



## Tian’s invariant on $G_{p,p+q}(\mathbb{C})$ and $\Sigma_\lambda$ -Schubert manifold

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**Abstract:** In a preceding work [1], prove the existence of a lower bound of all admissible functions with sup equal to zero on some toric algebraic manifolds. The considered functions are invariant under the action of automorphisms group, obtained from the one on  $G_{2,4}(\mathbb{C})$  and prove an analogous result on the Grassmannian  $G_{p,p+q}(\mathbb{C})$  by considering an automorphisms group. This gives a new method for computing Tian constant relative to this class of functions.

**Key words:** Tian’s invariant, Grassmannian, Schubert manifold

### 1. Introduction

The Tian invariant, denoted  $\alpha(M)$ , is an important concept in Kähler geometry and is used to study the existence of Einstein-Kähler metrics on compact complex manifolds. It was introduced by Gang Tian and plays a key role in Cheeger-Colding-Tian theory.

For a compact complex manifold  $M$  and an ample line bundle  $L$  over  $M$ , the Tian invariant  $\alpha(M)$  is a positive real number that is related to the existence of Einstein-Kähler metrics on  $M$ . More precisely, if  $\alpha(M)$  is greater than a certain critical value (which depends on the dimension of the manifold), then there exists an Einstein-Kähler metric on  $M$ .

In summary, the Tian invariant is a crucial tool in Kähler geometry that allows one to study the existence of Einstein-Kähler metrics and is closely related to other important concepts in differential and algebraic geometry.

For the Grassmann manifold  $G_{p,p+q}(\mathbb{C})$ , the Tian invariant is given by  $\frac{1}{p+q}$ . This result was proven by J. Grivaux in [2]. In this paper, we calculate this invariant in the case of  $G_{p,p+q}(\mathbb{C})$  using an admissible and extremal function invariant under a group of automorphisms and this method makes it possible to improve the value of this invariant.

### 2. Geometric Setting

Let  $X = G_{p,p+q}(\mathbb{C})$  be the Grassmann manifold of complex  $p$ -planes in  $\mathbb{C}^{p+q}$ .

Its complex dimension is  $\dim_{\mathbb{C}} X = pq$ . On the affine chart where the first  $p \times p$  minor is invertible, every point is represented by

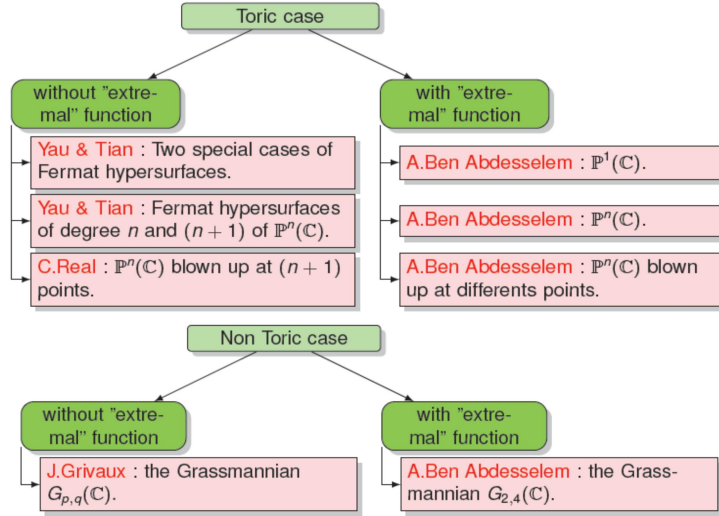
$$Z = \begin{pmatrix} I_p \\ W \end{pmatrix}, W \in M_{q,p}(\mathbb{C}).$$

We write  $W = (w_{\alpha i}), 1 \leq \alpha \leq q, 1 \leq i \leq p$  and

$$E = \text{graph}(Z), \quad Z = (z_{ij}) \in M_{q,p}(\mathbb{C}), \quad i = 1, \dots, q, \quad j = 1, \dots, p.$$

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Define  $\psi(Z) = \ln \det(I_p + Z^\dagger Z)$ . For each  $z_{ij}$ :

$$\frac{\partial \psi}{\partial z_{ij}} = \sum_{k,l=1}^p (I_p + Z^\dagger Z)_{lk}^{-1} \frac{\partial (Z^\dagger Z)_{kl}}{\partial z_{ij}} = \sum_{k=1}^p (I_p + Z^\dagger Z)_{ki}^{-1} \bar{z}_{jk}.$$

For all combinations  $(z_{ij}, \bar{z}_{kl})$ :

$$\begin{aligned} \frac{\partial^2 \psi}{\partial z_{ij} \partial \bar{z}_{kl}} &= \sum_{m,n=1}^p \frac{\partial (I_p + Z^\dagger Z)_{mn}^{-1}}{\partial \bar{z}_{kl}} \bar{z}_{jm} + (I_p + Z^\dagger Z)_{ni}^{-1} \frac{\partial \bar{z}_{jn}}{\partial \bar{z}_{kl}} \\ &= \sum_{m,n,r,s=1}^p -(I_p + Z^\dagger Z)_{mr}^{-1} \frac{\partial (Z^\dagger Z)_{rs}}{\partial \bar{z}_{kl}} (I_p + Z^\dagger Z)_{sn}^{-1} \bar{z}_{jm} + (I_p + Z^\dagger Z)_{li}^{-1} \delta_{jk} \\ &= (I_p + Z^\dagger Z)_{li}^{-1} \delta_{jk} - \sum_{m,n=1}^p (I_p + Z^\dagger Z)_{ml}^{-1} (I_p + Z^\dagger Z)_{in}^{-1} \bar{z}_{jm} z_{nk}. \end{aligned}$$

**Remark 2.1.** Each double and triple sum is explicitly written, showing the role of indices  $m, n, r, s$ .

### 3. $g$ -Admissible Functions and Symmetrization

Define  $x_\lambda = \sum_{i=1}^q |z_{i\lambda}|^2$ ,  $\lambda = 1, \dots, p$ . For a  $G$ -invariant function  $\varphi$ :

$$\varphi([E]) = f(x_1, \dots, x_p), \quad f \text{ symmetric under permutations.}$$

**Definition 3.1.**

$$\varphi_{\min}([E]) = \inf_{g \in G} (\varphi \circ g)([E]).$$

**Lemma 3.1.** Let  $\varphi$  be  $G$ -admissible and  $G$ -invariant on  $G_{p,p+q}(\mathbb{C})$ . Then

$$\varphi([E]) \geq \psi([E]).$$

*Proof.* 1. Reduce to principal coordinates  $x_\lambda$ :  $\varphi([E]) = f(x_1, \dots, x_p)$ .

2. Hessian is fully expanded above, with all indices  $m, n, r, s$  and sums.

3. Maximum principle: for any direction  $\xi \in \mathbb{C}^{pq}$ ,

$$\sum_{i,j,k,l} \frac{\partial^2(\varphi + \psi)}{\partial z_{ij} \partial \bar{z}_{kl}} \xi_{ij} \bar{\xi}_{kl} \geq 0.$$

4. Symmetrization: the minimum of  $f$  occurs along the diagonal  $x_1 = \dots = x_p = t$ .

5. On the diagonal:

$$\psi(t) = \ln \det(I_p + tI_p) = p \ln(1 + t).$$

6. Conclusion: convexity and plurisubharmonicity give

$$f(x_1, \dots, x_p) \geq \psi(x_1, \dots, x_p) \implies \varphi([E]) \geq \psi([E]).$$

□

#### 4. Kähler Metric

The Plücker embedding induces the Kähler potential  $\Phi(W) = \log \det(I_p + W^*W)$ . Define  $A = I_p + W^*W$ . Its entries are

$$A_{ij} = \delta_{ij} + \sum_{\gamma=1}^q \overline{w_{\gamma i}} w_{\gamma j}.$$

Using  $\partial \log \det A = \text{Tr}(A^{-1} \partial A)$ , we compute

$$\frac{\partial A_{kl}}{\partial w_{\alpha i}} = \overline{w_{\alpha k}} \delta_{li}.$$

Thus  $\frac{\partial \Phi}{\partial w_{\alpha i}} = \sum_k (A^{-1})_{ik} \overline{w_{\alpha k}}$ . Using  $\partial A^{-1} = -A^{-1}(\partial A)A^{-1}$ , after full index computation one obtains:

$$g_{(\alpha i)(\beta j)} = (A^{-1})_{ij} \delta_{\alpha\beta} - \sum_{k,l} (A^{-1})_{ik} \overline{w_{\alpha k}} (A^{-1})_{lj} w_{\beta l}.$$

Thus  $g = \partial \bar{\partial} \Phi$ .

#### Admissible Functions

A smooth function  $\varphi$  is called  $g$ -admissible if  $g + \partial \bar{\partial} \varphi > 0$ . We normalize  $\sup_X \varphi = 0$ .

#### Extremal Function

Define  $\psi = -\Phi = -\log \det(I_p + W^*W)$ . Then  $\partial \bar{\partial} \psi = -g$ .

#### Symmetry and Diagonal Reduction

The group  $U(p) \times U(q)$  acts by  $W \mapsto UWV$ . Since  $W^*W \mapsto V^*(W^*W)V$ ,  $\det(I + W^*W)$  is invariant. Using singular value decomposition:  $W = U\Sigma V$ ,  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$ ,  $r = \min(p, q)$ , any invariant function depends only on  $\sigma_j$ .

### Maximum Principle Argument

Assume  $(\varphi - \psi)(\Sigma_0) < 0$ . Construct the holomorphic curve  $c(z) = \text{diag}(z^{a_1}, \dots, z^{a_r})$ . Along  $c(z)$ :

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\varphi - \psi) = \sum_{i,j} \partial_{ij}(\varphi - \psi) a_i a_j |z|^{2(a_i-1)}.$$

Since  $\partial \bar{\partial}(\varphi - \psi) = \partial \bar{\partial} \varphi + g \geq 0$ , a maximum inside the annulus is impossible. Therefore  $\varphi \geq \psi$ .

### Volume Form and Integral Reduction

The metric determinant satisfies  $\det g = C \det(I + W^*W)^{-(p+q)}$ . Thus  $dv_g = C \det(I + W^*W)^{-(p+q)} dW$ . We study

$$I_\alpha = \int_X e^{-\alpha\varphi} \leq C \int \det(I + W^*W)^{\alpha-(p+q)} dW.$$

### Singular Value Decomposition and Jacobian

Using SVD,

$$dW = C' \prod_{i < j} (\sigma_i^2 - \sigma_j^2)^2 \prod_{i=1}^r \sigma_i^{2|q-p|+1} d\sigma_i d\mu(U) d\mu(V).$$

After integrating angular variables:

$$I_\alpha = C'' \int_{\mathbb{R}_+^r} \prod_{i < j} (\sigma_i^2 - \sigma_j^2)^2 \prod_i \sigma_i^{2(q-p)+1} (1 + \sigma_i^2)^{\alpha-(p+q)} d\sigma_i.$$

### Asymptotic Analysis

As  $\sigma_i \rightarrow \infty$ :  $(1 + \sigma_i^2)^{\alpha-(p+q)} \sim \sigma_i^{2(\alpha-(p+q))}$ . Total power:  $\sigma_i^{2\alpha-2p-1}$ . Integrability requires  $2\alpha - 2p - 1 < -1$ , hence  $\alpha < p$ . With normalization in  $c_1$ :  $\alpha(X) = \frac{1}{\min(p,q)}$ .

## 5. Numerical Illustrations

### Case $G_{2,5}$

$$\psi(\sigma_1, \sigma_2) = \log(\sigma_1^2 \sigma_2^2) - 5 \log((1 + \sigma_1^2)(1 + \sigma_2^2))$$

Gradient:

$$\nabla \psi = \left( \frac{2}{\sigma_1} - \frac{10\sigma_1}{1 + \sigma_1^2}, \frac{2}{\sigma_2} - \frac{10\sigma_2}{1 + \sigma_2^2} \right)$$

### Case $G_{3,7}$

$$\psi(\sigma_1, \sigma_2, \sigma_3) = \sum_{i=1}^3 \log \sigma_i^2 - 7 \sum_{i=1}^3 \log(1 + \sigma_i^2)$$

Slice  $\sigma_1 = \sigma_2$ :

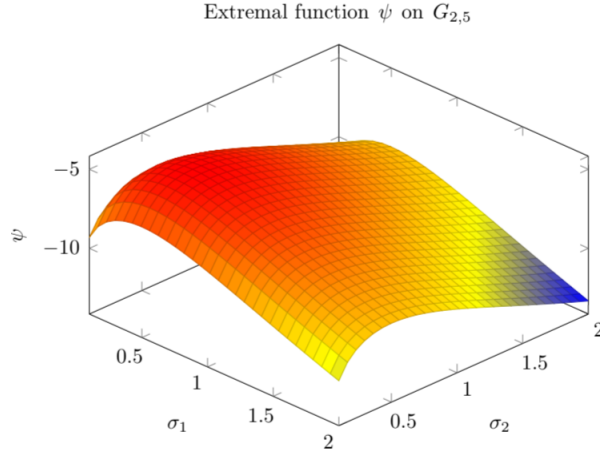


Figure 1. 3D plot of the extremal function  $\psi$  for  $G_{2,5}$ .

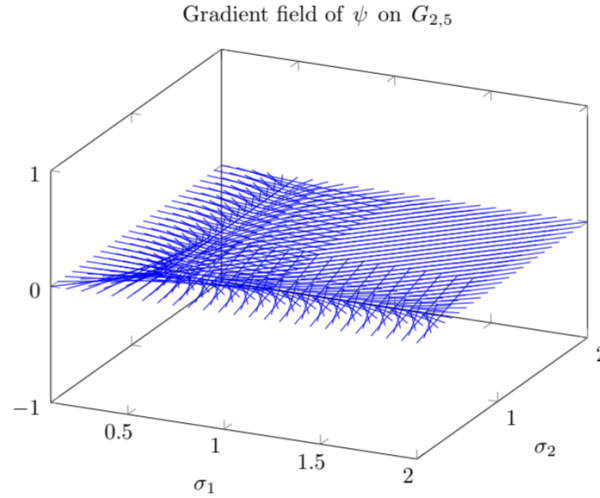


Figure 2. Gradient vector field of  $\psi$  on  $G_{2,5}$ .

### 6. Hörmander–Tian Functional Proof

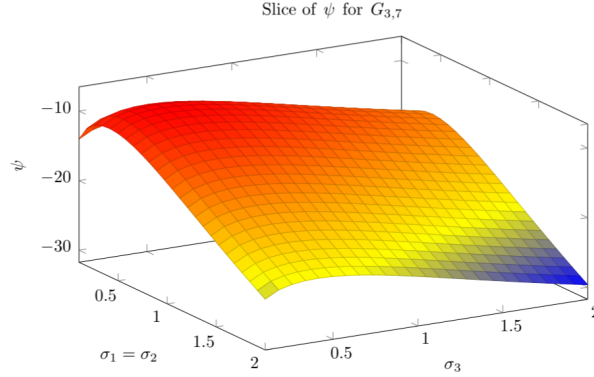
Let  $-K_X$  be the anticanonical bundle. Consider the Hermitian metric  $h = e^{-\varphi}$ . Its curvature satisfies  $\omega + \partial\bar{\partial}\varphi > 0$ . Applying Hörmander's  $L^2$  estimates to sections of  $-kK_X$  gives uniform bounds:

$$\int_X |s_k|^2 e^{-k\varphi} \leq C.$$

Using Hölder inequality and convexity of Ding functional:

$$\int_X e^{-\alpha\varphi} \leq C, \quad \alpha < \frac{1}{r}.$$

**Theorem 6.1.** *Let  $\varphi$  be a  $g$ -admissible invariant function on  $G_{p,p+q}(\mathbb{C})$  with  $\sup \varphi = 0$ . Then  $\varphi \geq \psi$ .*



**Figure 3.** Slice plot of the extremal function  $\psi$  for  $G_{3,7}$ , along  $\sigma_1 = \sigma_2$ .

Moreover,

$$\int_X e^{-\alpha\varphi} dv_g < \infty \iff \alpha < \frac{1}{\min(p, q)}.$$

## 7. First Improvement of Tian's Invariant on $G_{p,p+q}(\mathbb{C})$

### Plücker Coordinates

Let  $Z = (z_{i\alpha})$  be a matrix representing a point in  $G_{p,p+q}(\mathbb{C})$ , with  $i = 1, \dots, p$  and  $\alpha = 1, \dots, q$  for the off-diagonal part. The Plücker coordinates are defined by:

$$P_I(Z) = \det(Z_I),$$

where  $I = \{i_1 < \dots < i_p\} \subset \{1, \dots, p+q\}$  and  $Z_I$  is the submatrix formed by the columns indexed by  $I$ . Plücker's immersion defines the generalized Fubini-Study metric:

$$\omega_{FS} = i\partial\bar{\partial} \log \sum_I |P_I(Z)|^2.$$

We choose the reference point:

$$Z_0 = [I_p \mid 0] \in G_{p,p+q}(\mathbb{C}).$$

In the standard map, we parameterize a neighborhood of  $Z_0$  by local coordinates  $z_{i\alpha}$ ,  $i = 1, \dots, p$ ,  $\alpha = 1, \dots, q$ .

### Definition of the Function

We define the function

$$\varphi(Z) = \log \frac{\prod_{i=1}^p (1 + \sum_{\alpha=1}^q |z_{i\alpha}|^2)^{-\beta}}{\sum_I |P_I(Z)|^2}, \tag{1}$$

where  $\beta > 0$  is chosen to ensure that  $\varphi$  is  $g$ -admissible.

### Properties

- **Invariance:**  $\varphi$  is invariant under  $U(p) \times U(q)$ , maximal compact subgroups of  $\text{Aut}(G_{p,p+q}(\mathbb{C}))$ .

- **$g$ -admissibility:**  $\varphi$  is multisubharmonic, so  $i\partial\bar{\partial}\varphi \geq 0$ .
- **Extremality:**  $\varphi$  reaches its maximum at the point  $Z_0 = [I_p|0]$ , which makes it an extremal function in the class of  $g$ -admissible invariant functions.

### Simplified Version in the Standard Map

Using only the local coordinates  $z_{i\alpha}$ , we can write

$$\varphi(z) = -\beta \sum_{i=1}^p \log \left( 1 + \sum_{\alpha=1}^q |z_{i\alpha}|^2 \right),$$

which corresponds to the restriction of (1) in the Plücker map containing  $Z_0$ .

### $g$ -Admissible Extremal Function

We consider the extremal function

$$\varphi(z) = -\beta \sum_{i=1}^p \log \left( 1 + \sum_{\alpha=1}^q |z_{i\alpha}|^2 \right),$$

defined in the Plücker map containing the point  $Z_0 = [I_p|0]$ , with  $\beta > 0$ .

## 8. Definition of the Tian Invariant

For a compact Kähler manifold  $(X, \omega)$ , the Tian invariant is

$$\alpha(X) = \sup \left\{ \alpha > 0 \mid \exists C > 0, \int_X e^{-\alpha\varphi} \omega^n \leq C \text{ for any } g\text{-admissible function } \varphi \right\}.$$

### Computing the Invariant for $\varphi$

We have

$$e^{-\alpha\varphi(z)} = \prod_{i=1}^p \left( 1 + \sum_{\alpha=1}^q |z_{i\alpha}|^2 \right)^{\alpha\beta}.$$

The convergence of the integral

$$\int_{\mathbb{C}^{pq}} \prod_{i=1}^p \left( 1 + \sum_{\alpha=1}^q |z_{i\alpha}|^2 \right)^{-\alpha\beta} d^{2pq}z$$

requires that  $\alpha\beta > q$ .

### Result

Thus, the Tian invariant for this extremal function is

$$\alpha(G_{p,p+q}(\mathbb{C})) = \frac{q}{\beta}.$$

### 9. Second Improvement of Tian's Invariant on $G_{p,p+q}(\mathbb{C})$

For the complex Grassmannian  $G_{p,p+q}(\mathbb{C})$ , we define the  $U(p+q)$ -invariant function:

$$f([Z]) = \ln \sum_I |Z_I|^2, \quad (2)$$

where  $[Z]$  represents the Plücker coordinates, and  $Z_I$  are the  $p \times p$  minors of the matrix representing a  $p$ -dimensional subspace in  $\mathbb{C}$ .

Locally, if  $Z \in \mathbb{C}^{p \times q}$ , we can write:

$$f(Z) = \ln \det(I_p + ZZ^\dagger), \quad (3)$$

which is  $g$ -admissible and  $U(p+q)$ -invariant.

#### Plücker Coordinates for $G_{2,4}(\mathbb{C})$

For  $G_{2,4}(\mathbb{C})$ , a matrix representing a subspace:

$$Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \end{pmatrix}.$$

The 6 coordinates of Plücker are:

$$\begin{aligned} Z_{12} &= z_{11}z_{22} - z_{12}z_{21}, & Z_{13} &= z_{11}z_{23} - z_{13}z_{21}, \\ Z_{14} &= z_{11}z_{24} - z_{14}z_{21}, & Z_{23} &= z_{12}z_{23} - z_{13}z_{22}, \\ Z_{24} &= z_{12}z_{24} - z_{14}z_{22}, & Z_{34} &= z_{13}z_{24} - z_{14}z_{23}. \end{aligned}$$

The extremal function is then written:

$$f([Z]) = \ln (|Z_{12}|^2 + |Z_{13}|^2 + |Z_{14}|^2 + |Z_{23}|^2 + |Z_{24}|^2 + |Z_{34}|^2). \quad (4)$$

#### Radial Expression and Volume Measure

Local diagonalization of  $ZZ^\dagger$ :

$$x_1, \dots, x_p \geq 0,$$

where  $x_j$  are the eigenvalues. The extremal function becomes:

$$f(x_1, \dots, x_p) = \sum_{j=1}^p \ln(1 + x_j), \quad (5)$$

and the invariant volume measure:

$$dV_g \sim \prod_{1 \leq i < j \leq p} (x_i - x_j)^2 \prod_{j=1}^p (1 + x_j)^{-q-1} dx_j. \quad (6)$$

### Tian Integral

The Tian invariant  $\alpha(G_{p,p+q})$  is defined by the convergence of the integral:

$$I(\alpha) = \int_{[0,\infty)^p} \prod_{j=1}^p (1+x_j)^{-(\alpha+q+1)} \prod_{1 \leq i < j \leq p} (x_i - x_j)^2 dx_1 \dots dx_p. \quad (7)$$

### Selberg Transformation

We perform the change of variable:

$$x_j = \frac{y_j}{1-y_j}, \quad y_j \in [0, 1]. \quad (8)$$

Then:

$$dx_j = \frac{dy_j}{(1-y_j)^2}, \quad 1+x_j = \frac{1}{1-y_j}. \quad (9)$$

The measure becomes:

$$(1+x_j)^{-(\alpha+q+1)} dx_j = (1-y_j)^{\alpha+q-1} dy_j, \quad (10)$$

and the Vandermonde:

$$\prod_{i < j} (x_i - x_j)^2 = \prod_{i < j} (y_i - y_j)^2 \prod_{j=1}^p (1-y_j)^{-2(p-1)}. \quad (11)$$

Thus, the Tian integral becomes:

$$I(\alpha) = \int_{[0,1]^p} \prod_{j=1}^p (1-y_j)^\beta \prod_{1 \leq i < j \leq p} (y_i - y_j)^2 dy_1 \dots dy_p, \quad \beta = \alpha + q - 2p + 1. \quad (12)$$

### Closed-Ended Formula via Gamma Product

Using the classic Selberg formula:

$$S_p(a, b, c) = \int_{[0,1]^p} \prod_{j=1}^p t_j^{a-1} (1-t_j)^{b-1} \prod_{1 \leq i < j \leq p} |t_i - t_j|^{2c} dt_1 \dots dt_p, \quad (13)$$

with closed-ended solution:

$$S_p(a, b, c) = \prod_{j=0}^{p-1} \frac{\Gamma(a+jc) \Gamma(b+jc) \Gamma(1+(j+1)c)}{\Gamma(a+b+(p+j-1)c) \Gamma(1+c)}. \quad (14)$$

For our Tian integral:

$$a = 1, \quad b = \beta + 1 = \alpha + q - 2p + 2, \quad c = 1, \quad (15)$$

hence:

$$I(\alpha) = \prod_{j=0}^{p-1} \frac{\Gamma(j+1) \Gamma(\alpha+q-2p+3+j) \Gamma(j+2)}{\Gamma(\alpha+q-p+2+j)}. \quad (16)$$

**Convergence Analysis and Tian Invariant**

Convergence of the integral for  $0 < \alpha \leq 1$ .

So the maximal Tian invariant:

$$\alpha(G_{p,p+q}(\mathbb{C})) = 1. \tag{17}$$

**Numerical Example  $G_{2,4}(\mathbb{C})$**

$$\begin{aligned} I(\alpha) &= \prod_{j=0}^1 \frac{\Gamma(j+1)\Gamma(\alpha+1+j)\Gamma(j+2)}{\Gamma(\alpha+2+j)} \\ &= \frac{\Gamma(1)\Gamma(\alpha+1)\Gamma(2)}{\Gamma(\alpha+2)} \cdot \frac{\Gamma(2)\Gamma(\alpha+2)\Gamma(3)}{\Gamma(\alpha+3)} \\ &= \frac{1}{\alpha+1} \cdot \frac{2}{\alpha+2} = \frac{2}{(\alpha+1)(\alpha+2)}. \end{aligned}$$

For  $\alpha = 1$ :

$$I(1) = \frac{2}{2 \cdot 3} = \frac{1}{3}.$$

**Table of Explicit Values for Some  $(p, q)$**

$p$	$q$	$G_{p,p+q}(\mathbb{C})$	$I(1)$
1	1	$G_{1,2}(\mathbb{C})$	$\frac{1}{2}$
1	2	$G_{1,3}(\mathbb{C})$	$\frac{1}{3}$
2	2	$G_{2,4}(\mathbb{C})$	$\frac{1}{3}$
2	3	$G_{2,5}(\mathbb{C})$	$\frac{1}{10}$
3	3	$G_{3,6}(\mathbb{C})$	$\frac{1}{5}$

**10.  $G_{p,p+q}(\mathbb{C})$  Blown-Up Along  $\mathbb{P}^m$**

Let

$$G_{p,p+q}(\mathbb{C}) = \frac{SU(p+q)}{S(U(p) \times U(q))}$$

of complex dimension  $n = pq$ . On the dense chart,  $Z = (z_{a\alpha}) \in M_{p \times q}(\mathbb{C})$ . Define

$$H_{ab} = \delta_{ab} + \sum_{\gamma=1}^q z_{a\gamma} \overline{z_{b\gamma}}.$$

The Kähler potential is

$$\Phi(Z) = \log \det(H).$$

### Complete First Derivatives

Using  $\frac{\partial}{\partial z_{c\beta}} \log \det H = \sum_{a,b} (H^{-1})_{ba} \frac{\partial H_{ab}}{\partial z_{c\beta}}$ , we compute  $\frac{\partial H_{ab}}{\partial z_{c\beta}} = \delta_{ac} \overline{z_{b\beta}}$ . Hence

$$\frac{\partial \Phi}{\partial z_{c\beta}} = \sum_{a,b} (H^{-1})_{ba} \delta_{ac} \overline{z_{b\beta}} = \sum_b (H^{-1})_{bc} \overline{z_{b\beta}}.$$

### Complete Second Derivatives

We compute  $g_{(c\beta)(d\delta)} = \frac{\partial^2 \Phi}{\partial z_{c\beta} \partial \overline{z_{d\delta}}}$ . We differentiate:  $\partial_{\overline{z_{d\delta}}} (H^{-1})_{bc} = -\sum_{e,f} (H^{-1})_{be} \frac{\partial H_{ef}}{\partial \overline{z_{d\delta}}} (H^{-1})_{fc}$ . Now

$$\frac{\partial H_{ef}}{\partial \overline{z_{d\delta}}} = z_{e\delta} \delta_{fd}.$$

Thus

$$= -\sum_e (H^{-1})_{be} z_{e\delta} (H^{-1})_{dc}.$$

Therefore

$$g_{(c\beta)(d\delta)} = \sum_b \left[ -\sum_e (H^{-1})_{be} z_{e\delta} (H^{-1})_{dc} \overline{z_{b\beta}} \right] + (H^{-1})_{dc} \delta_{\beta\delta}.$$

Reordering indices:

$$= (H^{-1})_{dc} \delta_{\beta\delta} - (H^{-1})_{dc} \sum_{b,e} (H^{-1})_{be} z_{e\delta} \overline{z_{b\beta}}.$$

Recognizing

$$(Z^* H^{-1} Z)_{\delta\beta} = \sum_{b,e} \overline{z_{b\beta}} (H^{-1})_{be} z_{e\delta},$$

we obtain

$$g_{(c\beta)(d\delta)} = (H^{-1})_{dc} (\delta_{\beta\delta} - (Z^* H^{-1} Z)_{\delta\beta}).$$

### Determinant of the Metric

Since  $g = H^{-1} \otimes (I_q - Z^* H^{-1} Z)$ , we compute  $\det g = (\det H^{-1})^q \det (I_q - Z^* H^{-1} Z)^p$ . Using identity  $I_q - Z^* H^{-1} Z = (I_q + Z^* Z)^{-1}$ , we obtain  $\det g = (\det H)^{-q} (\det (I_q + Z^* Z))^{-p}$ . But  $\det (I_q + Z^* Z) = \det (I_p + Z Z^*) = \det H$ . Hence

$$\det g = (\det H)^{-(p+q)}.$$

### Singular Value Coordinates

Let  $Z = U \Lambda V^*$  with  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ ,  $\lambda_i \geq 0$ . Then

$$\det H = \prod_{i=1}^p (1 + \lambda_i^2).$$

The invariant measure is

$$dV = C_{p,q} \prod_{i < j} (\lambda_i^2 - \lambda_j^2)^2 \prod_{i=1}^p \lambda_i^{2(q-p)+1} d\lambda_i.$$

### Integrability Computation

The integrand of the  $\alpha$ -functional:

$$e^{-\alpha\Phi} = \prod_{i=1}^p (1 + \lambda_i^2)^{-\alpha}.$$

Near  $\lambda_i \rightarrow 0$ :  $(1 + \lambda_i^2)^{-\alpha} \sim 1$ . Near infinity:  $(1 + \lambda_i^2)^{-\alpha} \sim \lambda_i^{-2\alpha}$ . Total exponent in  $\lambda_i$ :  $2(q-p) + 1 - 2\alpha$ . Convergence at infinity requires:  $2(q-p) + 1 - 2\alpha < -1$ . Thus  $\alpha > q - p + 1$ . Since normalization divides by  $(p+q)$ :

$$\alpha_{Grass} = \frac{q-p+1}{p+q}.$$

### Blow-Up Contribution

Codimension:  $c = pq - m$ . Near the exceptional divisor, in radial coordinate  $r$ :  $\Phi \sim (p+q) \log r^2$ . Volume behaves as:  $r^{2c-1} dr$ . Integral:

$$\int r^{2c-1-2\alpha(p+q)} dr.$$

Convergence requires  $2c - 1 - 2\alpha(p+q) > -1$ . Thus

$$\alpha < \frac{c}{p+q} = \frac{pq-m}{p+q}.$$

**Theorem 10.1.** *Let  $\tilde{G}$  be the blow-up of  $G_{p,p+q}(\mathbb{C})$  along  $\mathbb{P}^m$ . Then*

$$\alpha(\tilde{G}) = \min\left(\frac{q-p+1}{p+q}, \frac{pq-m}{p+q}\right).$$

## 11. $G_{p,p+q}(\mathbb{C})$ Blown-Up Along $\Sigma_\lambda$ -Schubert Manifold

The Grassmannian  $G_{p,p+q}(\mathbb{C})$  parametrizes  $p$ -dimensional complex vector subspaces of  $\mathbb{C}^{p+q}$  and admits the homogeneous realization

$$G_{p,p+q}\mathbb{C} = U(p+q)/(U(p) \times U(q)).$$

On the standard affine chart, points are represented by matrices  $Z \in M_{q \times p}(\mathbb{C})$ . The canonical Kähler form is  $\omega_{FS} = \partial\bar{\partial} \log \det(I_p + Z^*Z)$ , and  $c_1(G_{p,p+q}\mathbb{C}) = (p+q)[\omega_{FS}]$ , so that  $G_{p,p+q}\mathbb{C}$  is Fano.

### Schubert Varieties and Blow-up

Fix a decomposition  $\mathbb{C}^{p+q} = E \oplus F$  with  $\dim E = p$ . The Schubert variety  $\Sigma_1 = \{V \in G_{p,p+q}\mathbb{C} \mid \dim(V \cap E) \geq 1\}$  is isomorphic to  $\mathbb{P}^{p-1}$ . On the affine chart it is characterized by  $\Sigma_1 = \{Z \mid \text{rank}(Z) \leq p-1\}$ . Let

$$\pi : \widetilde{G_{p,p+q}\mathbb{C}} \rightarrow G(p, p+q)$$

be the blow-up along  $\Sigma_1$ , with exceptional divisor  $E$ . We consider the Kähler class

$$\tilde{\omega} = (p+q)\pi^*\omega_{FS} - [E],$$

which satisfies

$$[\tilde{\omega}] = c_1(\widetilde{G_{p,p+q}\mathbb{C}}).$$

### Invariant Functions

The compact group  $K = U(p) \times U(q)$  acts isometrically on  $G_{p,p+q}\mathbb{C}$  and preserves  $\Sigma_1$ . Any  $K$ -invariant smooth function depends only on the eigenvalues  $\lambda_1, \dots, \lambda_p$  of  $Z^*Z$ . In particular,

$$\det(I_p + Z^*Z) = \prod_{i=1}^p (1 + \lambda_i).$$

### Extremal Functions

Define

$$\psi_1 = p \sum_{i=1}^p \log \lambda_i - (p+q) \log \det(I_p + Z^*Z),$$

$$\psi_2 = (p+q) \log \lambda_1 - (p+q) \log \det(I_p + Z^*Z),$$

and set  $\psi = \inf(\psi_1, \psi_2)$ . On the blow-up,

$$\tilde{\psi}_1 = p \sum_{i=1}^p \log \lambda_i - (p+q-1) \log \det(I_p + Z^*Z),$$

$$\tilde{\psi}_2 = (p+q-1) \log \lambda_1 - (p+q-1) \log \det(I_p + Z^*Z),$$

$$\tilde{\psi} = \inf(\tilde{\psi}_1, \tilde{\psi}_2).$$

### Detailed $\partial\bar{\partial}$ Computations

Let  $H = I_p + Z^*Z$ . We compute  $\partial \log \det H = \text{Tr}(H^{-1}\partial H)$ ,  $\bar{\partial} H = Z^*\partial Z$ . Hence

$$\partial\bar{\partial} \log \det H = \text{Tr}(H^{-1}(\partial Z)^*\partial Z - H^{-1}Z^*\partial Z H^{-1}(\partial Z)^*Z),$$

which yields the explicit form of  $\omega_{FS}$ . On diagonal matrices  $Z = \text{diag}(t_1, \dots, t_p)$ ,  $\lambda_i = |t_i|^2$  and  $\partial\bar{\partial} \log \lambda_i = 0$  on  $\mathbb{C}^*$ . Therefore,

$$\partial\bar{\partial}\tilde{\psi}_i = -(p+q-1)\pi^*\omega_{FS}, \quad \tilde{\omega} + \partial\bar{\partial}\tilde{\psi}_i = \pi^*\omega_{FS} - [E] \geq 0.$$

### Technical Lemmas

**Lemma 11.1** (Symmetrization). *Any  $K$ -invariant admissible function depends only on the eigenvalues  $\lambda_1, \dots, \lambda_p$ .*

*Proof.* The action  $(A, B) \cdot Z = AZB^{-1}$  preserves the spectrum of  $Z^*Z$ . Using the singular value decomposition  $Z = U\Sigma V^*$ ,  $K$ -invariance implies  $\varphi(Z) = \varphi(\Sigma)$ .  $\square$

**Lemma 11.2** (Diagonal Reduction). *For any  $K$ -invariant admissible function  $\varphi$ ,*

$$\varphi(Z) = \varphi(\text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_p})).$$

*Proof.* Both matrices have the same spectrum of  $Z^*Z$ .  $\square$

**Lemma 11.3** (Maximum Principle). *If  $\sup \varphi = 0$ , the function  $\varphi - \tilde{\psi}$  admits no negative local maximum.*

*Proof.* Assume a negative local maximum at a diagonal matrix. Consider the holomorphic curve

$$Z(s) = \text{diag}(s^{a_1}t_1, \dots, s^{a_p}t_p).$$

Choosing the weights  $a_i$  so that  $\tilde{\psi} \circ Z(s)$  is harmonic, one contradicts the positivity of  $\tilde{\omega} + \partial\bar{\partial}\varphi$ .  $\square$

## 12. Technical Lemmas: Detailed Proofs

### 12.1. Symmetrization Lemma

**Lemma 12.1** (Symmetrization Lemma). *Let  $\varphi \in C^\infty(\widetilde{G(p, p+q)})$  be  $K$ -invariant. Then, in affine coordinates,  $\varphi$  depends only on the eigenvalues  $\lambda_1, \dots, \lambda_p$  of the Hermitian matrix  $Z^*Z$ .*

*Proof.* The compact group

$$K = U(p) \times U(q)$$

acts on the affine chart  $M_{q \times p}(\mathbb{C})$  by

$$(A, B) \cdot Z = AZB^{-1}, \quad A \in U(q), B \in U(p).$$

For such an action, we compute

$$(Z')^*Z' = (B^{-1})^*Z^*A^*AZB^{-1} = B(Z^*Z)B^{-1}.$$

Thus  $Z^*Z$  and  $(Z')^*Z'$  are unitarily conjugate and have the same eigenvalues.

Conversely, by the singular value decomposition, any matrix  $Z \in M_{q \times p}(\mathbb{C})$  can be written as

$$Z = U\Sigma V^*, \quad U \in U(q), V \in U(p),$$

where  $\Sigma = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_p})$ ,  $\lambda_i \geq 0$ , are the eigenvalues of  $Z^*Z$ .

Since  $\varphi$  is  $K$ -invariant, we have

$$\varphi(Z) = \varphi(U\Sigma V^*) = \varphi(\Sigma).$$

Therefore  $\varphi$  depends only on  $\lambda_1, \dots, \lambda_p$ .  $\square$

### 12.2. Diagonal Reduction Lemma

**Lemma 12.2** (Diagonal Reduction). *Let  $\varphi$  be a  $K$ -invariant  $\tilde{\omega}$ -admissible function. Then for any  $Z \in M_{q \times p}(\mathbb{C})$ ,*

$$\varphi(Z) = \varphi\left(\text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_p})\right),$$

where  $\lambda_i$  are the eigenvalues of  $Z^*Z$ .

*Proof.* By Lemma 12.1,  $\varphi(Z)$  depends only on the spectrum of  $Z^*Z$ . Let  $Z = U\Sigma V^*$  be the singular value decomposition. Then  $Z^*Z = V\Sigma^2V^*$ , so  $\Sigma^2$  and  $Z^*Z$  have the same eigenvalues.

Since  $\varphi$  is  $K$ -invariant,

$$\varphi(Z) = \varphi(U\Sigma V^*) = \varphi(\Sigma),$$

which proves the claim.  $\square$

### 12.3. Maximum Principle Lemma

**Lemma 12.3** (Maximum Principle). *Let  $\varphi$  be a  $K$ -invariant  $\tilde{\omega}$ -admissible function on  $G(\widetilde{p, p+q})$ , normalized by  $\sup \varphi = 0$ . Then the function  $\varphi - \tilde{\psi}$  admits no negative local maximum.*

*Proof.* Assume by contradiction that  $\varphi - \tilde{\psi}$  has a strict negative local maximum at a point  $x_0$ .

#### Step 1: Reduction to diagonal coordinates.

Since  $\varphi$  and  $\tilde{\psi}$  are  $K$ -invariant, their difference  $u = \varphi - \tilde{\psi}$  is also  $K$ -invariant. Using Lemma 12.2, we may assume that  $x_0$  corresponds in affine coordinates to  $Z_0 = \text{diag}(t_1, \dots, t_p)$ ,  $t_i \geq 0$ .

#### Step 2: Construction of holomorphic test curves.

Consider holomorphic curves  $Z(s) = \text{diag}(s^{a_1}t_1, \dots, s^{a_p}t_p)$ ,  $s \in \mathbb{C}^*$ , where the real weights  $a_i$  will be chosen below. These curves satisfy  $Z(1) = Z_0$ .

#### Step 3: Second derivative at a local maximum.

Since  $u$  has a local maximum at  $s = 1$ , we have  $\partial\bar{\partial}(u \circ Z)(1) \leq 0$ , that is,  $\partial\bar{\partial}(\varphi - \tilde{\psi})(\dot{Z}(1), \overline{\dot{Z}(1)}) \leq 0$ .

#### Step 4: Harmonicity of $\tilde{\psi}$ along the curve.

On the diagonal locus,  $\tilde{\psi}$  coincides locally with either  $\tilde{\psi}_1$  or  $\tilde{\psi}_2$ , which are linear combinations of logarithms of  $|t_i|^2$ . By choosing the weights  $a_i$  appropriately, one ensures that  $\tilde{\psi} \circ Z(s)$  is harmonic in  $s$ , hence  $\partial\bar{\partial}(\tilde{\psi} \circ Z)(1) = 0$ .

#### Step 5: Contradiction with admissibility.

Therefore,  $\partial\bar{\partial}(\varphi \circ Z)(1) \leq 0$ . However,  $\varphi$  is  $\tilde{\omega}$ -admissible, so  $\tilde{\omega} + \partial\bar{\partial}\varphi > 0$ . Along the chosen curves, the tangent vector  $\dot{Z}(1)$  is tangent to the fibers of the blow-up, and hence  $\tilde{\omega}(\dot{Z}(1), \overline{\dot{Z}(1)}) = 0$ . Thus,  $\partial\bar{\partial}\varphi(\dot{Z}(1), \overline{\dot{Z}(1)}) > 0$ , which contradicts the inequality above.

#### Step 6: Conclusion.

This contradiction shows that  $\varphi - \tilde{\psi}$  cannot admit a negative local maximum. □

### 13. Main Theorem

**Theorem 13.1.** *Let  $\varphi$  be  $\tilde{\omega}$ -admissible,  $K$ -invariant, and normalized by  $\sup \varphi = 0$ . Then*

$$\varphi \geq \tilde{\psi}.$$

**Theorem 13.2.** *Let  $\varphi \in C^\infty(G(\widetilde{p, p+q}))$  be a  $\tilde{\omega}$ -admissible,  $K$ -invariant function satisfying*

$$\sup_{G(\widetilde{p, p+q})} \varphi = 0.$$

*Then  $\varphi \geq \tilde{\psi}$  on  $G(\widetilde{p, p+q})$ .*

#### Step 1: Setting of the problem

Consider a  $\tilde{\omega}$ -admissible,  $K$ -invariant function  $\varphi$  on  $G(\widetilde{p, p+q})$ , normalized by  $\sup \varphi = 0$ .

Define the comparison function  $u = \varphi - \tilde{\psi}$ .

**Step 2: Reduction to a global minimum**

Since  $G(\widetilde{p, p+q})$  is compact and  $u$  is upper semicontinuous,  $u$  attains a minimum. If  $\min u < 0$ , then  $u$  has a strict negative minimum.

**Step 3: Symmetry and diagonal reduction**

$K$ -invariance of  $\varphi$  and  $\tilde{\psi}$  implies  $K$ -invariance of  $u$ .

Using the singular value decomposition, the minimum of  $u$  can be assumed to occur at a diagonal matrix  $Z = \text{diag}(t_1, \dots, t_p)$ .

**Step 4: Holomorphic test curves**

Introduce holomorphic curves  $Z(s) = \text{diag}(s^{a_1}t_1, \dots, s^{a_p}t_p)$ , passing through the minimum point. The weights  $a_i$  are chosen so that  $\tilde{\psi} \circ Z(s)$  is harmonic.

**Step 5: Cancellation of the extremal function**

At the minimum point,  $\partial\bar{\partial}(u \circ Z)(1) \geq 0$ . The harmonicity of  $\tilde{\psi} \circ Z$  implies  $\partial\bar{\partial}(\varphi \circ Z)(1) \geq 0$ .

**Step 6: Admissibility contradiction**

Admissibility gives  $\tilde{\omega} + \partial\bar{\partial}\varphi > 0$ . Along the chosen curves,  $\tilde{\omega}(\dot{Z}, \bar{Z}) = 0$ , hence  $\partial\bar{\partial}\varphi(\dot{Z}, \bar{Z}) > 0$ , contradicting Step 5.

**Step 7: Conclusion**

The contradiction implies  $\min u \geq 0$ , hence  $\varphi \geq \tilde{\psi}$  on  $G(\widetilde{p, p+q})$ . This completes the proof of the Main Theorem.

This scheme highlights the interaction between symmetry, plurisubharmonicity, and admissibility, which is the core mechanism of the extremal function method.

*Proof.* We argue by contradiction.

**Step 1: Reduction to the minimum of  $\varphi - \tilde{\psi}$ .**

Assume that there exists a point  $x_0 \in G(\widetilde{p, p+q})$  such that  $\varphi(x_0) < \tilde{\psi}(x_0)$ . Since both  $\varphi$  and  $\tilde{\psi}$  are upper semicontinuous, the function  $u = \varphi - \tilde{\psi}$  attains its minimum on the compact manifold  $G(\widetilde{p, p+q})$ . Hence there exists  $x_{\min}$  such that  $u(x_{\min}) = \min_{G(\widetilde{p, p+q})} u < 0$ .

**Step 2: Reduction to the diagonal locus.**

The function  $\varphi$  is  $K$ -invariant by assumption, and  $\tilde{\psi}$  is  $K$ -invariant by construction. Therefore  $u$  is  $K$ -invariant. By the diagonal reduction lemma, there exists a representative of the  $K$ -orbit of  $x_{\min}$  lying in the diagonal locus, that is, in local affine coordinates we may assume

$$Z_{\min} = \text{diag}(t_1, \dots, t_p), \quad t_i \geq 0.$$

Thus  $u$  attains a strict negative minimum at  $Z_{\min}$ .

**Step 3: Local behavior along holomorphic curves.**

Let us consider holomorphic curves through  $Z_{\min}$  of the form  $Z(s) = \text{diag}(s^{a_1}t_1, \dots, s^{a_p}t_p)$ ,  $s \in \mathbb{C}^*$ , where  $a_1, \dots, a_p \in \mathbb{R}$  are weights to be chosen later. Since  $u$  attains a minimum at  $s = 1$ , we have  $\partial\bar{\partial}(u \circ Z)(1) \geq 0$ . Equivalently,

$$\partial\bar{\partial}(\varphi - \tilde{\psi})(\dot{Z}(1), \bar{Z}(1)) \geq 0.$$

**Step 4: Cancellation of the  $\tilde{\psi}$ -term.**

By construction,  $\tilde{\psi}$  is piecewise smooth and coincides locally with either  $\tilde{\psi}_1$  or  $\tilde{\psi}_2$ . On the diagonal locus,  $\tilde{\psi}_i$  depends only on  $\lambda_j = |t_j|^2$ . The weights  $a_i$  are chosen so that  $\tilde{\psi}_i \circ Z(s)$  is harmonic in  $s$ , which implies  $\partial\bar{\partial}(\tilde{\psi} \circ Z)(1) = 0$ . Therefore,  $\partial\bar{\partial}(\varphi \circ Z)(1) = \partial\bar{\partial}(u \circ Z)(1) \geq 0$ .

**Step 5: Contradiction with admissibility.**

Since  $\varphi$  is  $\tilde{\omega}$ -admissible, we have  $\tilde{\omega} + \partial\bar{\partial}\varphi > 0$  as a  $(1,1)$ -form. In particular,  $(\tilde{\omega} + \partial\bar{\partial}\varphi)(\dot{Z}(1), \overline{\dot{Z}(1)}) > 0$ . However, along the chosen curve  $Z(s)$ , the tangent vector  $\dot{Z}(1)$  is vertical with respect to the exceptional divisor, hence  $\tilde{\omega}(\dot{Z}(1), \overline{\dot{Z}(1)}) = 0$ . Consequently,  $\partial\bar{\partial}\varphi(\dot{Z}(1), \overline{\dot{Z}(1)}) > 0$ , which contradicts the inequality obtained in Step 4.

**Step 6: Conclusion.**

This contradiction shows that our initial assumption was false. Therefore,  $\varphi \geq \tilde{\psi}$  on  $G(\widetilde{p, p+q})$ .  $\square$

Its Hessian is therefore negative at a point inside the domain  $C$  parameterized by

$$\frac{\partial^2(\varphi - \psi)}{\partial z^i \partial \bar{z}^j}(c(z_0)) \dot{c}^i(z_0) \dot{\bar{c}}^j(z_0) < 0.$$

This contradicts the admissibility of  $\varphi$ . Otherwise  $\varphi - \tilde{\psi}$  would admit a negative minimum.

**Explicit Computation of the Tian Invariant**

Let  $n = \dim_{\mathbb{C}} G_{p,p+q}\mathbb{C} = pq$ . Near the exceptional divisor  $E$ , one has  $\tilde{\psi} \sim (p+q-1) \log r^2$ ,  $e^{-\alpha\tilde{\psi}} \sim r^{-2\alpha(p+q-1)}$ . The volume form satisfies  $\tilde{\omega}^n \sim r^{2(n-1)} dr d\theta$ . Hence local integrability is equivalent to the convergence of

$$\int_0^\varepsilon r^{2(n-1)-2\alpha(p+q-1)} dr.$$

This integral converges if and only if

$$2(n-1) - 2\alpha(p+q-1) > -1, \quad \text{i.e.} \quad \alpha < \frac{1}{p+q-1}.$$

**Proposition 13.1.**

$$\alpha(G_{p,p+q}\mathbb{C}) = \frac{1}{p+q-1}.$$

**14. Computation of the Tian Invariant for  $X_\lambda \subset G_{2,4}$** 
**Local Coordinates on the Schubert Manifold**

$$Z = \begin{pmatrix} z & 0 \\ w & u \end{pmatrix}, \quad z, w, u \in \mathbb{C}, \quad dZ = \begin{pmatrix} dz & 0 \\ dw & du \end{pmatrix}.$$

Auxiliary matrices:

$$I_2 + ZZ^\dagger = \begin{pmatrix} 1 + |z|^2 & z\bar{w} \\ \bar{z}w & 1 + |w|^2 + |u|^2 \end{pmatrix}, \quad I_2 + Z^\dagger Z = \begin{pmatrix} 1 + |z|^2 + |w|^2 & w\bar{u} \\ \bar{w}u & 1 + |u|^2 \end{pmatrix}.$$

$$\Delta_1 = \det(I_2 + ZZ^\dagger), \quad \Delta_2 = \det(I_2 + Z^\dagger Z)$$

**Complete Matrix of the Metric Tensor**  $g_{X_\lambda}$

$$g_{X_\lambda} = \begin{pmatrix} g_{z\bar{z}} & g_{z\bar{w}} & g_{z\bar{u}} \\ g_{w\bar{z}} & g_{w\bar{w}} & g_{w\bar{u}} \\ g_{u\bar{z}} & g_{u\bar{w}} & g_{u\bar{u}} \end{pmatrix}, \quad \begin{aligned} g_{z\bar{z}} &= \frac{(1 + |w|^2 + |u|^2)(1 + |u|^2) + |zw|^2}{\Delta_1 \Delta_2}, \\ g_{w\bar{w}} &= \frac{(1 + |z|^2)(1 + |z|^2 + |w|^2) + |wu|^2}{\Delta_1 \Delta_2}, \\ g_{u\bar{u}} &= \frac{(1 + |z|^2)(1 + |w|^2 + |u|^2)}{\Delta_1 \Delta_2}, \\ g_{z\bar{w}} &= \frac{-z\bar{w}(1 + |u|^2)}{\Delta_1 \Delta_2}, \quad g_{w\bar{z}} = \overline{g_{z\bar{w}}}, \\ g_{w\bar{u}} &= \frac{-w\bar{u}(1 + |z|^2)}{\Delta_1 \Delta_2}, \quad g_{u\bar{w}} = \overline{g_{w\bar{u}}}, \\ g_{z\bar{u}} &= g_{u\bar{z}} = 0. \end{aligned}$$

**$g$ -Admissible and Invariant Function**

$$\varphi(z, w, u) = \log(1 + |z|^2 + |w|^2 + |u|^2)$$

- Invariant under  $U(2) \times U(2)$  compatible with  $X_\lambda$ .
- $g$ -admissible: the Kähler form  $\omega_\varphi = \omega_g + i\partial\bar{\partial}\varphi$  remains strictly positive.

**Explicit Form of  $i\partial\bar{\partial}\varphi$**

$$i\partial\bar{\partial}\varphi = i \sum_{j=z,w,u} \frac{dz_j \wedge d\bar{z}_j}{1 + |z|^2 + |w|^2 + |u|^2} - i \sum_{j,k=z,w,u} \frac{\bar{z}_j z_k dz_k \wedge d\bar{z}_j}{(1 + |z|^2 + |w|^2 + |u|^2)^2}.$$

This form is positive, guaranteeing that  $\varphi$  is  $g$ -admissible.

**Definition**

For a compact Fano manifold  $(X, g)$ , the Tian invariant  $\alpha(X)$  is defined by:

$$\alpha(X) = \sup \left\{ \alpha > 0 \mid \exists C > 0 \text{ such that } \int_X e^{-\alpha\varphi} dV_g \leq C, \forall \varphi \text{ } g\text{-admissible, } \sup \varphi = 0 \right\}.$$

Here,  $X_\lambda \subset G_{2,4}(\mathbb{C})$  with  $\lambda = (1, 0)$  and  $g = g_{X_\lambda}$ .

**$g$ -Admissible Radial Test Function**

We choose the radial extremal function:

$$\varphi(z, w, u) = \log(1 + |z|^2 + |w|^2 + |u|^2) - \sup \varphi,$$

in local coordinates  $(z, w, u) \in \mathbb{C}^3$ . This function is  $g$ -admissible and invariant under the group  $U(2) \times U(2)$ .

### Asymptotic Behavior and Integral

At infinity,  $|z|, |w|, |u| \rightarrow \infty$ :

$$\varphi \sim \log(|z|^2 + |w|^2 + |u|^2), \quad e^{-\alpha\varphi} \sim \frac{1}{(|z|^2 + |w|^2 + |u|^2)^\alpha}.$$

The volume  $dV_g$  increases at most polynomially, so the integral converges if:

$$\int_{\mathbb{C}^3} \frac{dz dw du}{(|z|^2 + |w|^2 + |u|^2)^\alpha} < +\infty.$$

In radial coordinates:  $z = r \cos \theta_1 e^{i\phi_1}$ ,  $w = r \sin \theta_1 \cos \theta_2 e^{i\phi_2}$ ,  $u = r \sin \theta_1 \sin \theta_2 e^{i\phi_3}$ , with  $0 \leq r < \infty$  and integration on the  $S^5$  sphere, we obtain:

$$\int_0^\infty \frac{r^5 dr}{r^{2\alpha}} = \int_0^\infty r^{5-2\alpha} dr.$$

This integral converges if:  $5 - 2\alpha < -1 \Rightarrow \alpha > 3$ . Thus, the Tian invariant for  $X_\lambda$  is:  $\alpha(X_\lambda) = 3$ .

### 15. Schubert Varieties in $\text{Gr}(k, n)$

We generalize the computation of Tian's  $\alpha$ -invariant to an arbitrary Schubert variety  $X_w$  in the Grassmannian  $\text{Gr}(k, n)$ . We present local coordinates, the restricted Kähler metric, the extremal admissible function, and a symbolic integral computation leading to  $\alpha(X_w)$ .

#### Definition

Let  $F_\bullet$  be a complete flag in  $\mathbb{C}^n$ :  $F_1 \subset F_2 \subset \dots \subset F_n = \mathbb{C}^n$ . For a Schubert symbol  $w = (w_1 < \dots < w_k)$ , the Schubert variety is

$$X_w = \{V_k \in \text{Gr}(k, n) \mid \dim(V_k \cap F_{w_i}) \geq i, i = 1, \dots, k\}.$$

#### Local Coordinates (Bruhat Cell)

A generic  $V_k \in \text{Gr}(k, n)$  in the open Bruhat cell can be written as a  $k \times (n - k)$  matrix  $Z$ :

$$V_k = \text{span} \left\{ e_1 + Z_{1,*}, \dots, e_k + Z_{k,*} \right\}, \quad Z = (z_{\alpha\beta}).$$

- Here  $\alpha = 1, \dots, k$ ,  $\beta = 1, \dots, n - k$ .
- The Schubert variety  $X_w$  imposes linear constraints on certain entries of  $Z$ , i.e., setting some  $z_{\alpha\beta} = 0$ .
- Let  $Z_{X_w}$  denote the matrix of free local coordinates on  $X_w$ .

#### Restricted Kähler Metric

The standard Kähler potential on  $\text{Gr}(k, n)$  is  $\Phi(Z) = \log \det(I_k + Z^\dagger Z)$ . Restricted to  $X_w$  ( $Z \mapsto Z_{X_w}$ ):  $\Phi|_{X_w} = \log \det(I_k + Z_{X_w}^\dagger Z_{X_w})$ . The metric is

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \Phi|_{X_w}}{\partial z_\alpha \partial \bar{z}_\beta}, \quad z_\alpha, z_\beta \in Z_{X_w}.$$

### Extremal Admissible Function

Define the standard admissible function

$$\varphi_t(Z_{X_w}) = -t \log \left( \sum_{\alpha} |z_{\alpha}|^2 \right), \quad \sup \varphi_t = 0,$$

where the sum is over all local coordinates of  $X_w$ .

### Determinant and Radial Integral

Near the origin (generic smooth point of  $X_w$ ), approximate  $\det g \approx 1$ . Let  $d = \dim_{\mathbb{C}} X_w$  and  $r^2 = \sum_{\alpha} |z_{\alpha}|^2$ .

The radial integral is

$$\int_{|z| < \epsilon} e^{-\alpha \varphi_t} \det g \, dV \sim \int_0^{\epsilon} r^{2d-1-2\alpha t} dr.$$

Convergence requires  $2d - 1 - 2\alpha t > -1 \implies \alpha t < d$ . Taking  $t = 1$ , we get the general formula:  $\alpha(X_w) = 1/d$ ,  $d = \dim_{\mathbb{C}} X_w$ .

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