



CCM-Spaces and fixed point results of GC-contractive mappings

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Received: 21 Feb 2026

Accepted: 04 May 2026

Published Online: 30 May 2026

Abstract: In this paper, we obtain a fixed point result on CCM (Complete Cone Metric)-Spaces on GC (Generalized Contractive)-Condition defened on another function.

Key words: Cone metric space, contractive condition, fixed point, sequentially convergent.

AMS subject classifications: 46 J10; 46J15; 47H10.

1. Introduction and Preliminaries

Huang and Zhang [2] introduced the concept of cone metric space, where the set of real numbers is replaced by an ordered Banach space and obtained some fixed point results in cone metric space. Subsequently many authors were inspired by with these results, they have been extending these results in different ways (see for e.g. [1, 3-17]). Recently J.R. Morales, and E.Rojas [6] obtained some fixed point results on cone metric spaces and fixed point theorems of T-Kannan contractive mappings. In this paper we obtained a fixed point results on CCM-Spaces and on GC(Generalized Contractive) condition defened on another function.

The following are needed to obtain our main result which are due to [2,6].

1.1. Sections and subsections

Definition 1.1. Let S be a real Banach space and Q a subset of S . Q is called a cone if and only if:

- (a). Q is closed, non empty, and $Q \neq 0$.
- (b). $\alpha, \beta \in R, \alpha, \beta \geq 0, x, y \in Q$ implies $\alpha x + \beta y \in Q$.
- (c). $x \in Q$ and $-x \in Q$ implies $x = 0$.

Definition 1.2. Given a cone $Q \subset S$, we define a partial ordering \leq with respect to Q by : $x \leq y$ iff $y - x \in Q$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in$ interior of Q .

The cone Q is called normal if there is a number $L > 0$ such that for all $x, y \in S, 0 \leq x \leq y$ implies $\|x\| \leq L\|y\|$. In this case the least positive number L is called the normal constant of Q .

Definition 1.3. Let X be a non- empty set. Suppose the $\rho : X \rightarrow X$ satisfies the following: (a). $\rho(x, y) > 0$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.

(b). $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$.

(c). $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, for all $x, y, z \in X$. Then ρ is called a cone metric on X , and (X, ρ) is called a cone metric space.

Definition 1.4. Let (X, ρ) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $d \in S$, $d \gg 0$ there is N such that for all $n > N$, $\rho(x_n, x) \ll d$, then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. We denote this $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 1.5. Let (X, ρ) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $d \in S$, $d \gg 0$ there is N such that for all $n, m > N$, $\rho(x_n, x_m) \ll d$, then $\{x_n\}$ is said to be Cauchy sequence in X .

Definition 1.6. Let (X, ρ) be a cone metric space. If every Cauchy sequence is convergent, then X is called a CCM(Complete Cone Metric)-space.

Definition 1.7. Let (X, ρ) be a cone metric space. A mapping $A : X \rightarrow X$ is said to be sequentially convergent if we have, for every sequence $\{x_n\}$, if $\{Ax_n\}$ is convergence then $\{x_n\}$ is also convergence.

Definition 1.8. Let (X, ρ) be a cone metric space. A mapping $A : X \rightarrow X$ is said to be sub sequentially convergent if we have, for every sequence $\{x_n\}$, if $\{Ax_n\}$ is convergence then $\{x_n\}$ is also convergence subsequence.

2. Main Result

Theorem 2.1. Let (X, ρ) be a CCM-Space and P be a normal cone with normal constant M the mapping $A : X \rightarrow X$ be one to one continuous function and $B : X \rightarrow X$ a GC(Generalized Contraction):

$$\rho(BAx, ABY) \leq a\rho(Ax, Ay) + b[\rho(Ax, ABx) + \rho(Ay, ABY)].$$

Then

- (i). For every $x_0 \in X$, $\lim_{n \rightarrow \infty} \rho AB^n x_0, AB^{n+1} x_0 = 0$.
- (ii). There is $q \in X$ such that $\lim_{n \rightarrow \infty} \rho(AB^n x_0) = q$.
- (iii). If A is sub sequentially convergent then $(B^n x_0)$ has a convergent sub sequence.
- (iv). There is a unique $p \in X$ such that $Bp = p$.
- (v). If A is sequentially convergent then for each $x_0 \in X$ the iterative sequence $(B^n x_0)$ converges to p .

Proof. Let $x_0 \in X$ be an arbitrary point in X . Define iterative sequence x_n by $x_{n+1} = Bx_n = Bx_0$.

We have

$$\begin{aligned} \rho(Ax_n, Ax_{n+1}) &= \rho(ABx_{n-1}, ABx_n), \\ &\leq arho(Ax_{n-1}, Ax_n) + b[\rho(Ax_{n-1}, ABx_{n-1}) + \rho(Ax_n, ABx_n)], \\ &= arho(Ax_{n-1}, Ax_n) + b[\rho(Ax_{n-1}, Ax_n) + \rho(Ax_n, ABx_{n+1})], \\ &\leq (a + b)rho(Ax_{n-1}, Ax_n) + b\rho(Ax_n, ABx_{n+1}), \end{aligned}$$

$$\begin{aligned} (1 - b)\rho(Ax_n, Ax_{n+1}) &\leq (a + b)rho(Ax_{n-1}, Ax_n), \\ \rho(Ax_n, Ax_{n+1}) &\leq (a + b)/(1 - b)rho(Ax_{n-1}, Ax_n), \\ &\leq hrho(Ax_{n-1}, Ax_n), \text{ where } h = (a + b)/(1 - b) < 1. \end{aligned}$$

By repeating the same argument we get that

$$\rho(BAx_0, AB_{n+1}x_0) \leq h^n \rho(Ax_0, ABx_0). \tag{1}$$

From (1) we have

$$\|\rho(BAx_0, AB_{n+1}x_0)\| \leq h^n M \|\rho(Ax_0, ABx_0)\|.$$

Where M is a normal constant. letting $n \rightarrow \infty$ we get that

$$\lim_{n \rightarrow \infty} \|\rho(BAx_0, AB_{n+1}x_0)\| = 0. \quad (2)$$

By (1) for every $m, n \in N$, with $m > n$ we have

$$\begin{aligned} \rho(Ax_n, Ax_m) &\leq \rho(Ax_n, Ax_{n+1}) + \dots + \rho(Ax_{m-1}, Ax_m), \\ &\leq [h^n + \dots + h^{m-1}] \rho(Ax_0, ABx_0), \\ &= \frac{h^n}{1-h} \rho(Ax_0, ABx_0), \\ \rho(AB^n x_0, AB^m x_0) &\leq \frac{h^n}{1-h} \rho(Ax_0, ABx_0). \end{aligned} \quad (3)$$

From (3) we have

$$\|\rho(AB^n x_0, AB^m x_0)\| \leq \frac{h^n}{1-h} M \|\rho(Ax_0, ABx_0)\|,$$

where M is a normal constant of. letting $n \rightarrow \infty$ we get that

$$\lim_{n \rightarrow \infty} \|\rho(AB^n x_0, AB^m x_0)\| = 0.$$

In this we have Implies that $(AB^n x_0)$ is a Cauchy sequence in X . Since X is CCM-Space then there exists $q \in X$ such that

$$\lim_{n \rightarrow \infty} (AB^n x_0) = q. \quad (4)$$

Now A is sub sequentially convergent $(AB^n x_0)$ has a convergent subsequence. So there exists $p \in X$ and (x_{n_r}) such that

$$\lim_{r \rightarrow \infty} (B^{n_r} x_0) = p. \quad (5)$$

Since A is continuous and by (5) we obtain

$$\lim_{r \rightarrow \infty} (B^{n_r} x_0) = Ap. \quad (6)$$

By (4) and (6) we conclude that

$$Ap = q. \quad (7)$$

On the other hand

$$\begin{aligned} \rho(ABp, Ap) &\leq \rho(ABp, B^{n_r} x_0 A) + \rho(B^{n_r} x_0, AB^{n_r+1} x_0) + \rho(AB^{n_r+1} x_0, Ap), \\ &= \rho(ABp, B^{n_r-1} x_0 A) + h^{n_r} \rho(Ax_0, ABx_0) + \rho(AB^{n_r+1} x_0, Ap), \\ &\leq a \rho(Ap, AB^{n_r-1} x_0) + b [\rho(Ap, ABp) + \rho(AB^{n_r-1} x_0, AB^{n_r} x_0)] \\ &\quad + h^{n_r} \rho(Ax_0, ABx_0) + \rho(AB^{n_r+1} x_0, Ap), \end{aligned}$$

$$\begin{aligned} \rho(ABp, Ap) &\leq \frac{a}{1-b} \rho(Ap, AB^{n_r-1} x_0) + \frac{b}{1-b} \rho(AB^{n_r-1} x_0, AB^{n_r} x_0) \\ &\quad + h^{n_r} \frac{1}{1-b} \rho(Ax_0, ABx_0) + \frac{1}{1-b} \rho(AB^{n_r+1} x_0, Ap). \end{aligned}$$

Thus,

$$\begin{aligned} \|\rho(ABp, Ap)\| &\leq \frac{a}{1-b}M\|\rho(Ap, AB^{n_r-1}x_0)\| + \frac{b}{1-b}M\|\rho(AB^{n_r-1}x_0, AB^{n_r}x_0)\| \\ &\quad + h^{n_r} \frac{1}{1-b}M\|\rho(Ax_0, ABx_0)\| + \frac{1}{1-b}M\|\rho(AB^{n_r+1}x_0, Ap)\| \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned}$$

Where M is the normal constant of X . The convergence of above give us that

$$\rho(ABp, Ap) = 0.$$

Implies that $ABp = Ap$. Since B is one - to -one then $Bp = p$. Consequently B has a fixed point. Because B is a GC-Contraction.

$$\begin{aligned} \rho(ABp, ABq) &\leq a\rho(Ap, Aq) + b[\rho(Ap, ABp) + \rho(Aq, ABq)], \\ &\leq a\rho(ABp, Aq) + b[\rho(Ap, Ap) + \rho(Aq, ABq)], \\ &\leq a\rho(ABp, Aq) + b\rho(Aq, ABq). \end{aligned}$$

If q is another fixed point of B then from the injectivity of B we get that $Bp = Bq$ or which is the same the fixed point is unique. Finally if A is sequentially convergent by replacing (n) for (n_r) we conclude that

$$\lim_{n \rightarrow \infty} B^n x_0 = p.$$

This shows that $(B^n x_0)$ converges to the fixed point of B . This completes the proof of the theorem. □

Remark

If we take $a = 0$ in the above theorem we get the Theorem 3.1. of [6].

Conclusion

Our results are extended results and are also more general than the results of [6]

Acknowledgment

The author thankful to the reviewers to review this article and give the valuable suggestions for improvement of the article in the present form.

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