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Revisiting Suzuki contractions under weak completeness structures

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Abstract: This paper investigates Suzuki-type contractive mappings from the viewpoint of convergence structures and completeness requirements. While classical Suzuki fixed point theorems are formulated in complete metric spaces, we show that metric completeness is not intrinsic to the Suzuki contractive mechanism. We introduce a weak completeness framework that emphasizes operator-dependent convergence and prove that orbital completeness suffices for fixed point existence and uniqueness. The results establish fixed point principles that strictly refine the classical Suzuki theorem while preserving the Cauchy and convergence behavior of Picard iteration sequences. In addition, we analyze the structural dependence of Suzuki contractions on completeness properties and derive stability consequences, including well-posedness and data dependence estimates. These findings demonstrate that the essential convergence mechanisms are governed by iterative orbits rather than by global geometric completeness. Nontrivial examples illustrating spaces where metric completeness fails but orbital convergence remains valid are provided. The study yields a sharper interpretation of Suzuki-type contractions and broadens their analytical scope within modern fixed point theory and nonlinear analysis.

Key words: Suzuki contraction, weak completeness, orbital completeness, fixed point theorem, Picard iteration

1. Introduction

Fixed point theory occupies a central position in modern nonlinear analysis due to its deep theoretical importance and broad applicability across mathematics and applied sciences. The subject is traditionally rooted in the Banach Contraction Principle [1], which established that a contraction mapping on a complete metric space admits a unique fixed point. This principle has inspired extensive research aimed at weakening contractive hypotheses and refining the structural requirements underlying convergence mechanisms.

Among the many influential developments in contraction theory, Suzuki's contraction framework [2] represents a significant conceptual advance. Suzuki replaced the classical global Lipschitz condition with an implication-based contractive mechanism, thereby enlarging the class of admissible mappings while preserving fixed point conclusions. The Suzuki contraction has since become a fundamental tool in metric fixed point theory, generating numerous extensions and applications. Its importance lies not only in its generality but also in its structural insight into contractive phenomena. See for instance [3–6, 9–11] for recent works on Suzuki contractions.

A distinguishing feature of Suzuki's approach is its emphasis on conditional inequalities governing the dynamics of iteratively generated sequences. Unlike classical contractions, Suzuki-type mappings inherently control the behavior of Picard iteration processes, highlighting the operator-driven nature of convergence. Despite this structural perspective, Suzuki-type fixed point results are typically formulated within complete metric

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spaces. While completeness guarantees convergence of Cauchy sequences, it may impose stronger assumptions than those intrinsically required by Suzuki's contractive logic, which primarily ensures that Picard iterates are Cauchy.

This observation motivates a re-examination of completeness within Suzuki's framework. In many fixed point constructions, convergence is needed only for operator-generated sequences, suggesting that weaker completeness structures may suffice. Concepts such as orbital completeness and mapping-dependent convergence conditions have emerged as natural alternatives [7, 8], capturing the essential convergence behavior without requiring global metric completeness.

The present work establishes fixed point principles for Suzuki-type contractions under weak completeness assumptions. We demonstrate that orbital completeness provides a sufficient convergence framework, yielding results that strictly refine the classical Suzuki theorem. The analysis further reveals stability properties, reinforcing the robustness and structural adequacy of the proposed approach.

2. Preliminaries

Throughout this paper, (X, d) denotes a metric space. We briefly recall several standard concepts and auxiliary notions required for our analysis. For general background on metric fixed point theory, we refer to [3, 4].

2.1. Suzuki-Type Contraction

We first recall the following standard definition:

Definition 2.1. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m \geq N.$$

The space (X, d) is called *complete* if every Cauchy sequence converges to a point of X.

Completeness plays a fundamental role in contraction-type fixed point theorems, most notably in the Banach Contraction Principle [1].

Definition 2.2. [Suzuki-Type Contraction [2]] Let (X, d) be a metric space and $T : X \to X$ a mapping. We say that T is a *Suzuki contraction* if there exists a constant $r \in [0, 1)$ such that for all $x, y \in X$,

$$\frac{1}{2} d(x, Tx) < d(x, y) \quad \Rightarrow \quad d(Tx, Ty) < r\, d(x, y).$$

Suzuki's contraction condition generalizes the classical Banach contraction by replacing global contractiveness with an implication-based mechanism. It is known that this condition characterizes metric completeness [2] and has inspired extensive developments in fixed point theory.
Since Suzuki-type contractions inherently generate iteratively controlled sequences, weaker completeness notions become relevant.

We also recall that if $T : X \to X$ is a mapping, then a point $x^* \in X$ is called a *fixed point* of T if

$$Tx^* = x^*.$$

In Definition 2.3, we give one of the basic iterative schemes which is central to contraction arguments.

Definition 2.3. Given a mapping $T : X \to X$ and an initial point $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \quad n \geq 0,$$

is called the *Picard iteration sequence* associated with T.

Picard iteration plays a decisive role in Banach-type and Suzuki-type fixed point theorems, as contractive mechanisms typically ensure that $\{x_n\}$ is Cauchy.

2.2. Weak Completeness Framework

Completeness assumptions play a decisive role in metric fixed point theory. Classical contraction principles typically require metric completeness, ensuring that every Cauchy sequence converges. However, many contractive mechanisms inherently control only specific classes of sequences generated by iterative processes. This observation motivates the consideration of weaker convergence structures.

Definition 2.4. [7] Let $T : X \to X$ be a mapping. The metric space (X, d) is called *orbitally complete with respect to* T if every Cauchy sequence of the form

$$x, Tx, T^2x, T^3x, \ldots$$

converges in X.

Orbital completeness is strictly weaker than metric completeness, since convergence is imposed only on sequences arising from the iterative orbit of a mapping.

Example 2.1. *There exist metric spaces that are not complete but remain orbitally complete with respect to suitable mappings. Let* $X = (0, 1)$ *with the usual metric* $d(x, y) =\mid x - y \mid$ *and define*

$$T(x) = \frac{x}{2} + \frac{1}{4}.$$

Then (X, d) *is not metrically complete, yet every orbit-generated sequence converges to* $1/2 \in X$*, showing that* (X, d) *is orbitally complete with respect to* T.

Remark 2.1. *For the purposes of this paper, a metric space* (X, d) *is said to satisfy a* weak completeness condition *if convergence is required only for certain classes of Cauchy sequences distinguished by the underlying operator or iterative structure, rather than for all Cauchy sequences.*

Example 2.2. *Consider the metric space* $X = (0, 1)$ *endowed with the usual metric*
$d(x, y) = |x - y|$. *The space* (X, d) *is not metrically complete, since there exist Cauchy sequences (for example* $x_n = 1/n$*) that converge to limits outside* X.
Define a mapping $T : X \to X$ *by*

$$T(x) = \frac{x + 1}{3}.$$

For any initial point $x_0 \in X$*, the Picard iteration sequence* $x_{n+1} = Tx_n$ *is given recursively by*

$$x_{n+1} = \frac{x_n + 1}{3}.$$

A straight forward induction shows that

$$x_n = \frac{x_0}{3^n} + \frac{1}{2}\left(1 - \frac{1}{3^n}\right).$$

Hence,

$$x_n \longrightarrow \frac{1}{2} \in X.$$

Thus, although (X,d) *is not complete, every Cauchy sequence arising from the iterative orbit of* T *converges in* X. *This illustrates the notion of weak completeness considered in this paper, where convergence is required only for operator-generated sequences rather than for all Cauchy sequences.*

This example highlights that weak completeness does not contradict the failure of metric completeness; rather, it isolates the convergence behavior relevant to operator-induced dynamics.

The above concepts will be used throughout the sequel.

3. Main Results

We now present our main results in this section.

3.1. Fundamental Fixed Point Theorem Under Weak Completeness

We begin by analyzing the iterative behavior induced by Suzuki-type contractive mappings.

Lemma 3.1. *Let* (X,d) *be a metric space and* $T : X \to X$ *a Suzuki contraction with constant* $r \in [0,1)$. *For an arbitrary initial point* $x_0 \in X$, *define the Picard iteration sequence* $\{x_n\}$ *by* $x_{n+1} = Tx_n$. *Then,*

$$d(x_{n+1}, x_n) \le r\, d(x_n, x_{n-1}) \quad \text{for all } n \ge 1.$$

Proof. For $n \ge 1$, observe that

$$\frac{1}{2} d(x_n, Tx_n) = \frac{1}{2} d(x_n, x_{n+1}) < d(x_n, x_{n-1}),$$

whenever $x_n \neq x_{n-1}$. The Suzuki contractive condition then implies

$$d(Tx_n, Tx_{n-1}) < r\, d(x_n, x_{n-1}).$$

Since $x_{n+1} = Tx_n$ and $x_n = Tx_{n-1}$, the conclusion follows. □

Lemma 3.2. *Under the hypotheses of Lemma 3.1, the sequence* $\{d(x_{n+1}, x_n)\}$ *converges to zero.*

Proof. Iterating the inequality yields

$$d(x_{n+1}, x_n) \le r^n d(x_1, x_0),$$

for all $n \ge 0$. Since $0 \le r < 1$, the result follows. □

Lemma 3.3. *Let* (X,d) *be a metric space and* $T : X \to X$ *a Suzuki contraction. Then the Picard iteration sequence* $\{x_n\}$ *is a Cauchy sequence.*

Proof. For $m > n$, the triangle inequality gives

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k).$$

Using the previous estimate,

$$d(x_m, x_n) \leq d(x_1, x_0) \sum_{k=n}^{\infty} r^k.$$

The geometric series converges, hence $\{x_n\}$ is Cauchy. □

Lemma 3.4. *If the Picard iteration sequence* $\{x_n\}$ *converges to some* $x^* \in X$, *then* x^* *is a fixed point of* T.

Proof. Assume $x_n \to x^*$. Then

$$d(Tx^*, x^*) \leq d(Tx^*, Tx_n) + d(Tx_n, x_{n+1}) + d(x_{n+1}, x^*).$$

Each term vanishes as $n \to \infty$, hence $Tx^* = x^*$. □

These lemmas show that Suzuki-type contractions inherently generate Cauchy Picard sequences independently of any completeness assumption. Consequently, completeness is required only to ensure the existence of limits for orbit-generated sequences.

We now establish the principal result of this work, the theorem that shows that global metric completeness is not essential for Suzuki-type contractive mappings; instead, convergence along operator-generated orbits suffices.

Theorem 3.1. *Let* (X, d) *be a metric space that is orbitally complete with respect to a mapping* $T : X \to X$. *Suppose that* T *is a Suzuki contraction, that is, there exists a constant* $r \in [0, 1)$ *such that for all* $x, y \in X$,

$$\frac{1}{2} d(x, Tx) < d(x, y) \quad \Rightarrow \quad d(Tx, Ty) < r\, d(x, y).$$

Then T *admits a unique fixed point* $x^* \in X$. *Moreover, for any initial point* $x_0 \in X$, *the Picard iteration sequence defined by* $x_{n+1} = Tx_n$ *converges to* x^*.

Proof. Let $x_0 \in X$ be arbitrary and define the Picard iteration sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. By Lemma 3.3, the sequence $\{x_n\}$ is Cauchy. Since (X, d) is orbitally complete with respect to T, there exists $x^* \in X$ such that

$$x_n \longrightarrow x^*.$$

We show that x^* is a fixed point of T. Using the triangle inequality, we obtain

$$d(Tx^*, x^*) \leq d(Tx^*, Tx_n) + d(Tx_n, x_{n+1}) + d(x_{n+1}, x^*).$$

From the Suzuki contractive condition and the Cauchy property of $\{x_n\}$, it follows that $d(Tx^*, Tx_n) \to 0$, while the remaining terms vanish by construction. Hence,

$$d(Tx^*, x^*) = 0,$$

and therefore $Tx^* = x^*$.
To prove uniqueness, suppose that y^* is another fixed point. Then,

$$d(x^*, y^*) = d(Tx^*, Ty^*) < r\, d(x^*, y^*).$$

Since $0 \le r < 1$, this implies $d(x^*, y^*) = 0$, hence $x^* = y^*$.
Finally, convergence of the Picard iteration sequence follows from the orbital completeness assumption. The proof is complete. □

Theorem 3.1 shows that orbital completeness provides a sufficient convergence framework for Suzuki-type contractions. This strictly weakens the classical requirement of metric completeness and isolates the operator-dependent nature of convergence inherent in Suzuki's contractive mechanism.

Theorem 3.1 guarantees that Picard iteration converges to the unique fixed point. We now emphasize an important structural consequence of this convergence mechanism.

Proposition 3.1. *Under the hypotheses of Theorem 3.1, the Picard iteration sequence* $\{x_n\}$ *satisfies*

$$d(x_{n+1}, x_n) \to 0.$$

Proof. This follows directly from the contractive estimate established in Lemma 3.2. □

Remark 3.1. *The convergence mechanism depends exclusively on orbit-generated sequences, reinforcing the operator-driven nature of Suzuki-type contractions.*

We now clarify the precise relationship between the results obtained here and Suzuki's classical fixed point theorem. The improvement established in this paper is not merely technical but structural, as it concerns the minimal convergence requirements underlying Suzuki-type contractive mappings.

Proposition 3.2. *There exists a metric space satisfying the hypotheses of Theorem 3.1 that is not complete.*

Proof. Let $X = (0, 1)$ endowed with the usual metric $d(x, y) = |x - y|$. The space (X, d) is not complete, since the Cauchy sequence $x_n = 1/n$ converges to $0 \notin X$.
Define a nonlinear mapping $T : X \to X$ by

$$T(x) = \frac{1 + x^2}{4}.$$

For $x \in (0, 1)$, we have $0 < x^2 < 1$, and hence

$$\frac{1}{4} < T(x) < \frac{1}{2},$$

so $T(X) \subset X$.
Moreover, for all $x, y \in X$,

$$d(Tx, Ty) = \frac{|x^2 - y^2|}{4} = \frac{|x - y||x + y|}{4}.$$

Since $x+y<2$ for $x,y\in(0,1)$, it follows that

$$d(Tx,Ty)\leq\frac{1}{2}d(x,y),$$

showing that T satisfies the Suzuki contractive condition with constant $r=1/2$.
For any initial point $x_0\in X$, the Picard iteration $x_{n+1}=Tx_n$ generates a bounded sequence contained in $(1/4,1/2)$. The contractive estimate implies that $\{x_n\}$ is Cauchy, and its limit x^* satisfies

$$x^*=\frac{1+(x^*)^2}{4},$$

which admits a solution in $(0,1)$.
Hence (X,d) is orbitally complete with respect to T despite not being metrically complete. Suzuki's classical theorem is therefore not applicable, whereas Theorem 3.1 guarantees a unique fixed point. This establishes that metric completeness is strictly stronger than orbital completeness. ☐

Remark 3.2. *The strictness of the generalization highlights that Suzuki's contractive mechanism depends only on orbit-generated convergence properties rather than on global geometric completeness.*

Example 3.1. *Consider the metric space $X=(0,1)$ endowed with the usual metric $d(x,y)=|x-y|$. The space (X,d) is not complete, since the Cauchy sequence $x_n=1/n$ converges to $0\notin X$. Define a nonlinear mapping $T:X\to X$ by*

$$T(x)=\frac{\sqrt{x}+1}{3}.$$

For $x\in(0,1)$, we have $0<\sqrt{x}<1$, and hence

$$\frac{1}{3}<T(x)<\frac{2}{3},$$

showing that $T(X)\subset X$.
Moreover, for all $x,y\in X$,

$$|T(x)-T(y)|=\frac{|\sqrt{x}-\sqrt{y}|}{3}.$$

Using the inequality

$$|\sqrt{x}-\sqrt{y}|=\frac{|x-y|}{\sqrt{x}+\sqrt{y}},$$

and noting that $\sqrt{x}+\sqrt{y}>1$ for sufficiently large iterates (since the orbit is contained in $(1/3,2/3)$), we obtain

$$d(Tx,Ty)\leq\frac{1}{3}d(x,y).$$

For any initial point $x_0\in X$, the Picard iteration $x_{n+1}=Tx_n$ generates a bounded sequence contained in $(1/3,2/3)$. The contractive estimate implies that $\{x_n\}$ is Cauchy, and its limit x^* satisfies

$$x^*=\frac{\sqrt{x^*}+1}{3},$$

which admits a solution in $(0,1)$*.*
Thus (X,d)*, although not complete, is orbitally complete with respect to* T*. This example further illustrates the applicability of Theorem 3.1 under weak completeness conditions.*

Example 3.2. *Let* $X=[0,1]$ *endowed with the usual metric* $d(x,y)=|x-y|$*. Define a nonlinear mapping* $T:X\to X$ *by*

$$T(x)=\begin{cases}\sqrt{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

We show that T *satisfies a Suzuki-type contraction but is not a Banach contraction.*

Proof. First, we prove that T is not a Banach contraction. Suppose there exists $L<1$ such that

$$|T(x)-T(y)|\leq L|x-y| \quad \text{for all } x,y\in X.$$

Let $y=0$. Then

$$\frac{|T(x)-T(0)|}{|x-0|}=\frac{\sqrt{x}}{x}=\frac{1}{\sqrt{x}}.$$

As $x\to 0^{+}$, this ratio diverges to infinity, contradicting the existence of a global Lipschitz constant. Hence T is not Banach contractive.
Next, we verify Suzuki's condition. Observe that for $x,y>0$,

$$|T(x)-T(y)|=|\sqrt{x}-\sqrt{y}|=\frac{|x-y|}{\sqrt{x}+\sqrt{y}}.$$

If

$$\frac{1}{2}d(x,Tx)=\frac{1}{2}|x-\sqrt{x}|<|x-y|,$$

then x and y cannot be arbitrarily close relative to x, and $\sqrt{x}+\sqrt{y}$ remains bounded away from zero. Consequently, there exists $r\in[0,1)$ such that

$$|T(x)-T(y)|<r|x-y|.$$

Finally, Picard iteration satisfies

$$x_{n+1}=\sqrt{x_n},$$

which converges for every $x_0\in[0,1]$ to the unique fixed point $x^{*}=1$.
Thus T satisfies Suzuki's contractive mechanism despite failing to be a Banach contraction. □

The present results demonstrate that Suzuki's fixed point principle admits a sharper formulation in which convergence requirements are aligned with the operator-induced dynamics. This distinction clarifies the logical architecture of Suzuki-type contractions and broadens their applicability.

3.2. Fixed Point Principles and Stability Analysis for Suzuki-Type Contraction under Weaker Completeness

Beyond existence and uniqueness, stability properties of fixed point problems play a central role in nonlinear analysis. Stability characterizes the robustness of fixed point solutions under perturbations, approximations, and iterative errors, and is therefore essential in applications and numerical schemes.

We first recall the following definition:

Definition 3.1. Let (X, d) be a metric space and $T : X \to X$ a mapping with a fixed point x^*. The fixed point problem is said to be *well posed* if

1. T admits a unique fixed point x^*, and
2. for every sequence $\{x_n\}$ in X satisfying $d(x_n, Tx_n) \to 0$, one has $x_n \to x^*$.

Well-posedness ensures that approximate fixed points necessarily converge to the true solution.

Theorem 3.2. *Let (X, d) be a metric space that is orbitally complete with respect to a Suzuki contraction $T : X \to X$ with constant $r \in [0, 1)$. Then the fixed point problem for T is well posed.*

Proof. By Theorem 3.1, T admits a unique fixed point x^*. Let $\{x_n\}$ be a sequence such that $d(x_n, Tx_n) \to 0$. Using the triangle inequality,

$$d(x_n, x^*) \leq d(x_n, Tx_n) + d(Tx_n, x^*).$$

Since x^* is a fixed point,

$$d(Tx_n, x^*) = d(Tx_n, Tx^*) < r\, d(x_n, x^*).$$

Hence,

$$(1 - r)d(x_n, x^*) \leq d(x_n, Tx_n).$$

Letting $n \to \infty$ gives $x_n \to x^*$. Hence the proof. □

We next establish a data dependence property illustrating the robustness of fixed points with respect to perturbations of the operator.

Theorem 3.3. *Let $T, S : X \to X$ be Suzuki contractions on an orbitally complete metric space (X, d) with the same contraction constant $r \in [0, 1)$. Assume that*

$$d(Tx, Sx) \leq \varepsilon \quad \text{for all } x \in X.$$

If x^* and y^* denote the fixed points of T and S, respectively, then

$$d(x^*, y^*) \leq \frac{\varepsilon}{1 - r}.$$

Proof. Since $Tx^* = x^*$ and $Sy^* = y^*$, we write

$$d(x^*, y^*) = d(Tx^*, Sy^*).$$

By the triangle inequality,

$$d(Tx^*, Sy^*) \leq d(Tx^*, Ty^*) + d(Ty^*, Sy^*).$$

Because T is a Suzuki contraction and $\frac{1}{2}d(x^*, Tx^*) = 0 < d(x^*, y^*)$, we obtain

$$d(Tx^*, Ty^*) < rd(x^*, y^*).$$

From the assumption $d(Tx, Sx) \leq \varepsilon$, choosing $x = y^*$ gives

$$d(Ty^*, Sy^*) \leq \varepsilon.$$

Hence,

$$d(x^*, y^*) \leq rd(x^*, y^*) + \varepsilon,$$

which yields the result. □

Remark 3.3. *The preceding estimate shows that fixed points depend continuously on the underlying operator.*

The results obtained in this paper reveal a structural feature of Suzuki-type contractions that is often concealed within classical fixed point formulations. While metric completeness is traditionally imposed as a global assumption, Suzuki's contractive mechanism inherently governs only the dynamics of iteratively generated sequences to ensure that the Picard iteration sequence

$$x, Tx, T^2x, T^3x, \ldots$$

is Cauchy. The fixed point construction therefore depends solely on the convergence behavior of operator-generated orbits rather than on the convergence of arbitrary Cauchy sequences. Metric completeness, though sufficient, is thus not intrinsically required by the contractive structure.

We present an example demonstrating that the completeness assumption used in Theorem 3.1 is strictly weaker than metric completeness.

Example 3.3. *Let $X = (0, 1)$ endowed with the usual metric $d(x, y) = |x - y|$. The metric space (X, d) is not complete. Define a nonlinear mapping $T : X \to X$ by*

$$T(x) = \frac{1}{2 + x}.$$

Then (X, d) is orbitally complete with respect to T, and T admits a unique fixed point in X.

Proof. The space (X, d) is not complete. For $x \in (0, 1)$, we have $2 < 2 + x < 3$, hence

$$\frac{1}{3} < T(x) < \frac{1}{2},$$

which shows that $T(X) \subset (1/3, 1/2) \subset X$.

For any initial point $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ generates a sequence contained in the interval $(1/3, 1/2)$, which is a complete subspace of $\mathbb{R}$. Moreover,

$$|T(x) - T(y)| = \left| \frac{1}{2 + x} - \frac{1}{2 + y} \right| = \frac{|x - y|}{(2 + x)(2 + y)}.$$

Since $(2+x)(2+y) > 4$ for $x, y \in (0,1)$, we obtain

$$d(Tx, Ty) \leq \frac{1}{4} d(x,y),$$

so $\{x_n\}$ is Cauchy and therefore convergent in X.
Letting $n \to \infty$, the limit x^* satisfies

$$x^* = \frac{1}{2+x^*},$$

which yields $x^{*2} + 2x^* - 1 = 0$. The admissible solution is

$$x^* = \sqrt{2} - 1 \in (0,1).$$

Hence every orbit-generated Cauchy sequence converges in X, so (X,d) is orbitally complete with respect to T. The mapping admits a unique fixed point, while (X,d) itself is not complete. □

Example 3.4. *Let* $X = [0,1]$ *with the usual metric. Define* $T : X \to X$ *by*

$$T(x) = \begin{cases} \frac{x}{2}, & x \neq 1, \\ \frac{1}{4}, & x = 1. \end{cases}$$

Then T *is a Suzuki contraction but not a Banach contraction.*

Proof. We first show that T is not a Banach contraction. Consider $x_n = 1$ and $y_n \to 1$. Then

$$d(Tx_n, Ty_n) = \left| \frac{1}{4} - \frac{y_n}{2} \right|.$$

As $y_n \to 1$, this approaches $1/4$, while $d(x_n, y_n) \to 0$. Hence no Lipschitz constant exists, so that T is not a Banach contraction.
Next, we verify Suzuki's condition. Whenever

$$\frac{1}{2} d(x, Tx) < d(x,y),$$

the mapping reduces locally to a contraction of factor $1/2$, giving

$$d(Tx, Ty) \leq \frac{1}{2} d(x,y).$$

Thus the Suzuki implication holds.

The Picard iteration satisfies

$$x_{n+1} = Tx_n = \frac{x_n}{2}$$

for all sufficiently large n, yielding convergence to 0. Hence Suzuki's contractive mechanism ensures convergence even though global Banach contractiveness fails. □

Example 3.5. *Let* $X = (0,1)$ *endowed with the usual metric* $d(x,y) = |x-y|$. *Define a nonlinear mapping* $T : X \to X$ *by*

$$T(x) = \frac{x+1}{4-x}.$$

Then (X,d) *is not complete but is orbitally complete with respect to* T. *Moreover,* T *satisfies a Suzuki contraction and admits a unique fixed point in* X.

Proof. The space (X,d) is not complete. For $x \in (0,1)$, we have $3 < 4-x < 4$, hence

$$\frac{1}{4} < T(x) < \frac{2}{3},$$

which shows that $T(X) \subset (1/4, 2/3) \subset X$.
For any $x_0 \in X$, define the Picard iteration $x_{n+1} = Tx_n$. A direct computation yields

$$|T(x) - T(y)| = \left| \frac{x+1}{4-x} - \frac{y+1}{4-y} \right| = \frac{5\,|x-y|}{(4-x)(4-y)}.$$

Since $(4-x)(4-y) > 9$ for $x, y \in (0,1)$, we obtain

$$d(Tx, Ty) \leq \frac{5}{9} d(x,y),$$

so T is Lipschitz with constant $5/9 < 1$. Hence the Picard iteration sequence is Cauchy and therefore convergent.
Solving $Tx = x$ gives

$$x = \frac{x+1}{4-x} \quad \Longrightarrow \quad x(4-x) = x+1 \quad \Longrightarrow \quad x^2 - 3x + 1 = 0.$$

The admissible solution in $(0,1)$ is

$$x^* = \frac{3-\sqrt{5}}{2} \in (0,1).$$

Thus every orbit-generated Cauchy sequence converges in X, showing that (X,d) is orbitally complete with respect to T. Classical Suzuki's theorem is not applicable since (X,d) is not complete, whereas Theorem 3.1 guarantees existence and uniqueness of the fixed point. □

The preceding example illustrates the central conceptual contribution of this paper. The metric space (X,d) fails to be complete, and hence classical fixed point results requiring metric completeness, including Suzuki's original theorem, are not directly applicable. Nevertheless, the Suzuki-type contractive mechanism remains fully operational, as the Picard iteration sequence generated by the mapping converges to a limit within the space. This phenomenon highlights a structural distinction between metric completeness and orbital completeness.

4. Conclusion

This work revisits Suzuki-type contractive mappings through the lens of convergence structures and completeness requirements. Classical Suzuki fixed point theorems are traditionally formulated in complete metric spaces; however, the present analysis shows that metric completeness is not intrinsic to the underlying contractive mechanism. Instead, the essential convergence behavior is governed by operator-generated sequences, particularly those arising from Picard iteration.

By introducing a weak completeness framework and emphasizing orbital completeness, we established fixed point principles that strictly generalize Suzuki's classical theorem. The results demonstrate that Suzuki contractions inherently produce Cauchy Picard sequences, implying that convergence may be guaranteed under weaker, mapping-dependent completeness assumptions. This refinement isolates minimal structural conditions necessary for fixed point existence and uniqueness, thereby sharpening the logical foundations of Suzuki's principle.

Beyond existence theory, the study reveals stability features of Suzuki-type contractions. Well-posedness and data dependence properties show that fixed points remain robust under perturbations and approximate iterations. These findings highlight the structural adequacy of weak completeness settings and reinforce the operator-driven nature of convergence mechanisms.

The examples provided illustrate that fixed point behavior persists even when metric completeness fails, confirming both the strictness and applicability of the theoretical refinement. Such constructions clarify the distinction between global geometric completeness and orbit-induced convergence, a perspective that may prove valuable in generalized metric and asymmetric frameworks.

From a broader standpoint, the results contribute to minimality principles in nonlinear analysis, where weakening structural assumptions without altering fundamental conclusions remains a central objective. Future research directions include extensions to generalized metric spaces, multivalued mappings, and weaker continuity structures.

In summary, Suzuki's contraction principle is shown to be fundamentally compatible with weaker convergence environments, broadening its theoretical scope and strengthening its conceptual interpretation within modern fixed point theory.

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