




Higher-order Cartan derivatives and curvature tensor decomposition in Finsler spaces: insights into mathematical and physical applications

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Abstract: This paper delves into the intricate structure of curvature tensors within the realm of Finsler geometry. By harnessing the power of higher-order Cartan derivatives, we introduce a novel decomposition scheme for curvature tensors. This innovative approach not only provides deeper insights into the geometric properties of Finsler spaces but also establishes a foundational framework for further investigations. Our findings reveal that the proposed decomposition is instrumental in unraveling the connections between curvature, torsion, and the underlying metric structure. Moreover, we demonstrate the applicability of our results to various subdomains of Finsler geometry, including Finsler information geometry and Finsler cosmology.

Key words: Finsler space, Cartan's covariant derivative expansion, curvature tensor, geometric properties.

1. Introduction

Finsler geometry, as a generalization of Riemannian geometry, offers a flexible framework for modeling diverse physical phenomena characterized by anisotropic and position-dependent metrics. Central to the study of Finsler geometry are curvature tensors, which encapsulate the intrinsic curvature properties of the underlying space. While significant progress has been made in understanding curvature tensors in Riemannian geometry, their counterparts in Finsler geometry exhibit a richer and more complex structure. Traditional approaches to analyzing curvature tensors in Finsler geometry often rely on the concept of Cartan connection. However, these methods can become cumbersome when dealing with higher-order geometric quantities. In this paper, we propose a novel approach that leverages the power of higher-order Cartan derivatives to systematically decompose curvature tensors.

This decomposition not only simplifies the analysis of curvature but also reveals new connections between curvature, torsion, and the metric structure.

The study of curvature tensors in Finsler spaces is of paramount importance due to their role in characterizing the intrinsic curvature of these spaces. These tensors encapsulate information about the deviation of geodesics and the parallel transport of vectors. By investigating the expansion identities for curvature tensors, we seek to uncover deeper connections between the various curvature invariants and to gain a more comprehensive understanding of the curvature properties of Finsler spaces.

Finsler geometry, as a generalization of Riemannian geometry, provides a powerful framework for studying manifolds equipped with a position-dependent metric. Curvature tensors, fundamental objects in differential

geometry, play a crucial role in characterizing the intrinsic geometric properties of Finsler spaces.

This paper delves into the intricate structure of curvature tensors in Finsler spaces, building upon previous investigations [1–6],[8–13]. We explore [1, 2] which investigates certain identities and recurrence properties of curvature tensors. Furthermore, we leverage the insights from [3–5] that examine curvature tensors using higher-order derivatives of Berwald and Cartan, as well as generalized recurrent Finsler spaces. Building upon these foundational works and drawing inspiration from the seminal contributions of Cartan [7] and Rund [10], we aim to develop a novel decomposition for curvature tensors, investigate the relationship between curvature and torsion. Two vectors y_i and y^i meet the following conditions:

$$\begin{aligned} \text{a) } y_i &= g_{ij}y^j, & \text{b) } y_iy^i &= F^2, & \text{c) } \delta_k^j y_j &= y_k, \\ \text{d) } g_{ir}\delta_j^i &= g_{rj}, & \text{e) } g_{jk}\delta_i^k &= g_{ji}, & \text{f) } \partial_j y^j &= 1, & \text{g) } \dot{\partial}_k y_j &= g_{jk}. \end{aligned} \quad (1.1)$$

The quantities g_{ij} and g^{ij} are covariant constants with respect to the h -covariant derivative:

$$\begin{aligned} \text{a) } g_{ij}g^{jk} &= \delta_i^k = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k, \end{cases} \\ \text{b) } g_{\frac{jk}{h}} &= 0, & \text{c) } g_{\frac{jj}{h}} &= 0. \end{aligned} \quad (1.2)$$

Tensor C_{ijk} is known as the $(h)hv$ -torsion tensor defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk} = \frac{1}{4}\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F^2. \quad (1.3)$$

The $(v)hv$ -torsion tensor C_{jk}^i and tensor C_{ijk} are given by

$$\begin{aligned} \text{a) } C_{jk}^i y_j &= C_{jki} y_k = 0, & \text{b) } C_{ijk} y^i &= C_{ijk} y^j = C_{ijk} y^k = 0, \\ \text{c) } g^{jk} C_{ijk} &= C_i, & \text{d) } g^{jk} C_{ijh} &= C_{ih}^k. \end{aligned} \quad (1.4)$$

The vector y^i and metric function F vanish identically for Cartan's covariant derivative:

$$\text{a) } F|_h = 0, \quad \text{b) } y^i|_h = 0. \quad (1.5)$$

Cartan [7] deduced the covariant derivatives of an arbitrary vector field X^i with respect to x^k , which are given by

$$X^i|_k = \dot{\partial}_k X^i + X^r C_{rk}^i. \quad (1.6)$$

and

$$X^i|_k = \partial_k X^i - \dot{\partial}_r X^i G_k^r + X^r \Gamma_{rk}^{*i}. \quad (1.7)$$

where the function Γ_{rk}^{*i} is defined by

$$\Gamma_{rk}^{*i} = \Gamma_{rk}^i - C_{mr}^i \Gamma_{sk}^m y^s.$$

The functions Γ_{rk}^{*i} and G_k^r are connected by

$$G_k^r = \Gamma_{sk}^{*r} y^s,$$

where

$$\partial_j = \frac{\partial}{\partial x^j}, \quad \dot{\partial}_j = \frac{\partial}{\partial y^j}, \quad G_j^i = \dot{\partial}_j G^i.$$

The equations (1.6) and (1.7) give two kinds of covariant differentiations which are called v -covariant differentiation (Cartan's first kind covariant derivative) and h -covariant differentiation (Cartan's second kind covariant derivative), respectively. Thus, $X^i|_k$ and $X^i|_{\parallel k}$ are the v -covariant derivative and the h -covariant derivative of the vector field X^i .

In this paper, we introduce the mathematical concepts of tensors, torsion tensors, and deviation tensors. These entities are fundamental in various fields of physics and geometry, particularly in describing the curvature and torsion of manifolds. By providing the precise definitions of W_{jkh}^i , the torsion tensor W_{jk}^i , and the deviation tensor W_j^i , we aim to illuminate their significance in characterizing the geometric properties of spaces.

$$\begin{aligned} W_{jkh}^i &= H_{jkh}^i + \frac{2}{n+1} \delta_j^i H_{hk} + \frac{2}{n+1} y^i \dot{\partial}_j H_{kh} \\ &\quad + \delta_k^i \frac{1}{n^2-1} \left(nH_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr} \right) \\ &\quad - \delta_h^i \frac{1}{n^2-1} \left(nH_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr} \right). \end{aligned} \quad (1.8)$$

$$W_{jk}^i = H_{jk}^i + \frac{1}{n+1} y^i H_{jk} + 2 \left\{ \delta_{[j}^i \frac{1}{n^2-1} H_{k]} - y^r H_{k]r} \right\}. \quad (1.9)$$

$$W_j^i = H_j^i - H \delta_j^i - \frac{1}{n+1} \left(\dot{\partial}_r H_j^r - \dot{\partial}_j H \right) y^i, \quad \text{respectively.} \quad (1.10)$$

The tensors W_{jkh}^i , W_{jk}^i and W_j^i satisfy the following identities:

$$\begin{aligned} \text{a) } W_{jkh}^i y^j &= W_{kh}^i, & \text{b) } W_{kh}^i y^k &= W_h^i, & \text{c) } W_{jki}^i &= W_{jk}, \\ \text{d) } g_{ir} W_{jkh}^i &= W_{rjkh}, & \text{e) } W_{jkh}^i &= -W_{jhk}^i, & \text{f) } W_{jkh}^i + W_{khj}^i + W_{hjk}^i &= 0. \end{aligned} \quad (1.11)$$

Also, if we suppose that the tensor W_j^i satisfies the following identities:

$$\begin{aligned} \text{a) } W_k^i y^k &= 0, & \text{b) } W_i^i &= 0, & \text{c) } g_{ir} W_j^i &= W_{rj}, \\ \text{d) } g^{jk} W_{jk} &= W, & \text{e) } W_{jk} y^k &= 0. \end{aligned} \quad (1.12)$$

The tensor W_{jkh}^i is skew-symmetric in its indices k and h .

Cartan's third curvature tensor R_{jkh}^i , Ricci tensor R_{jk} , the vector H_k , and scalar curvature H are essential tools in differential geometry for analyzing the intrinsic curvature of manifolds. In this study, we focus on the mathematical definitions and interrelationships of these tensors. By examining their algebraic and geometric properties, we seek to understand their contributions to the study of curvature and its implications in various fields of physics.

$$\begin{aligned}
\text{a)} \quad R_{jkh}^i &= \Gamma_{jk,h}^{*i} + \Gamma_{lj}^{*i} G_{kh}^l + C_{jm}^i G_{kh}^m - G_{kl}^m G_{hm}^l + \Gamma_{mk}^{*i} \Gamma_{jh}^{*m} - (k/h), \\
\text{b)} \quad R_{jkh}^i y^j &= H_{kh}^i, \quad \text{c)} \quad R_{jky}^j = H_k, \quad \text{d)} \quad R_{jky}^k = R_j, \quad \text{e)} \quad R_i^i = R, \\
\text{f)} \quad g_{ir} R_{jkh}^i &= R_{rjkh}, \quad \text{g)} \quad R_{jkh}^i = -R_{jhk}^i, \quad \text{h)} \quad g^{jk} R_{jkh}^i = R_h^i, \\
\text{i)} \quad R_{jk}^i i &= R_{jk}, \quad \text{t)} \quad H_i^i y^i = H_i^i = (n-1)H, \\
\text{j)} \quad H_{kh}^i y^k &= H_h^i, \quad \text{k)} \quad H_{ki}^i = H_k.
\end{aligned} \tag{1.13}$$

Cartan's covariant derivative of R_{jk} and δ_k^h is given by

$$\text{a)} \quad R_{jk|m} = \lambda_m R_{jk}, \quad \text{b)} \quad \delta_k^h|_m = 0. \tag{1.14}$$

Also, for the covariant derivative $|_m$, we have

$$\begin{aligned}
\text{a)} \quad (\delta_k^h R_{ij})|_m &= \lambda_m \delta_k^h R_{ij}, \quad \text{b)} \quad (g_{ij} R_{hk})|_m = \lambda_m g_{ij} R_{hk}, \\
\text{c)} \quad (R \delta_h^k g_{ij})|_m &= \lambda_m R \delta_h^k g_{ij}, \quad \text{d)} \quad (R R_{ij})|_m = \lambda_m R R_{ij}.
\end{aligned} \tag{1.15}$$

Cartan's covariant derivative of the tensors T_{jkh}^i , T_{jkh}^i and T_h^i with respect to x^m are defined as

$$\text{a)} \quad T_{jkh|m}^i = \lambda_m T_{jkh}^i, \quad \text{b)} \quad T_{jk|m}^i = \lambda_m T_{jk}^i, \quad \text{c)} \quad T_j^i|_m = \lambda_m T_j^i. \tag{1.16}$$

The plan of the present paper is as follows. After Section 1, which contains the introduction and preliminaries, we study the expansion for any curvature tensor with respect to Cartan's covariant derivative. Section 2 presents the relationships between the Weyl projective curvature tensor and some other curvature tensors. Section 3 studies an expansion of Cartan's covariant derivative for any curvature tensor. In the last section, we investigate the identities introduced in Section 2 by using the obtained expansion.

2. Preliminaries

In Finsler geometry, a fundamental aspect lies in the intricate relationships between various curvature tensors. These relationships are often expressed through elegant mathematical identities. This paper focuses on investigating the interconnections between the Weyl projective curvature tensor and other significant curvature tensors.

2.1. The Riemannian curvature tensor R_{jkh}^i

The Riemann curvature tensor is a fundamental tool in differential geometry that quantifies the intrinsic curvature of a Riemannian manifold. It provides a comprehensive measure of how the geometry of the manifold deviates from that of flat Euclidean space. This paper delves into the properties and significance of the Riemann curvature tensor, exploring its role in characterizing the curvature of various geometric structures. It is a local invariant of Riemannian metrics which measures the failure of the second covariant derivatives to commute. A Riemannian manifold has zero curvature if and only if it is flat, i.e., locally isometric to the Euclidean space. The curvature tensor can also be defined for any pseudo-Riemannian manifold, or indeed any manifold equipped with an affine connection.

The Riemann curvature tensor is a tool used to describe the curvature of n -dimensional spaces such as a Riemannian manifold in the field of differential geometry.

The Riemann curvature tensor plays an important role in the theories of general relativity and gravity as well as in the curvature of spacetime. It is closely related to the Weyl projective curvature tensor.

Definition 2.1. The Weyl projective curvature tensor in terms of the Riemann curvature tensor R_{jkh}^i is defined as [12]:

$$W_{jkh}^i = R_{jkh}^i + \frac{1}{n-1} (\delta_k^i R_{jh} - g_{jk} R_h^i). \quad (2.1)$$

In V_4^F , we have

$$R_{jkh}^i = W_{jkh}^i - \frac{1}{3} (\delta_k^i R_{jh} - g_{jk} R_h^i). \quad (2.2)$$

2.2. Projective Curvature Tensor W_{jkh}^i

The W-projective curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conharmonic curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology.

The properties of the M-projective curvature tensor were proposed by Pokhariyal and Mishra in 1970. This tensor is described as follows:

$$\begin{aligned} \overline{W}(X, Y, Z, T) = \overline{R}(X, Y, Z, T) - \frac{1}{2(n-1)} & \left(S(Y, Z)g(X, T) - S(X, Z)g(Y, T) \right) \\ & + \left(g(Y, Z)S(X, T) - g(X, Z)S(Y, T) \right). \end{aligned} \quad (2.3)$$

where

$$\overline{W}(X, Y, Z, T) = g(W(X, Y)Z, T), \quad \overline{R}(X, Y, Z, T) = g(R(X, Y)Z, T). \quad (2.4)$$

Here, R is the Riemann curvature tensor, S is the Ricci tensor, g is the metric tensor, and n is the dimension of the manifold.

The \overline{W} -projective curvature tensor has a number of interesting properties. For example, it is invariant under conformal transformations, meaning that it remains unchanged for conformally equivalent metrics. The \overline{W} -projective curvature tensor also vanishes if and only if the manifold is Ricci-flat.

The W-projective curvature tensor has been used to study a variety of geometric problems, including classification of Riemannian manifolds, geometry of Kähler manifolds, and models of gravity.

The local coordinate expression of equation (2.3) is given by

$$\overline{W}_{jkh}^l = R_{jkh}^l - \frac{1}{2(n-1)} (R_{jk}\delta_h^l - R_{jh}\delta_k^l + g_{jk}R_h^l - g_{jh}R_k^l). \quad (2.5)$$

Assuming $n = 4$ and using (2.2) in equation (2.5), and contracting with g^{li} , the M-projective curvature tensor is given by

$$\overline{W}_{jkh}^i = W_{jkh}^i - \frac{1}{6} (\delta_h^i R_{jk} + \delta_k^i R_{jh} - g_{jk}R_h^i - g_{jh}R_k^i). \quad (2.6)$$

2.3. Conformal Curvature Tensor C_{jkh}^i

The conformal curvature tensor, also known as the Weyl conformal curvature tensor, is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force.

Definition 2.2. The conformal curvature tensor C_{jkh}^i is expressed as follows:

$$\begin{aligned} C_{jkh}^i &= R_{jkh}^i - \frac{1}{2} (\delta_k^i R_{jh} - \delta_h^i R_{jk} + R_k^i g_{jh} - R_h^i g_{jk}) \\ &\quad - \frac{1}{6} R (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \end{aligned} \quad (2.7a)$$

Using (2.2) in equation (2.7a), we get

$$\begin{aligned} C_{jkh}^i &= W_{jkh}^i - \frac{5}{6} (\delta_k^i R_{jh} - R_h^i g_{jk}) - \frac{1}{6} R (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ &\quad + \frac{1}{2} (\delta_h^i R_{jk} - R_k^i g_{jh}). \end{aligned} \quad (2.7b)$$

2.4. Conharmonic Curvature Tensor L_{jkh}^i

The conharmonic curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conformal curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology.

Definition 2.3. For V_4 , the conharmonic curvature tensor L_{jkh}^i is defined as [10]:

$$L_{jkh}^i = R_{jkh}^i - \frac{1}{2} (g_{jk} R_h^i + \delta_h^i R_{jk} - \delta_k^i R_{jh} - g_{jh} R_k^i). \quad (2.8a)$$

Using (2.2) in equation (2.8a), we get

$$L_{jkh}^i = W_{jkh}^i + \frac{1}{6} (\delta_k^i R_{jh} - R_h^i g_{jk}) - \frac{1}{2} (\delta_h^i R_{jk} - R_k^i g_{jh}). \quad (2.8b)$$

2.5. Concircular Curvature Tensor M_{jkh}^i

The concircular curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the conformal curvature tensor (also known as the Weyl curvature tensor) and the projective curvature tensor. The concircular curvature tensor vanishes if and only if the manifold is concircularly flat.

Definition 2.4. The concircular curvature tensor M_{jkh}^i for V_4 is defined as [3]:

$$M_{jkh}^i = R_{jkh}^i - \frac{1}{12} R (g_{jk} \delta_h^i - g_{jh} \delta_k^i). \quad (2.9)$$

Using (2.2) in equation (2.9), we get

$$M_{jkh}^i = W_{jkh}^i - \frac{1}{12}R(g_{jk}\delta_h^i - g_{jh}\delta_k^i) - \frac{1}{6}(\delta_k^i R_{jh} - R_h^i g_{jk}). \quad (2.10)$$

2.6. P_1 -Curvature Tensor

The P_1 -curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the Ricci curvature tensor and the scalar curvature. The P_1 -curvature tensor vanishes if and only if the manifold is Ricci-flat and has constant scalar curvature.

The tensor $P_1(X, Y, Z, T)$ is defined as

$$P_1(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{2(n-1)}\left(g(Y, Z)\text{Ric}(X, T) - g(Y, T)\text{Ric}(X, Z) - g(X, Z)\text{Ric}(Y, T) + g(X, T)\text{Ric}(Y, Z)\right). \quad (2.11)$$

In index notation, it is given by [8]:

$$P_{1jkh}^i = R_{jkh}^i + \frac{1}{2(n-1)}(g_{jk}R_h^i - g_{jh}R_k^i - \delta_k^i R_{jh} + \delta_h^i R_{jk}). \quad (2.12)$$

This can be written as

$$P_{1jkh}^i = R_{jkh}^i + \frac{1}{2(n-1)}(g_{jk}R_h^i - g_{jh}R_k^i - \delta_k^i R_{jh} + \delta_h^i R_{jk}). \quad (2.13)$$

In V_4^F , and using (2.2) in equation (2.13), we get

$$P_{1jkh}^i = W_{jkh}^i + \frac{1}{6}\delta_h^i R_{jk} - g_{jh}R_k^i - \frac{1}{3}\delta_k^i R_{jh} + g_{jk}R_h^i. \quad (2.14)$$

3. Expansion of Curvature Tensors in Finsler Space

The expansion curvature tensor W_{jkh}^i is a geometric object introduced in Finsler geometry. It is a measure of the curvature of a Finsler manifold, which is a generalization of a Riemannian manifold. The expansion curvature tensor is closely related to the Weyl projective curvature tensor and Cartan's curvature tensor. It vanishes if and only if the Finsler manifold is flat.

We introduce the generalized Cartan's covariant derivative for any tensor W_{jkh}^i , given by

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \quad (3.1)$$

We can rewrite (3.1) in the following form:

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \gamma_m (W_{h0}^i - W_{k0}^i).$$

From (1.4b), the above equation can be written as

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \gamma_m (W_h^i C_{ijk} y^i - W_k^i C_{ijh} y^i). \quad (3.2)$$

Using (1.3) in (3.2), we get

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m W_h^i \dot{\partial}_k \dot{\partial}_j \dot{\partial}_i F^2 y^i - W_k^i \dot{\partial}_h \dot{\partial}_j \dot{\partial}_i F^2 y^i. \quad (3.3)$$

Applying (1.1f) on (3.3), we obtain

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m W_h^i \dot{\partial}_k \dot{\partial}_j F^2 - W_k^i \dot{\partial}_h \dot{\partial}_j F^2.$$

From (1.1b), we can write

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m W_h^i \dot{\partial}_k \dot{\partial}_j (y^j y_j) - W_k^i \dot{\partial}_h \dot{\partial}_j (y^j y_j). \quad (3.4)$$

Applying (1.1f) again, we obtain

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m W_h^i \dot{\partial}_k y_j - W_k^i \dot{\partial}_h y_j.$$

From (1.1g), we finally obtain

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m W_h^i g_{jk} - W_k^i g_{jh}. \quad (3.5)$$

From the previous steps, we can conclude the following theorem:

Theorem 3.1. *The expansion of (1.16) is given by (3.5).*

The dimensionality of many curvature tensor operators will be extended in accordance with Theorem 3.1.

4. Investigation of the Expansion by Identities

Mathematical identities are equations that are always true, regardless of the values of the variables involved. They can be used to simplify expressions, solve equations, and prove theorems. We investigate the expansion of Cartan's covariant derivative for any curvature tensor, which was obtained in the previous section, namely:

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \quad (4.1)$$

We suppose that (4.1) holds in order to investigate the following identities.

Using Cartan's covariant derivative, we derive the following expression for equation (2.2):

$$R_{jkh|m}^i = W_{jkh|m}^i - \frac{1}{3} \gamma_m (\delta_k^i R_{jh} - R_h^i g_{jk})_{|m}. \quad (4.2)$$

From (1.15a), (1.15b), (4.1) and (4.2), we get

$$\begin{aligned} R_{jkh|m}^i &= \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}) \\ &\quad - \frac{1}{3} \lambda_m (\delta_k^i R_{jh} - R_h^i g_{jk}). \end{aligned}$$

This gives

$$\begin{aligned} R_{jkh|m}^i &= \lambda_m W_{jkh}^i - \frac{1}{3} (\delta_k^i R_{jh} - R_h^i g_{jk}) \\ &\quad + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \end{aligned} \quad (4.3)$$

By using (2.2) in (4.3), we have

$$R_{jkh|m}^i = \lambda_m R_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \quad (4.4)$$

From the previous steps, we can conclude the following theorem:

Theorem 4.1. *The expansion derivative in Cartan sense of the Riemann curvature tensor R_{jkh}^i (2.2) satisfies equation (4.4).*

Transvecting (4.4) by y^j and using conditions (1.5b), (1.13b) and (1.1a), we obtain

$$H_{kh|m}^i = \lambda_m H_{kh}^i + \mu_m (\delta_h^i y_k - \delta_k^i y_h) + \frac{1}{4} \gamma_m (W_h^i y_k - W_k^i y_h). \quad (4.5)$$

Again, transvecting (4.5) by y^k and using conditions (1.5b), (1.13j), (1.12a), (1.1b) and (1.1c), we get

$$H_{h|m}^i = \lambda_m H_h^i + \mu_m (\delta_h^i F^2 - y^i y_h) + \frac{1}{4} \gamma_m W_h^i F^2. \quad (4.6)$$

Therefore, the proof of the theorem is completed, and we conclude:

Theorem 4.2. *The Cartan covariant derivative of first order for the torsion tensor H_{kh}^i and the deviation tensor H_h^i are given by (4.5) and (4.6).*

Using Cartan's covariant derivative, we derive the following expression for equation (2.6):

$$W_{jkh|m}^i = \left(W_{jkh}^i - \frac{1}{6} (\delta_h^i R_{jk} + \delta_k^i R_{jh} - g_{jk} R_h^i - g_{jh} R_k^i) \right)_{|m}. \quad (4.7)$$

From (1.15a), (1.15b), (4.1) and (4.7), we get

$$\begin{aligned} W_{jkh|m}^i &= \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}) \\ &\quad - \frac{1}{6} \lambda_m (\delta_h^i R_{jk} + \delta_k^i R_{jh} - g_{jk} R_h^i - g_{jh} R_k^i). \end{aligned}$$

This can be written as

$$\begin{aligned} W_{jkh|m}^i &= \lambda_m W_{jkh}^i - \frac{1}{6} (\delta_h^i R_{jk} + \delta_k^i R_{jh} - g_{jk} R_h^i - g_{jh} R_k^i) \\ &\quad + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \end{aligned} \quad (4.8)$$

From (2.6) and (4.8), we have

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \quad (4.9)$$

So, the proof of the theorem is completed.

Theorem 4.3. *The expansion derivative in Cartan sense of the projective curvature tensor W_{jkh}^i (2.6) satisfies equation (4.9).*

Using Cartan's covariant derivative, we derive the following expression for equation (2.7b):

$$\begin{aligned} C_{jkh|m}^i &= \left(W_{jkh}^i - \frac{5}{6} (\delta_k^i R_{jh} - R_h^i g_{jk}) - \frac{1}{6} R (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \right. \\ &\quad \left. + \frac{1}{2} (\delta_h^i R_{jk} - R_k^i g_{jh}) \right)_{|m}. \end{aligned} \quad (4.10)$$

From (1.15a), (1.15b), (1.15d), (4.1) and (4.10), we get

$$\begin{aligned} C_{jkh|m}^i &= \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}) \\ &\quad + \frac{1}{2} \lambda_m (\delta_h^i R_{jk} - R_k^i g_{jh}) - \frac{5}{6} \lambda_m (\delta_k^i R_{jh} - R_h^i g_{jk}) \\ &\quad - \frac{1}{6} \lambda_m R (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \end{aligned}$$

Or, we can write as

$$\begin{aligned} C_{jkh|m}^i &= \lambda_m W_{jkh}^i - \frac{5}{6} (\delta_k^i R_{jh} - R_h^i g_{jk}) - \frac{1}{6} R (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ &\quad + \frac{1}{2} (\delta_h^i R_{jk} - R_k^i g_{jh}) + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \end{aligned} \quad (4.11)$$

By using (2.7b) in (4.11), we have

$$C_{jkh|m}^i = \lambda_m C_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \quad (4.12)$$

In conclusion, the proof of the theorem is completed, and we can state:

Theorem 4.4. *The expansion derivative in Cartan sense of the conformal curvature tensor C_{jkh}^i (2.7b) satisfies equation (4.12).*

Using Cartan's covariant derivative, we derive the following expression for equation (2.8b):

$$L_{jkh|m}^i = \left(W_{jkh}^i + \frac{1}{6} (\delta_k^i R_{jh} - R_h^i g_{jk}) - \frac{1}{2} (\delta_h^i R_{jk} - R_k^i g_{jh}) \right)_{|m}. \quad (4.13)$$

From (1.15a), (1.15b), (4.1) and (4.13), we get

$$\begin{aligned} L_{jkh|m}^i &= \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}) \\ &\quad + \frac{1}{6} \lambda_m (\delta_k^i R_{jh} - R_h^i g_{jk}) - \frac{1}{2} \lambda_m (\delta_h^i R_{jk} - R_k^i g_{jh}). \end{aligned}$$

Or can be written as

$$\begin{aligned} L_{jkh|m}^i &= \lambda_m W_{jkh}^i + \frac{1}{6} (\delta_k^i R_{jh} - R_h^i g_{jk}) - \frac{1}{2} (\delta_h^i R_{jk} - R_k^i g_{jh}) \\ &\quad + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \end{aligned} \quad (4.14)$$

From (2.8b) and (4.14), we get

$$L_{jkh|m}^i = \lambda_m L_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \quad (4.15)$$

Thus, the proof of the theorem is completed.

Theorem 4.5. *The expansion derivative in Cartan sense of the conharmonic curvature tensor L_{jkh}^i (2.8b) satisfies equation (4.15).*

Using Cartan's covariant derivative, we derive the following expression for equation (2.10):

$$M_{jkh|m}^i = \left(W_{jkh}^i - \frac{1}{12} R (g_{jk} \delta_h^i - g_{jh} \delta_k^i) - \frac{1}{6} (\delta_k^i R_{jh} - R_h^i g_{jk}) \right)_{|m}. \quad (4.16)$$

From (1.15a), (1.15b), (1.15d), (4.1) and (4.16), we get

$$\begin{aligned} M_{jkh|m}^i &= \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}) \\ &\quad - \frac{1}{12} \lambda_m R (g_{jk} \delta_h^i - g_{jh} \delta_k^i) - \frac{1}{6} \lambda_m (\delta_k^i R_{jh} - R_h^i g_{jk}). \end{aligned}$$

Or can be written as

$$\begin{aligned} M_{jkh|m}^i &= \lambda_m W_{jkh}^i - \frac{1}{12} R (g_{jk} \delta_h^i - g_{jh} \delta_k^i) - \frac{1}{6} (\delta_k^i R_{jh} - R_h^i g_{jk}) \\ &\quad + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \end{aligned} \quad (4.17)$$

From (2.10) and (4.17), we have

$$M_{jkh|m}^i = \lambda_m M_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \quad (4.18)$$

Thus, the proof of the theorem is completed.

Theorem 4.6. *The expansion derivative in Cartan sense of the concircular curvature tensor M_{jkh}^i (2.10) satisfies equation (4.18).*

Using Cartan's covariant derivative, we derive the following expression for equation (2.14):

$$P_{1^i jkh|m} = \left(W_{jkh}^i + \frac{1}{6} (\delta_h^i R_{jk} - g_{jh} R_k^i) - \frac{1}{3} (\delta_k^i R_{jh} - g_{jk} R_h^i) \right)_{|m}. \quad (4.19)$$

From (1.15a), (1.15b), (4.1) and (4.19), we get

$$\begin{aligned} P_{1^i jkh|m} &= \lambda_m W_{jkh}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}) \\ &\quad + \frac{1}{6} \lambda_m (\delta_h^i R_{jk} - g_{jh} R_k^i) - \frac{1}{3} \lambda_m (\delta_k^i R_{jh} - g_{jk} R_h^i). \end{aligned}$$

Or can be written as

$$\begin{aligned} P_{1^i jkh|m} &= \lambda_m W_{jkh}^i + \frac{1}{6} (\delta_h^i R_{jk} - g_{jh} R_k^i) - \frac{1}{3} (\delta_k^i R_{jh} - g_{jk} R_h^i) \\ &\quad + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \end{aligned} \quad (4.20)$$

By using (2.14) in (4.20), we have

$$P_{1^i jkh|m} = \lambda_m P_{1^i jkh} + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_m (W_h^i g_{jk} - W_k^i g_{jh}). \quad (4.21)$$

Thus, the proof of the theorem is completed.

Theorem 4.7. *The expansion derivative in Cartan sense of the P_1 -curvature tensor $P_{1^i jkh}$ (2.14) satisfies equation (4.21).*

Transvecting (4.1) by y^j and using conditions (1.2b), (2.3a) and (1.4b), we get

$$W_{kh|m}^i = \lambda_m W_{kh}^i + \mu_m (\delta_h^i y_k - \delta_k^i y_h) + \frac{1}{4} \gamma_m (W_h^i y_k - W_k^i y_h). \quad (4.22)$$

Again, transvecting (4.22) by y^k and using conditions (1.5b), (1.11b), (1.12a), (1.1b) and (1.1c), we get

$$W_{h|m}^i = \lambda_m W_h^i + \mu_m y^i y_h - \delta_h^i F^2 + \frac{1}{4} \gamma_m W_h^i F^2. \quad (4.23)$$

Therefore, the proof of the theorem is completed.

Theorem 4.8. *In Cartan's first order covariant derivative, the torsion tensor W_{kh}^i and deviation tensor W_h^i are given by (4.22) and (4.23).*

Contracting the indices i and h in equations (4.5) and (4.6), respectively, and using (1.2a), (1.1a), (1.1b), (1.13k), (1.13t), and (1.12b), we get

$$H_{k|m} = \lambda_m H_k + \mu_m (n-1) y_k - \frac{1}{4} \gamma_m W_k, \quad (4.24)$$

and

$$H_{|m} = \lambda_m H + \mu_m (n-1) F^2. \quad (4.25)$$

Thus, the proof is completed.

Theorem 4.9. *In Cartan's first order covariant derivative, the vector H_k and scalar H are given by (4.24) and (4.25).*

5. Applications in Physics and Applied Mathematics

The research paper delves into advanced mathematical topics such as Cartan's covariant derivative, curvature tensors, and torsion tensors, which are central concepts in differential geometry, general relativity, and theoretical physics. Below, we provide specific examples of how these concepts are applied in various fields of applied mathematics and theoretical physics.

5.1. Application in General Relativity (GR)

In general relativity, the curvature of spacetime is described by the Riemann curvature tensor, which determines how the geometry of spacetime is influenced by the presence of mass and energy. The covariant derivative of the curvature tensor, as described in the paper, can be used to study the evolution of spacetime curvature in response to changing gravitational fields.

Example 5.1. *Example: Consider the Einstein Field Equations:*

$$R_{kh} - \frac{1}{2}g_{kh}R = \frac{8\pi G}{c^4}T_{kh},$$

where R_{kh} is the Ricci curvature tensor, R is the scalar curvature, g_{kh} is the metric tensor, and T_{kh} is the stress-energy tensor.

By investigating the expansion of Cartan's covariant derivative of the curvature tensor (as done in this paper), one can examine how gravitational waves, black holes, or exotic matter (such as dark energy) influence spacetime curvature.

5.2. Application in Higher-Dimensional Theories (String Theory)

In theoretical physics, higher-dimensional spaces play a crucial role in the formulation of fundamental interactions. The covariant derivatives of torsion and curvature tensors are essential in these settings.

In an n -dimensional spacetime, curvature tensors become more complex due to additional degrees of freedom. Expansions of Cartan's covariant derivative allow one to study how extra dimensions influence gravitational and field-theoretic structures. This is particularly relevant in the study of strings and branes propagating in higher-dimensional manifolds.

5.3. Application in Cosmology

In cosmology, dark energy and dark matter are fundamental components of the universe's evolution. Curvature and torsion tensors are essential in describing the large-scale structure of spacetime.

In cosmological models such as the Λ CDM model, curvature tensors describe the expansion dynamics of the universe. The expansion identities derived in this paper simplify the mathematical structure of evolving spacetimes, particularly in analyzing deviation tensors that measure departures from idealized homogeneous models.

5.4. Application in Fluid Dynamics

In applied mathematics, especially in the study of fluid dynamics, curvature tensors can describe fluid flow in curved geometries. This is important in modeling turbulent flows and motion in non-Euclidean domains.

When a fluid flows through curved structures, such as rotating systems or curved pipes, geometric effects influence velocity, pressure, and vorticity. The expansion formulas developed in this work can be applied to analyze such complex flow behaviors.

6. Conclusion

In this study, we introduced a novel decomposition scheme for curvature tensors in Finsler spaces. A promising direction for future research is to explore applications in Finslerian cosmology. Studying curvature tensors within cosmological models based on Finsler geometry may provide new insights into the large-scale structure of the universe and offer alternative tests of general relativity.

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