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Lacunary Cesaro-Kirk iteration for fixed point problems

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Abstract: We propose a lacunary Cesàro–Kirk iterative process for fixed point approximation of asymptotically nonexpansive mappings in Banach spaces. The scheme is built by replacing classical Cesàro averages with lacunary block means and embedding them into a multi–step Kirk-type iteration via convex combinations of shifted iterates. We establish boundedness of the generated sequence under a summability condition on the lacunary asymptotic deviations induced by the asymptotic parameters of the mapping, and show that the distance to any fixed point admits a limit. Assuming an appropriate asymptotic regularity, we obtain strong convergence in the compact constraint case, and weak convergence in uniformly convex Banach spaces satisfying Opial's condition under a demiclosedness hypothesis. We also investigate the lacunary statistical convergence behavior of the proposed iteration and establish a connection between strong convergence and lacunary statistical convergence to fixed points. Our results yield lacunary analogues of several Cesàro and ergodic fixed point approximation schemes.

Key words: Asymptotically nonexpansive mapping, fixed point, lacunary sequence, Cesàro mean, Kirk iteration, iterative approximation, convergence

1. Introduction

Fixed point theory is one of the most active and influential areas in nonlinear analysis. One of the most fundamental results in this field is the Banach contraction principle [2]. This theorem serves as a cornerstone for the development of fixed point theory and has inspired extensive research over the past century. Numerous extensions and applications have been established in various branches of mathematics and applied sciences. In parallel with the development of fixed point theory, several classes of mappings and iterative approximation techniques have been introduced and investigated [3, 4, 11, 13, 15, 18, 22, 24].

Among these classes of mappings, asymptotically nonexpansive mappings play a particularly important role. These mappings were introduced by Goebel and Kirk in 1972 as a natural generalization of nonexpansive mappings. They proved that every asymptotically nonexpansive self–mapping defined on a nonempty closed convex subset of a uniformly convex Banach space possesses a fixed point [8, 12]. Since then, asymptotically nonexpansive mappings have been widely studied in the context of iterative approximation of fixed points.

A variety of iterative algorithms have been proposed for approximating fixed points of nonlinear operators. One of the most influential methods is the Halpern iteration, introduced by Halpern [9] for a nonexpansive mapping $S : C \to C$

$$u_{n+1} = \alpha_n \iota + (1 - \alpha_n) S u_n, \quad n \geq 0,$$

where $\iota, u_0 \in C$ and $\{\alpha_n\} \subset [0, 1]$. Over the past several decades, many authors have studied the convergence properties of this scheme under different assumptions. Lions [14] proved strong convergence of the iteration under

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suitable choices of α_n, while Wittmann [23] obtained further convergence results under additional conditions. Reich [17] established strong convergence in Hilbert spaces, and Shioji and Takahashi [19] extended these results to uniformly convex Banach spaces with uniformly Gâteaux differentiable norms.

In recent decades, significant progress has been made in the study of nonexpansive mappings and their generalizations. Several new classes of operators have been introduced and extensively investigated in both Banach and metric space settings. These developments have produced a rich body of literature in fixed point theory and approximation methods; see [4, 16, 18, 22, 24] and the references therein.

Another important direction of research concerns ergodic approximation methods. In this framework, Cesàro averages of iterates are used to stabilize the behavior of iterative sequences. In particular, Baillon [1] proved a nonlinear ergodic theorem stating that if C is a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ is a nonexpansive mapping with $F(S) \neq \varnothing$, then for each $u \in C$ the sequence defined by

$$S_n u = \frac{1}{n+1} \sum_{j=0}^{n} S^j u$$

converges weakly to a fixed point of S. Recently, Cona and Şimşek [6] studied the convergence of a Kirk-type iteration generated by classical Cesàro means for asymptotically nonexpansive mappings.

While classical Cesàro averaging considers averages of all iterates, another important summability concept is based on lacunary sequences. Lacunary methods originate from summability theory and provide block-type averaging processes determined by rapidly increasing index sequences. Such techniques allow the study of convergence behavior through sparse substructures of a sequence and have proven useful in various areas including summability theory, ergodic theory, and sequence spaces.

The advantages of lacunary averaging over classical Cesàro averaging are twofold. First, lacunary methods capture the asymptotic behavior of a sequence through sparse blocks, which can lead to faster stabilization when the sequence exhibits oscillatory behavior between blocks. Second, from a computational perspective, lacunary averages require significantly fewer iterations to be updated, making them more efficient in scenarios where function evaluations are costly. These properties make lacunary approaches particularly attractive for fixed point approximation problems in large-scale or infinite-dimensional settings.

Motivated by these developments, the aim of this paper is to introduce a new iterative approximation method that combines lacunary Cesàro means with a multi–step Kirk type iteration for asymptotically nonexpansive mappings. More precisely, we study the convergence properties of an iterative process generated via lacunary block averages of operator iterates.

The remainder of the paper is organized as follows. In Section 2, we present the preliminary concepts and notations used throughout the paper. In Section 3, we introduce the lacunary Cesàro–Kirk iteration and establish boundedness and convergence results for asymptotically nonexpansive mappings. The final section provides illustrative examples and numerical experiments supporting the theoretical results.

2. Preliminaries

Let E be a Banach space. Throughout this paper, if (u_n) is a sequence in E, then $u_n \to u$ denotes the strong convergence of (u_n) to u, and $u_n \rightharpoonup u$ denotes the weak convergence of (u_n) to u.

Let C be a nonempty subset of E.

Definition 2.1. [7] A sequence $\theta = \{k_r\}$ of nonnegative integers is called a lacunary sequence if

$$k_0 = 0, \qquad h_r = k_r - k_{r-1} \to \infty \quad (r \to \infty).$$

The intervals determined by θ are defined by

$$I_r = (k_{r-1}, k_r], \qquad r \geq 1.$$

Definition 2.2. Let E be a Banach space. The space E is said to be uniformly convex if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta_\varepsilon$$

whenever $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$.

Definition 2.3. Let L be a normed space and let $B = \{u \in L : \|u\| = 1\}$ denote the unit sphere. The norm of L is said to be Gâteaux differentiable at $u \in B$ if the limit

$$\lim_{t \to 0} \frac{\|u + tv\| - \|u\|}{t}$$

exists for every $v \in L$. If the norm is Gâteaux differentiable at each point of B, then the norm of L is said to be Gâteaux differentiable.

Summability methods are widely used to study convergence behavior of sequences and series. Among these methods, the Cesàro summability method plays a central role in summability theory. In recent years, lacunary summability methods have also attracted considerable attention due to their ability to describe the convergence behavior of sequences via block averages determined by lacunary sequences.

Definition 2.4. Let C be a nonempty set and $S : C \to C$ be a mapping. If there exists $u \in C$ such that

$$Su = u,$$

then u is called a fixed point of S. The set of all fixed points of S is denoted by $F(S)$.

Definition 2.5. Let (E, d) be a metric space and $S : E \to E$ be a mapping. If there exists $\lambda \in (0,1)$ such that

$$d(Su, Sv) \leq \lambda d(u, v)$$

for all $u, v \in E$, then S is called a contraction mapping.

Definition 2.6. Let E be a normed space and let $C \subseteq E$. A mapping $S : C \to C$ is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\| \qquad (u, v \in C).$$

Definition 2.7. Let E be a normed space and C be a nonempty closed convex subset of E. A mapping $S : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{t_n\} \subset [1, \infty)$ with $t_n \to 1$ as $n \to \infty$ such that

$$\|S^n u - S^n v\| \leq t_n \|u - v\|$$

for all $u, v \in C$ and $n \geq 1$.

Definition 2.8. Let $\theta = \{k_r\}$ be a lacunary sequence. For $u \in C$, the lacunary Cesàro mean associated with the mapping S is defined by

$$S_{\theta,r}(u) = \frac{1}{h_r} \sum_{j \in I_r} S^j u.$$

Iterative approximation methods play a central role in fixed point theory. In general, an iterative scheme can be expressed as

$$\begin{cases} u_0 \in C, \\ u_{n+1} = f(S, u_n), \qquad n = 0, 1, 2, \dots \end{cases}$$

Definition 2.9. The iteration defined by

$$u_{n+1} = Su_n, \qquad n \geq 0,$$

is called the Picard iteration. Equivalently, $u_n = S^n u_0$ for all $n \geq 0$.

Definition 2.10. Let $k \geq 1$ and let $\alpha_i \geq 0$ $(0 \leq i \leq k)$ satisfy

$$\sum_{i=0}^{k} \alpha_i = 1.$$

The iteration

$$u_{n+1} = \sum_{i=0}^{k} \alpha_i S^i u_n$$

is called the Kirk iteration.

Definition 2.11. Let C be a subset of a Banach space E and $S : C \to C$. We say that $I - S$ is demiclosed at 0 if whenever $u_n \rightharpoonup u$ and $\|u_n - Su_n\| \to 0$, then $u \in F(S)$.

Remark 2.1. *For asymptotically nonexpansive mappings, demiclosedness of $I - S$ at 0 is typically ensured under additional geometric assumptions on E and/or regularity of S. Since our weak convergence theorem only requires this property as a hypothesis, we explicitly assume it whenever needed.*

The following lemmas will be useful in the sequel.

Lemma 2.1. *(See, e.g., [24, Lemma 2.2] or standard discrete Grönwall-type inequalities.) Let $(a_n), (b_n), (c_n)$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + b_n) a_n + c_n.$$

If $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.

3. Results

Ergodic theory, originating from the works of von Neumann and Birkhoff, plays a central role in the interplay between dynamical systems, functional analysis, and summability theory. In the setting of fixed point approximation, ergodic methods often rely on averaging the iterates of an operator in order to stabilize the behavior of iterative schemes.

In the classical Cesàro framework one considers the averages

$$S_n u = \frac{1}{n+1} \sum_{j=0}^{n} S^j u,$$

and studies the convergence of $(S_n u)$ to a fixed point of S. In this paper we adopt a lacunary approach: instead of averaging over all indices up to n, we average over lacunary blocks determined by a lacunary sequence $\theta = \{k_r\}$ with intervals $I_r = (k_{r-1}, k_r]$ and block lengths $h_r = k_r - k_{r-1} \to \infty$.

3.1. Lacunary Cesàro–Kirk iteration

Let E be a Banach space, C be a nonempty closed convex subset of E, and $S : C \to C$ be an asymptotically nonexpansive mapping with parameters $\{t_n\} \subset [1, \infty)$, $t_n \to 1$. For $u \in C$, define the lacunary Cesàro mean by

$$S_{\theta,r}(u) = \frac{1}{h_r} \sum_{j \in I_r} S^j u.$$

Let $k \geq 1$, $\alpha_i \geq 0$ $(0 \leq i \leq k)$ with $\alpha_1 > 0$ and $\sum_{i=0}^{k} \alpha_i = 1$. We consider the lacunary Cesàro–Kirk scheme

$$u_{r+1} = \alpha_0 u_r + \sum_{i=1}^{k} \alpha_i \left(\frac{1}{h_r} \sum_{j \in I_r} S^{j+i} u_r \right), \qquad r \geq 0. \tag{1}$$

3.2. Boundedness

For $r \geq 1$, define the (weighted) lacunary asymptotic deviation by

$$\widetilde{\delta}_r^{\theta} := \sum_{i=0}^{k} \alpha_i \left(\frac{1}{h_r} \sum_{j \in I_r} (t_{j+i} - 1) \right). \tag{2}$$

Remark 3.1. *The quantity $\widetilde{\delta}_r^{\theta}$ measures, in a lacunary block sense, how far the asymptotic parameters (t_n) deviate from the ideal nonexpansive case $t_n \equiv 1$. It is called* weighted *because it aggregates the block deviations of the shifted indices $j+i$ through the convex weights α_i coming from the Kirk combination.*

More precisely, $\frac{1}{h_r} \sum_{j \in I_r} (t_{j+i} - 1)$ is the average excess expansion on the block I_r for the iterate-shift i, and $\widetilde{\delta}_r^{\theta}$ is the convex mixture of these excess expansions across $i = 0, \ldots, k$. Thus the summability condition $\sum_{r \geq 1} \widetilde{\delta}_r^{\theta} < \infty$ ensures that the accumulated excess expansion along lacunary blocks is finite, which yields a quasi-Fejér type control of the distances $(\|u_r - p\|)$.

Lemma 3.1. *Let E be a Banach space and let C be a nonempty closed convex subset of E. Suppose that $S : C \to C$ is an asymptotically nonexpansive mapping with parameters $\{t_n\}$ satisfying $t_n \to 1$. Let (u_r) be generated by (1). If*

$$\sum_{r=1}^{\infty} \widetilde{\delta}_r^{\theta} < \infty,$$

then the sequence (u_r) *is bounded. Moreover, for each* $p \in F(S)$*, the limit*

$$\lim_{r\to\infty} \|u_r - p\|$$

exists.

Proof. Fix $p \in F(S)$. By asymptotic nonexpansiveness,

$$\|S^{j+i}u_r - S^{j+i}p\| \le t_{j+i}\|u_r - p\| \qquad (j \in I_r,\ 0 \le i \le k).$$

Using (1), convexity, and the above inequality, we obtain

$$\|u_{r+1} - p\| \le \sum_{i=0}^{k} \alpha_i \left(\frac{1}{h_r} \sum_{j\in I_r} t_{j+i} \right) \|u_r - p\| = \sum_{i=0}^{k} \alpha_i \left(1 + \frac{1}{h_r} \sum_{j\in I_r} (t_{j+i} - 1) \right) \|u_r - p\|$$

$$= \left(1 + \widetilde{\delta}_r^{\theta}\right) \|u_r - p\|.$$

Set $a_r = \|u_r - p\|$, $b_r = \widetilde{\delta}_r^{\theta}$, $c_r = 0$. Then

$$a_{r+1} \le (1 + b_r)a_r + c_r.$$

Since $\sum_r b_r < \infty$, Lemma 2.1 yields that $\lim_{r\to\infty} a_r$ exists, in particular (a_r) is bounded, hence (u_r) is bounded. □

3.3. Asymptotic regularity

The asymptotic regularity condition $\|u_r - Su_r\| \to 0$ plays a crucial role in the convergence theorems. The following lemma provides a useful representation and an estimate that can be used when asymptotic regularity is available.

Lemma 3.2. *Let* E*,* C*,* S*, and* (u_r) *be as in Lemma 3.1. Define the lacunary Cesàro mean*

$$S_{\theta,r}(u) := \frac{1}{h_r} \sum_{j\in I_r} S^j u \qquad (u \in C).$$

Then the sequence (u_r) *generated by* (1) *satisfies, for every* $r \ge 0$*,*

$$u_{r+1} = \alpha_0 u_r + \sum_{i=1}^{k} \alpha_i\, S_{\theta,r}\big(S^i u_r\big).$$

In particular,

$$\|u_{r+1} - S_{\theta,r}(u_r)\| \le \alpha_0 \|u_r - S_{\theta,r}(u_r)\| + \sum_{i=1}^{k} \alpha_i \big\|S_{\theta,r}(S^i u_r) - S_{\theta,r}(u_r)\big\|.$$

Proof. By definition,

$$S_{\theta,r}(S^i u_r) = \frac{1}{h_r} \sum_{j\in I_r} S^{j+i} u_r.$$

Substituting this identity into (1) yields

$$u_{r+1} = \alpha_0 u_r + \sum_{i=1}^{k} \alpha_i \Big(\frac{1}{h_r} \sum_{j \in I_r} S^{j+i} u_r \Big) = \alpha_0 u_r + \sum_{i=1}^{k} \alpha_i \, S_{\theta,r}(S^i u_r).$$

Hence

$$u_{r+1} - S_{\theta,r}(u_r) = \alpha_0 \big(u_r - S_{\theta,r}(u_r) \big) + \sum_{i=1}^{k} \alpha_i \big(S_{\theta,r}(S^i u_r) - S_{\theta,r}(u_r) \big).$$

Note that the $i = 0$ contribution in (1) is $\alpha_0 u_r$, hence in $u_{r+1} - S_{\theta,r}(u_r)$ it becomes $\alpha_0 \big(u_r - S_{\theta,r}(u_r) \big)$. Taking norms and using the triangle inequality, we obtain

$$\|u_{r+1} - S_{\theta,r}(u_r)\| \le \alpha_0 \|u_r - S_{\theta,r}(u_r)\| + \sum_{i=1}^{k} \alpha_i \|S_{\theta,r}(S^i u_r) - S_{\theta,r}(u_r)\|.$$

□

Remark 3.2. *In practice, the condition* $\lim_{r\to\infty} \frac{1}{h_r} \sum_{j \in I_r} (t_{j+i} - t_j) = 0$ *is automatically satisfied if* $t_n \to 1$ *sufficiently fast and the lacunary blocks are not too large. For example, if* $t_n = 1 + \frac{c}{n^\beta}$ *with* $\beta > 0$*, then this condition holds for any lacunary sequence with* $h_r = O(k_r)$*.*

3.4. Strong convergence under compactness

Remark 3.3. *The asymptotic regularity condition*

$$\|u_r - S u_r\| \to 0$$

appearing in Theorems 3.1 and 3.2 is imposed as an additional hypothesis. It is not claimed here to follow automatically from the lacunary Cesàro–Kirk iteration defined by (1). In general, such a property may require extra assumptions on the mapping S*, on the parameters of the iterative scheme, or on the lacunary sequence* θ*. Therefore, Theorems 3.1 and 3.2 should be understood as conditional convergence results obtained under the asymptotic regularity assumption.*

Theorem 3.1. *Let* E *be a Banach space and let* C *be a nonempty compact convex subset of* E*. Let* $S : C \to C$ *be an asymptotically nonexpansive mapping with parameters* $\{t_n\} \subset [1, \infty)$ *satisfying* $t_n \to 1$*. Let* (u_r) *be generated by (1). Assume that*

$$\sum_{r=1}^{\infty} \widetilde{\delta}_r^{\theta} < \infty \quad \text{and} \quad \|u_r - S u_r\| \to 0 \quad \text{(see Remark 3.3).}$$

Then (u_r) *converges strongly to some* $u^* \in F(S)$*.*

Proof. Fix any $p \in F(S)$. By Lemma 3.1, the limit

$$\lim_{r\to\infty} \|u_r - p\|$$

exists, and in particular (u_r) is bounded. Since $u_r \in C$ and C is compact, there exists a subsequence (u_{r_m}) and a point $u^* \in C$ such that

$$u_{r_m} \to u^* \quad \text{(strongly)}.$$

We claim that $u^* \in F(S)$. Indeed, by asymptotic nonexpansiveness, taking $n = 1$ we have

$$\|Su - Sv\| \leq t_1\|u - v\| \qquad (u, v \in C),$$

so S is Lipschitz and hence continuous on C. Therefore,

$$Su_{r_m} \to Su^*.$$

On the other hand, by the asymptotic regularity assumption,

$$\|Su_{r_m} - u_{r_m}\| \to 0.$$

Hence

$$\|Su^* - u^*\| \leq \|Su^* - Su_{r_m}\| + \|Su_{r_m} - u_{r_m}\| + \|u_{r_m} - u^*\| \to 0,$$

which implies $Su^* = u^*$. Thus $u^* \in F(S)$.

Finally, by Lemma 3.1 (applied with $p = u^* \in F(S)$), the limit $\lim_{r\to\infty} \|u_r - u^*\|$ exists. Since a subsequence satisfies $\|u_{r_m} - u^*\| \to 0$, the only possible value of this limit is 0. Therefore $u_r \to u^*$ strongly. □

Remark 3.4. *The asymptotic regularity assumption $\|u_r - Su_r\| \to 0$ is imposed with respect to S (not to the lacunary mean $S_{\theta,r}$). In general, $\|u_r - Su_r\| \to 0$ does not automatically imply $\|u_r - S_{\theta,r}(u_r)\| \to 0$, nor conversely, without additional conditions on (t_n) and the lacunary blocks. Lemma 3.2 provides sufficient conditions for this property.*

3.5. Weak convergence under Opial's condition

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E satisfying Opial's condition. Let $S : C \to C$ be asymptotically nonexpansive with $t_n \to 1$. Let (u_r) be generated by (1). Assume that*

$$\sum_{r=1}^{\infty} \widetilde{\delta}_r^{\theta} < \infty, \quad \|u_r - Su_r\| \to 0 \quad \textit{(see Remark 3.3)}, \quad \textit{and} \quad I - S \textit{ is demiclosed at } 0.$$

Then (u_r) converges weakly to some $u^* \in F(S)$.

Proof. By Lemma 3.1, (u_r) is bounded. Since E is uniformly convex, it is reflexive, hence (u_r) has weakly convergent subsequences. Let $u_{r_m} \rightharpoonup u^*$.

Because $\|u_{r_m} - Su_{r_m}\| \to 0$ and $I - S$ is demiclosed at 0, we conclude that $u^* \in F(S)$.

We next prove that the weak limit is unique. Suppose, to the contrary, that there exists another subsequence (u_{r_j}) and a point $v \in E$ such that

$$u_{r_j} \rightharpoonup v, \qquad v \neq u^*.$$

Again, by demiclosedness and $\|u_{r_j} - Su_{r_j}\| \to 0$, we have $v \in F(S)$.

By Lemma 3.1, for every $p \in F(S)$ the limit

$$\ell(p) := \lim_{r\to\infty} \|u_r - p\|$$

exists. In particular, $\ell(u^*)$ and $\ell(v)$ exist.

Now apply Opial's condition to the subsequence $u_{r_m} \rightharpoonup u^*$. Since $v \neq u^*$, Opial's condition yields

$$\liminf_{m\to\infty} \|u_{r_m} - u^*\| < \liminf_{m\to\infty} \|u_{r_m} - v\|.$$

But because the full limits exist (not only liminf), we have

$$\lim_{m\to\infty} \|u_{r_m} - u^*\| = \ell(u^*), \qquad \lim_{m\to\infty} \|u_{r_m} - v\| = \ell(v),$$

and thus

$$\ell(u^*) < \ell(v). \tag{1}$$

Similarly, apply Opial's condition to the subsequence $u_{r_j} \rightharpoonup v$. Since $u^* \neq v$, we obtain

$$\liminf_{j\to\infty} \|u_{r_j} - v\| < \liminf_{j\to\infty} \|u_{r_j} - u^*\|.$$

Using again existence of the full limits,

$$\lim_{j\to\infty} \|u_{r_j} - v\| = \ell(v), \qquad \lim_{j\to\infty} \|u_{r_j} - u^*\| = \ell(u^*),$$

we get

$$\ell(v) < \ell(u^*). \tag{2}$$

Inequalities (1) and (2) contradict each other. Therefore, the weak cluster point of (u_r) is unique; hence the entire sequence (u_r) converges weakly to $u^* \in F(S)$. □

4. Lacunary Statistical Cesàro–Kirk Iteration

In this section we connect the lacunary Cesàro–Kirk scheme (1) with the notion of *lacunary statistical convergence* determined by a given lacunary sequence $\theta = \{k_r\}$. The motivation is twofold. First, lacunary statistical convergence provides a natural intermediate notion between pointwise (strong) convergence and purely qualitative asymptotic information, and it is particularly suitable when the sequence exhibits occasional irregular deviations that are sparse inside each lacunary block. Second, since our iteration itself is built from lacunary block means, it is conceptually consistent to record stability of the method under a block-based convergence concept.

Recall that lacunary statistical convergence measures the "density of bad indices" within each lacunary block $I_r = (k_{r-1}, k_r]$ rather than among all indices $\{1, \ldots, n\}$. Thus, it can be viewed as a robustness notion: if the iterates (u_r) approach a limit up to a set of indices of asymptotically negligible proportion in each block, then the method is still considered convergent in the lacunary statistical sense.

Definition 4.1 (Lacunary statistical convergence)**.** Let $\theta = \{k_r\}$ be a lacunary sequence with $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$, and $I_r = (k_{r-1}, k_r]$. A sequence (x_n) in a normed space E is said to be *lacunary statistically convergent* to $L \in E$ (with respect to θ) if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{\, n \in I_r : \ \|x_n - L\| \geq \varepsilon \,\}\right| = 0.$$

In this case we write $st_\theta\text{-}\lim x_n = L$.

The next proposition records the basic implication that strong convergence dominates lacunary statistical convergence, independently of the choice of θ. This is useful in our setting because Theorem 3.1 yields strong convergence under compactness and asymptotic regularity, hence it automatically provides a lacunary statistical convergence interpretation of the proposed method.

Proposition 4.1. *If $x_n \to L$ strongly in E, then $st_\theta\text{-}\lim x_n = L$ for every lacunary sequence θ.*

Proof. Fix $\varepsilon > 0$. Since $x_n \to L$, there exists N such that $\|x_n - L\| < \varepsilon$ for all $n \geq N$. Hence, for every r with $k_{r-1} \geq N$,

$$\{\, n \in I_r : \ \|x_n - L\| \geq \varepsilon \,\} = \varnothing,$$

so its cardinality is 0. Therefore

$$\frac{1}{h_r}\left|\{\, n \in I_r : \ \|x_n - L\| \geq \varepsilon \,\}\right| = 0 \quad \text{for all sufficiently large } r,$$

which yields $st_\theta\text{-}\lim x_n = L$. □

Definition 4.2 (Lacunary statistical Cesàro–Kirk convergence)**.** Let (u_r) be generated by (1). We say that (u_r) converges *lacunary statistically* to a fixed point $p \in F(S)$ if

$$st_\theta\text{-}\lim u_r = p.$$

Corollary 4.1. *Under the assumptions of Theorem 3.1, there exists $u^* \in F(S)$ such that*

$$st_\theta\text{-}\lim u_r = u^*.$$

Proof. By Theorem 3.1, $u_r \to u^*$ strongly for some $u^* \in F(S)$. The conclusion follows from Proposition 4.1. □

Remark 4.1. *Corollary 4.1 shows that the lacunary Cesàro–Kirk method admits a lacunary statistical convergence interpretation in the strong convergence regime (compact constraint case). In particular, even if one is interested in a weaker convergence concept that is compatible with block-type averaging, the method remains stable whenever strong convergence holds.*

Remark 4.2. *The converse implication is false in general: lacunary statistical convergence does not necessarily imply strong convergence, since a sequence may deviate from its limit on a sparse subset of indices within each block while still failing to converge in norm. Moreover, weak convergence does not in general imply lacunary statistical convergence. Therefore, in the present work we record the implication in the strong convergence regime obtained in Theorem 3.1.*

Theorem 4.1. *Let E be a Banach space and let $C \subset E$ be nonempty, closed and convex. Let $S : C \to C$ be asymptotically nonexpansive and let (u_r) be generated by (1). Suppose that the assumptions of Lemma 3.1 hold, so that for each $p \in F(S)$ the limit*

$$\lim_{r\to\infty} \|u_r - p\|$$

exists. If there exists $p \in F(S)$ such that

$$st_\theta\text{-}\lim u_r = p,$$

then $u_r \to p$ strongly.

Proof. Let $d_r = \|u_r - p\|$ and let $\ell = \lim_{r\to\infty} d_r$, which exists by Lemma 3.1. Assume $\ell > 0$ and choose $\varepsilon = \ell/2$. Then there exists R such that $d_r \geq \varepsilon$ for all $r \geq R$. Hence, for every m with $k_{m-1} \geq R$, we have $r \geq R$ for all $r \in I_m$ and thus

$$\{\, r \in I_m : \ \|u_r - p\| \geq \varepsilon \,\} = I_m,$$

so

$$\frac{1}{h_m}\big|\{\, r \in I_m : \ \|u_r - p\| \geq \varepsilon \,\}\big| = 1,$$

which contradicts $st_\theta\text{-}\lim u_r = p$. Consequently $\ell = 0$, i.e. $\|u_r - p\| \to 0$, so $u_r \to p$ strongly. □

5. Numerical Examples

In this section we illustrate the convergence behavior of the proposed lacunary Cesàro–Kirk iteration and compare it with the classical Cesàro–Kirk iteration.

Example 5.1. *Consider the linear mapping*

$$S(u) = \frac{u+2}{3}, \qquad u \in \mathbb{R}.$$

The fixed point of S is obtained from

$$u = \frac{u+2}{3},$$

which yields

$$u = 1.$$

Hence the unique fixed point is $p = 1$.

Note that this mapping is actually a contraction with Lipschitz constant $1/3$, and therefore it is also asymptotically nonexpansive with $t_n = 1$ for all n. While this is a simple test case, it allows us to verify the basic convergence properties of the proposed iteration before considering more complex mappings.

Let the initial value be

$$u_0 = 5.$$

For the lacunary sequence we choose

$$k_r = 2^r, \qquad r \geq 1,$$

with intervals

$$I_r = (k_{r-1}, k_r], \qquad h_r = k_r - k_{r-1}.$$

For numerical computation we take

$$\alpha_0 = 0.4, \qquad \alpha_1 = 0.6.$$

The classical Cesàro–Kirk iteration is defined by

$$u_{n+1} = \alpha_0 u_n + \alpha_1 \frac{1}{n+1} \sum_{j=0}^{n} S^j(u_n),$$

while the lacunary Cesàro–Kirk iteration is given by

$$u_{r+1} = \alpha_0 u_r + \alpha_1 \frac{1}{h_r} \sum_{j \in I_r} S^j(u_r).$$

Table 1 presents the first few iterates of both methods.

Iteration	Cesàro–Kirk	Lacunary Cesàro–Kirk
0	5.0000	5.0000
1	3.4000	3.1333
2	2.2800	1.8849
3	1.6353	1.3548
4	1.3012	1.1419
5	1.1385	1.0568

Table 1. Comparison of Cesàro–Kirk and lacunary Cesàro–Kirk iterations for $S(u) = \frac{u+2}{3}$.

Figure 1. Convergence behavior of Cesàro–Kirk and lacunary Cesàro–Kirk iterations for the mapping $S(u) = \frac{u+2}{3}$. Both sequences converge to the fixed point $p = 1$, with the lacunary version showing faster initial convergence.

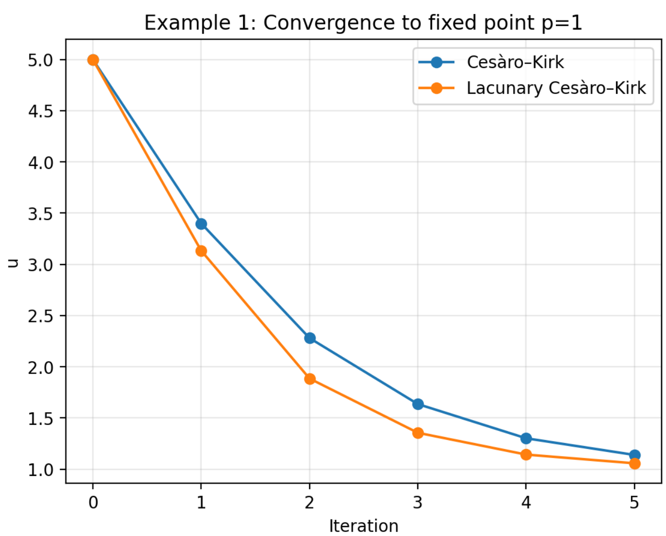


Figure 1 illustrates the convergence behavior. The lacunary averaging stabilizes the iteration and leads to faster convergence to the fixed point.

Example 5.2. *Let $E = \mathbb{R}^2$ equipped with the Euclidean norm and let*

$$C = [-2, 2] \times [-2, 2] \subset \mathbb{R}^2.$$

Define $S : C \to C$ by

$$S(x, y) = \left(\tfrac{1}{2}x + \tfrac{1}{2}, \tfrac{1}{2}y - \tfrac{1}{2}\right).$$

Then S is a contraction with Lipschitz constant $1/2$, hence it is nonexpansive and therefore asymptotically nonexpansive with $t_n \equiv 1$.

The unique fixed point $p = (x^*, y^*)$ satisfies

$$(x^*, y^*) = \left(\tfrac{1}{2}x^* + \tfrac{1}{2}, \tfrac{1}{2}y^* - \tfrac{1}{2}\right),$$

hence $x^* = 1$ and $y^* = -1$, i.e., $p = (1, -1)$.

We apply the lacunary Cesàro–Kirk iteration (1) with $k = 1$:

$$u_{r+1} = \alpha_0 u_r + \alpha_1 \left(\frac{1}{h_r} \sum_{j \in I_r} S^{j+1}(u_r) \right) = \alpha_0 u_r + \alpha_1 S_{\theta,r}(S u_r), \qquad r \geq 1,$$

where

$$u_0 = (2, 2), \qquad \alpha_0 = 0.4, \qquad \alpha_1 = 0.6,$$

and we choose the lacunary sequence $k_r = 2^r$ $(r \geq 1)$, so that $I_r = (2^{r-1}, 2^r]$ and $h_r = 2^{r-1}$.

For comparison, we also compute the classical Cesàro–Kirk iteration (with the same $k = 1$ shift convention):

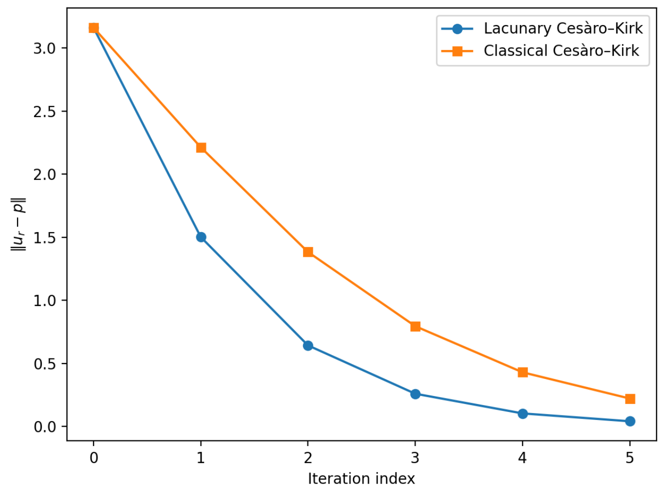
$$u_{n+1} = \alpha_0 u_n + \alpha_1 \frac{1}{n+1} \sum_{j=0}^{n} S^{j+1}(u_n), \qquad n \geq 0.$$

Table 2 reports the distances $\|u_r - p\|$ for both schemes.

r	$\|u_r - p\|$ (lacunary)	$\|u_r - p\|$ (classical)
0	3.1623	3.1623
1	1.5021	2.2136
2	0.6431	1.3835
3	0.2601	0.7955
4	0.1041	0.4301
5	0.0416	0.2220

Table 2. Comparison of lacunary and classical Cesàro–Kirk iterations for $S(x, y) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y - \frac{1}{2})$ with $k_r = 2^r$, $\alpha_0 = 0.4$, $\alpha_1 = 0.6$.

Figure 2. Convergence of classical and lacunary Cesàro–Kirk iterations toward the fixed point $p = (1, -1)$.



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