* $\mathcal{A}$ *and* $\mathcal{B}$ *are IFpi consistent. Then* $\mathcal{A} \cup \mathcal{B}, \mathcal{A} \cap \mathcal{B}$ *are IFpi consistent. If* $\mathcal{F} = P(X)$, *then* $\mathcal{A} \wedge \mathcal{B}$ *is IFpi consistent too.*

Proof. Similar and analogous to the previous lemma. ☐

Suppose now that $(X, \mathcal{I}, \mathcal{F})$ is an IF structure. Assume that $\mathcal{A}, \mathcal{B} \in$ IFpi$\mathbf{LN}$.

Now consider $\mathcal{A} \cap \mathcal{B} = (A \cap B, A^{\sim} \cup B^{\sim})$. We know that $A \subseteq A^{\sim}$ and $B \subseteq B^{\sim}$. Thus, $A \cap B \subseteq A^{\sim} \cup B^{\sim}$. Hence, the class of IPpi$\mathbf{LN}$ sets is closed under $\cap$.

Assume now that $\mathcal{F} = P(X)$. Now we have a guarantee that $\mathcal{A} \wedge \mathcal{B}$ is an IFpi set. But $A \cap B \subseteq A^{\sim} \cap B^{\sim}$. Thus, the class of IFpi$\mathbf{LN}$ sets is closed under $\wedge$ (when $\mathcal{F} = P(X)$).

On the other hand, we do not have any guarantee that $A \cup B \subseteq A^{\sim} \cap B^{\sim}$. Hence, we cannot say that $\mathcal{A} \cup \mathcal{B}$ belongs to IFpi$\mathbf{LN}$.

However, if $\mathcal{I} = P(X)$, then $\mathcal{A} \vee \mathcal{B} \in$ IFpi$\mathbf{LN}$ because $A \cup B \subseteq A^{\sim} \cup B^{\sim}$.

The reader is encouraged to formulate analogus considerations with respect to IFpi$\mathbf{RN}$ class.

6. The notion of point

A natural step is to define the notion of point in our framework. The approach presented below is inspired mostly by Çoker and his "intuitionistic points" (see [4]).

Definition 6.1. Suppose that $(X, \mathcal{I}, \mathcal{F})$ is an IF structure. Let $m \in X$ and assume that $\tilde{m} = (\{m\}, \{m\}^c)$. Then we say that $\tilde{m}$ is an *IFpi point* (in X).

We already know (e.g. from Lemma 5.1) that such points are properly defined IFpi sets (no matter what is the particular IF structure).

Following Çoker (and other sources, e.g. [7] and [8]) we shall agree on the following definition.

Definition 6.2. Suppose that $(X, \mathcal{I}, \mathcal{F})$ is an IF structure and $\tilde{m}$ is an IF point in X. Assume that $\mathcal{A} = (A, A^{\sim})$ is an IFpi set on X. We say that $\tilde{m} \in \mathcal{A}$ if and only if $m \in A$. We read this as: $\tilde{m}$ *belongs* to $\mathcal{A}$ (or *is contained* in $\mathcal{A}$). The collection of all IFpi points that belong to $\mathcal{A}$ will be denoted as $\mathsf{P}(\mathcal{A})$.

Obviously, each IFpi point is both IFpi consistent and IFpi quasi-boolean.

Example 6.1. *Let* $X = \{a, b, c, d\}$, $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ *and* $\mathcal{F} = \{\{a, b, c\}, X\}$. *Consider* $\mathcal{A} = (\{a, b\}, \{b, c\})$. *Now* $\mathsf{P}(\mathcal{A}) = \{\tilde{a}, \tilde{b}\}$. *Besides,* $\tilde{a} \cup \tilde{b} = (\{a\}, \{b, c, d\}) \cup (\{b\}, \{a, c, d\}) = (\{a, b\}, \{c, d\}) \neq \mathcal{A}$. *Hence,* $\mathcal{A}$ *cannot be expressed as the union of its IFpi points.*

Note that if $A = \emptyset$, then there are no IFpi points in $\mathcal{A}$.

The example above allows us to define certain class of IFpi sets.

Definition 6.3. Assume that $(X, \mathcal{I}, \mathcal{F})$ is an IF structure. Let $\mathcal{A}$ be an IFpi set on X. We shall say that $\mathcal{A}$ is an *IFpi p-u* ("IFpi point-union") set if $\mathsf{P}(\mathcal{A}) \neq \emptyset$ and $\mathcal{A}$ can be expressed as the union of its IFpi points.

The following lemma describes IFpi p-u sets.

Lemma 6.1. *Assume that* $(X, \mathcal{I}, \mathcal{F})$ *is an IF structure and* $\mathcal{A} = (A, A^{\sim})$ *is an IFpi p-u set. Then* $\mathcal{A} \in$ *IFpi*$\mathbf{B} \cap$ *IFpi*$\mathbf{C}$.

Proof. Obviously, $A = \bigcup_{i \in J} \{m_i\}$ for some $J \neq \emptyset$ (where for each $m_i \in X$ we have $m_i \in A$). On the other hand, $A^{\sim} = \bigcap_{i \in J} \{m_i\}^c$.

Now suppose that $A \cup A^{\sim} \neq X$. Thus, there is some $a \in X$ such that $a \notin A \cup A^{\sim}$. It means that $a \notin A$ and $a \notin A^{\sim}$. In particular, $a \notin \bigcup_{i \in J} \{m_i\}$. Hence, for every m_i (such that $m_i \in A$) we have $a \in \{m_i\}^c$. But then $a \in \bigcap_{i \in J} \{m_i\}^c = A^{\sim}$. This is contradiction.

Now assume that $A \cap A^{\sim} \neq \emptyset$. Thus, there is some $a \in X$ such that $a \in A \cap A^{\sim}$. On the one hand, it means that $a \in \bigcup_{i \in J} \{m_i\}$. Hence, $a = m_i$ for some $i \in J$. On the other hand, $a \in \bigcap_{i \in J} \{m_i\}^c$. Thus, for every $i \in J$, $a \in \{m_i\}^c$. This means that (for every $i \in J$) $a \neq m_i$. This is a contradiction. □

We can define the following relation:

Definition 6.4. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two IFpi sets on $X \neq \emptyset$. We say that $\mathcal{A} \leq \mathcal{B}$ if and only if $\mathsf{P}(\mathcal{A}) \subseteq \mathsf{P}(\mathcal{B})$. When $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$ then we write that $\mathcal{A} \approx \mathcal{B}$.

In practice, it is clear that $\mathcal{A} \leq \mathcal{B}$ if $A \subseteq B$. However, this does not imply that $B^{\sim} \subseteq A^{\sim}$. Thus, the fact that $\mathcal{A} \leq \mathcal{B}$ does not imply that $A \subseteq B$. Obviously, the converse is true.

Çoker (see [4]) introduced *vanishing* intuitionistic points. They are of the form $(\emptyset, \{m\}^c)$. Things are slightly more complicated in our environment. For example, take $X = \{a, b, c\}$, $\mathcal{I} = \{\emptyset\}$ and $\mathcal{F} = \{\{a, c\}, X\}$. Now take the following object: $(\emptyset, \{a\}^c) = (\emptyset, \{b, c\})$. The problem is that this object is not an IFpi set (in this structure) because $\emptyset \cup \{b, c\} = \{b, c\} \notin \mathcal{F}$. Hence, the usage of vanishing points in our framework should be restricted.

7. Conclusion

In this paper we introduced the general concept of IF pseudo-intuitionistic sets on X. We defined algebraic operations on these objects and we analyzed some of their properties. We compared our non-classical framework with those previously existing (that are similar but still different). Moreover, we distingushed several specific subclasses of IFpi sets. We explained the interpretation of proposed objects, trying also to highlight some possible applications and perspectives.

Now one can define and investigate topological IFpi structures (and thus also infra or supra topologies together with minimal and weak structures, anti-topologies etc.). In particular, it would be valuable to examine the behaviour of consistent, nested, complement invariant and quasi-boolean IFpi sets in the context of openess, closure, density, nowhere density, connectedness and other properties of this kind.

Moreover, we would like to recall the difference between picture interior (closure) and picture *point* interior (closure) that was presented in [7] and [8]. This distinction could be implemented (after necessary adjustments) in the framework of IFpi sets.

The aim of the present paper was to establish the foundations for creating more advanced structures.

References

[1] Abdulsada, D. Ali, Al-Swidi, L. A. A., Hadi, M. H., *On NCT-set theory*, Neutrosophic Sets and Systems, vol. 68, 2024.

[2] Atanassov K., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20 (1986), 87-96.

[3] Chae, G. B., Kim, J., Lee, J. G., Hur, K., *Interval-valued intuitionistic sets and their application to topology*, Annals of Fuzzy Mathematics and Informatics, vol 21 (2021), no. 1, pp. 1 - 28.

[4] Çoker D., *A note on intuitionistic sets and intuitionistic points*, Tr. J. of Mathematics, 20 (1996), 343-351.

[5] Çoker D., *An introduction to intuitionistic topological spaces*, BUSEFAL 81 (2000), 51-56.

[6] Cuong, B. C., *Picture fuzzy sets*, Journal of Computer Science and Cybernetics 2014; 30 (4): 409 - 420.

[7] Durga, S., Vidyarani, L., Vigneshwaran, M., Witczak, T., *An introduction to picture topological spaces*, Asia Mathematika, vol. 7, issue 3 (2023).

[8] Durga, S., Vidyarani, L., Vigneshwaran, M., Witczak, T., *Properties of frontier, exterior and border in picture topological spaces*, Turkish Journal of Mathematics, 49 (2025), pp. 96 - 116.

[9] Jo, D., Saleh, S., Lee, J. G., Hur, K., Xueyou, C., *Topological structures via interval-valued neutrosophic crisp sets*, Symmetry, vol. 21 (2020).

[10] Kim, J. H., Lim P. K., Lee J. G., Hur, K., *Intuitionistic topological spaces*, Annals of Fuzzy Mathematic and Informatics, vol. 15(2), pp. 101-122, 2018.

[11] Kim, J. H., Yun Y. B., Lee J. G., Hur, K., *Topological structures based on interval-valued sets*, Annals of Fuzzy Mathematic and Informatics, vol. 20, pp. 273-295, 2020.

[12] Salama A. A., Smarandache F., Kroumov V., *Neutrosophic crisp sets and neutrosophic crisp topological spaces*, Neutrosophic Sets and Systems, 25-30, Vol. 2, 2014.

[13] Salama A. A., Smarandache F., Alblowi S. A., *New neutrosophic crisp topological concepts*, Neutrosophic Sets and Systems, Vol. 4, 2014.

[14] Seif, A. M., *Multi-fuzzifying topology*, Asia Mathematika, vol. 8, issue 3 (2024), pp. 42 - 58.

[15] Subha, V. S., Dhanalakshmi, D., *Characterization of near-ring by interval valued Picture fuzzy ideals*, Asia Mathematika, vol. 5, issue 3 (2021), pp. 14 - 21.

[16] Witczak, T., *A note on the algebra of triple sets*, Asia Mathematika, vol. 7, issue 2 (2023), pp. 48 - 56.

[17] Witczak, T., *On the algebra of possibly paraconsistent sets*, Neutrosophic Sets and Systems, vol. 73 (2024).

[18] Witczak, T., *Restricted ideal pseudo-intuitionistic sets*, Asia Mathematika, vol. 9, issue 2 (2025), pp. 71 - 80.

[19] Yong-jin, L., *Weak rough numbers*, `https://www.sinoss.net/uploadfile/2010/1130/1421.pdf`.