

Interval-valued topological spaces associated with Λ_{rg} -closed sets

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Abstract: In this paper, a new class of sets, called *interval-valued Λ -regular generalized closed sets*, is introduced in interval-valued topological spaces. The basic definitions and fundamental properties of these sets are investigated in detail. The corresponding *interval-valued Λ -regular generalized open sets* are also defined and studied. Furthermore, several relationships between these newly introduced sets and the existing classes of interval-valued closed and open sets are established. It is shown that interval-valued Λ -regular generalized closed sets constitute a weaker class than interval-valued closed sets. Similarly, interval-valued Λ -regular generalized open sets form a weaker class than interval-valued open sets. Appropriate examples are provided to illustrate and support the obtained results. This study extends the theory of generalized sets in interval-valued topological spaces and contributes to the further development of interval-valued topology.

Key words: interval-valued Λ_r -set, interval-valued λ_r -closed, interval-valued Λ_g -closed and Λ_{rg} -closed

1. Introduction

In 2026, I. Rajasekaran *et al.* [4] introduced the concept of interval-valued Λ_r -sets in interval-valued topological spaces. They also defined interval-valued λ_r -closed sets and interval-valued λ_r -open sets and investigated some of their basic properties. These concepts extended several classical notions in interval-valued topology and provided a broader framework for further study. Motivated by these developments, in this paper we introduce a new class of sets called interval-valued Λ_{rg} -closed sets and interval-valued Λ_{rg} -open sets in interval-valued topological spaces. Various fundamental properties and characterizations of these newly defined sets are studied in detail. Relationships between these sets and existing interval-valued closed and interval-valued open sets are also obtained. It is shown that interval-valued Λ_{rg} -closed sets and interval-valued Λ_{rg} -open sets are weaker forms of interval-valued closed sets and interval-valued open sets, respectively. These results contribute to the development of generalized structures in interval-valued topology.

2. Preliminaries

Definition 2.1. [5] Let X be a non-empty set. Then the form $[\mathcal{M}^-, \mathcal{M}^+] = \{\mathcal{S} \subset X : \mathcal{M}^- \subset \mathcal{S} \subset \mathcal{M}^+\}$ is called an interval-valued set (briefly, an $\mathcal{IVS}$) in X , if $\mathcal{M}^-, \mathcal{M}^+ \subset X$ and $\mathcal{M}^- \subset \mathcal{M}^+$.

In this case, $\mathcal{M}^-$ [resp. $\mathcal{M}^+$] represents the set of minimum [resp. maximum] memberships of elements of X to $[\mathcal{M}^-, \mathcal{M}^+]$.

Here, $[\emptyset, \emptyset]$ [resp. $[X, X]$] is called the interval-valued empty [resp. whole] set in X and is denoted by $\tilde{\emptyset}$ [resp. $\tilde{X}$]. The set of all $\mathcal{IVS}$ s in X is denoted by $\mathcal{IVS}(X)$.

Definition 2.2. [3] Let X be a non-empty set, and let $[\mathcal{M}^-, \mathcal{M}^+]$ and $[\mathcal{S}^-, \mathcal{S}^+]$ be $\mathcal{IVS}$ s in X . Then

1. $[\mathcal{M}^-, \mathcal{M}^+] \subset [\mathcal{S}^-, \mathcal{S}^+] \iff \mathcal{M}^- \subset \mathcal{S}^- \text{ and } \mathcal{M}^+ \subset \mathcal{S}^+$
2. $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \iff [\mathcal{M}^-, \mathcal{M}^+] \subset [\mathcal{S}^-, \mathcal{S}^+] \text{ and } [\mathcal{S}^-, \mathcal{S}^+] \subset [\mathcal{M}^-, \mathcal{M}^+]$
3. $([\mathcal{M}^-, \mathcal{M}^+])^c = [(\mathcal{M}^+)^c, (\mathcal{M}^-)^c]$
4. $[\mathcal{M}^-, \mathcal{M}^+] \cup [\mathcal{S}^-, \mathcal{S}^+] = [\mathcal{M}^- \cup \mathcal{S}^-, \mathcal{M}^+ \cup \mathcal{S}^+]$
5. $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{S}^-, \mathcal{S}^+] = [\mathcal{M}^- \cap \mathcal{S}^-, \mathcal{M}^+ \cap \mathcal{S}^+]$

Definition 2.3. [3] Let X be a non-empty set and let τ be a non-empty family of $\mathcal{IVS}$ s on X . Then τ is called an interval-valued topology (briefly, $\mathcal{IVT}$) on X if it satisfies the following axioms:

1. $\tilde{\emptyset}, \tilde{X} \in \tau$,
2. $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{S}^-, \mathcal{S}^+] \in \tau$ for any $[\mathcal{M}^-, \mathcal{M}^+], [\mathcal{S}^-, \mathcal{S}^+] \in \tau$.
3. $\bigcup_{j \in J} \mathcal{S}_j \in \tau$ for any family $(\mathcal{S}_j)_{j \in J}$ of members of τ .

In this case, $\mathcal{T}^X$ is called an interval-valued topological space (briefly, $\mathcal{IVTS}$) and each member of τ is called an interval-valued open set (briefly, $\mathcal{IVOs}$) in X . The complement of an $\mathcal{IVS}$ $[\mathcal{M}^-, \mathcal{M}^+]$ is called an interval-valued closed set (briefly, $\mathcal{IVCs}$) in X .

An interval-valued topological space is denoted by $\mathcal{IVTS}$ or $\mathcal{T}^X$.

Definition 2.4. [3] Let $\mathcal{T}^X$ be an interval-valued topological space with respect to X , and let $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \tilde{X}$. Then

1. the interval-valued interior of $[\mathcal{M}^-, \mathcal{M}^+]$ is defined as the union of all interval-valued open subsets of $[\mathcal{M}^-, \mathcal{M}^+]$ and is denoted by $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$.

That is, $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$ is the largest interval-valued open subset of $[\mathcal{M}^-, \mathcal{M}^+]$.

2. the interval-valued closure of $[\mathcal{M}^-, \mathcal{M}^+]$ is defined as the intersection of all interval-valued closed sets containing $[\mathcal{M}^-, \mathcal{M}^+]$ and is denoted by $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$.

That is, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ is the smallest interval-valued closed set containing $[\mathcal{M}^-, \mathcal{M}^+]$.

Definition 2.5. [4] A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $\mathcal{T}^X$ is called an interval-valued regular-open (resp. $\mathcal{IVr}$ -open) set if $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))$.

The complement of such a set is called an interval-valued regular-closed (resp. $\mathcal{IVr}$ -closed) set.

Definition 2.6. [4] Let $[\mathcal{M}^-, \mathcal{M}^+]$ be a subset of an interval-valued topological space $\mathcal{T}^X$. The interval-valued regular-kernel of the set $[\mathcal{M}^-, \mathcal{M}^+]$, denoted by $\mathcal{IVr-ker}([\mathcal{M}^-, \mathcal{M}^+])$, is the intersection of all $\mathcal{IVr}$ -open supersets of $[\mathcal{M}^-, \mathcal{M}^+]$.

Definition 2.7. [4] A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $\mathcal{T}^X$ is called

1. an interval-valued Λ_r -set (resp. $\mathcal{IV}\Lambda_r$ -set) if $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}r\text{-ker}([\mathcal{M}^-, \mathcal{M}^+])$.
2. an interval-valued λ_r -closed (briefly, $\mathcal{IV}\lambda_r$ -closed) set if $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap [\mathcal{K}^-, \mathcal{K}^+]$, where $[\mathcal{S}^-, \mathcal{S}^+]$ is an $\mathcal{IV}\Lambda_r$ -set and $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}$ -closed.

Definition 2.8. [2]

A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of $\mathcal{IVS}(X)$ is called an interval-valued generalized closed (briefly, $\mathcal{IV}g$ -closed) set if $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, whenever $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$ and $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}$ -open.

Definition 2.9. [4]

A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $\mathcal{T}^X$ is called an interval-valued regular generalized-closed (briefly, $\mathcal{IV}rg$ -closed) set if $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$ whenever $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$ and $[\mathcal{S}^-, \mathcal{S}^+]$ is $\mathcal{IV}r$ -open.

Lemma 2.1. [4] Let $\mathcal{T}^X$ be an interval-valued topological space. Then the following hold:

1. every $\mathcal{IV}$ -closed set is $\mathcal{IV}\lambda_r$ -closed.
2. every $\mathcal{IV}\Lambda_r$ -set is $\mathcal{IV}\lambda_r$ -closed.

3. On interval-valued Λ -rg-closed sets

Definition 3.1. A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $\mathcal{T}^X$ is called an $\mathcal{IV}\lambda_r$ -open set if $[\mathcal{M}^-, \mathcal{M}^+]^c = \tilde{X} - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\lambda_r$ -closed.

Example 3.1. Let $X = \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}$. Then

$$\mathcal{T}^X = \{\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], \tilde{X}\}$$

is an interval-valued topology on X . The $\mathcal{IV}$ -closed sets are

$$\tilde{\emptyset}, [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], \tilde{X}.$$

The $\mathcal{IV}r$ -open sets are

$$\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], \tilde{X}.$$

Take the $\mathcal{IV}\Lambda_r$ -set $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}]$ (which is $\mathcal{IV}r$ -open, hence trivially its own $\mathcal{IV}r$ -kernel) and the $\mathcal{IV}$ -closed set $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$. Then

$$[\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}] \cap [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] = [\emptyset, \{\mathfrak{r}^2, \mathfrak{r}^4\}],$$

so by Definition 2.7(2), $[\emptyset, \{\mathfrak{r}^2, \mathfrak{r}^4\}]$ is $\mathcal{IV}\lambda_r$ -closed. Note that $[\emptyset, \{\mathfrak{r}^2, \mathfrak{r}^4\}]$ is not among the $\mathcal{IV}$ -closed sets listed above.

Consequently its complement

$$([\emptyset, \{\mathfrak{r}^2, \mathfrak{r}^4\}])^c = [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X]$$

is $\mathcal{IV}\lambda_r$ -open. Since $[\{\mathfrak{r}^1, \mathfrak{r}^3\}, X] \notin \mathcal{T}^X$, this set is $\mathcal{IV}\lambda_r$ -open but not $\mathcal{IV}$ -open, showing that the class of $\mathcal{IV}\lambda_r$ -open sets is strictly larger than the class of $\mathcal{IV}$ -open sets.

Definition 3.2. A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $\mathcal{T}^X$ is called an interval-valued regular generalized closed (resp. $\mathcal{IV}\Lambda$ -rg-closed) set if

$$\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+], \text{ whenever } [\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+] \text{ and } [\mathcal{K}^-, \mathcal{K}^+] \text{ is } \mathcal{IV}\lambda_r\text{-open.}$$

$[\mathcal{M}^-, \mathcal{M}^+]$ is called $\mathcal{IV}\Lambda$ -rg-open if its complement $[\mathcal{M}^-, \mathcal{M}^+]^c = \tilde{X} - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Example 3.2. Consider the interval-valued topological space $\mathcal{T}^X$ from Example 3.1.

The $\mathcal{IV}r$ -open sets are $\tilde{\emptyset}$, $[\emptyset, \{\mathfrak{r}^2\}]$, $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}]$, $\tilde{X}$, so that the $\mathcal{IV}\lambda_r$ -open sets are

$$\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \tilde{X}.$$

1. $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Let $[\mathcal{M}^-, \mathcal{M}^+] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$.

First compute its closure: $\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$, since $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is the smallest $\mathcal{IV}$ -closed set containing $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$.

Now list every $\mathcal{IV}\lambda_r$ -open set $[\mathcal{K}^-, \mathcal{K}^+]$ with $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$. Checking against the list above,

this requires $\{\mathfrak{r}^1\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2\} \subseteq \mathcal{K}^+$; the qualifying sets are $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$, $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$, $[\{\mathfrak{r}^1, \mathfrak{r}^3\}, X]$, $\tilde{X}$.

For each of these, we check $\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$:

- $[\mathcal{K}^-, \mathcal{K}^+] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$: $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$.
- $[\mathcal{K}^-, \mathcal{K}^+] = [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$: $\{\mathfrak{r}^1\} \subseteq \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq X$.
- $[\mathcal{K}^-, \mathcal{K}^+] = [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X]$: $\{\mathfrak{r}^1\} \subseteq \{\mathfrak{r}^1, \mathfrak{r}^3\}$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq X$.
- $[\mathcal{K}^-, \mathcal{K}^+] = \tilde{X}$: trivially holds.

Since $\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$ holds for every $\mathcal{IV}\lambda_r$ -open set containing $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$, it follows by Definition 3.3 that $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

2. $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open.

By definition, $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open iff its complement

$$([\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}])^c = [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$$

is $\mathcal{IV}\Lambda$ -rg-closed.

Note $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is already $\mathcal{IV}$ -closed, so

$$\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}].$$

Now find every $\mathcal{IV}\lambda_r$ -open set $[\mathcal{K}^-, \mathcal{K}^+]$ containing $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$, i.e. with $\{\mathfrak{r}^1, \mathfrak{r}^2\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq \mathcal{K}^+$. Checking the list of $\mathcal{IV}\lambda_r$ -open sets, **only** $\tilde{X} = [X, X]$ satisfies $\{\mathfrak{r}^1, \mathfrak{r}^2\} \subseteq \mathcal{K}^-$ (every other listed set has $\mathcal{K}^-$ missing at least one of $\mathfrak{r}^1, \mathfrak{r}^2$).

For $[\mathcal{K}^-, \mathcal{K}^+] = \tilde{X}$, clearly

$$\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq \tilde{X}.$$

Since this is the only case to check and it holds, $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed. Hence, by definition, $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open.

Lemma 3.1. *In an interval-valued topological space $\mathcal{T}^X$, every $\mathcal{IV}$ -open set is $\mathcal{IV}\Lambda$ -rg-open.*

Proof.

Let $[\mathcal{M}^-, \mathcal{M}^+]$ be a $\mathcal{IV}$ -open set in $\mathcal{T}^X$, and suppose $[\mathcal{M}^-, \mathcal{M}^+]^c \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, where $[\mathcal{K}^-, \mathcal{K}^+]$ is an arbitrary $\mathcal{IV}\lambda_r$ -open set. We must show

$$\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c) \subseteq [\mathcal{K}^-, \mathcal{K}^+].$$

Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}$ -open, its complement $[\mathcal{M}^-, \mathcal{M}^+]^c$ is $\mathcal{IV}$ -closed by definition. A set that is already $\mathcal{IV}$ -closed coincides with its own closure, so

$$\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c) = [\mathcal{M}^-, \mathcal{M}^+]^c.$$

Combining this with the initial containment,

$$\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c) = [\mathcal{M}^-, \mathcal{M}^+]^c \subseteq [\mathcal{K}^-, \mathcal{K}^+].$$

As $[\mathcal{K}^-, \mathcal{K}^+]$ was an arbitrary $\mathcal{IV}\lambda_r$ -open set containing $[\mathcal{M}^-, \mathcal{M}^+]^c$, this shows that $[\mathcal{M}^-, \mathcal{M}^+]^c$ satisfies the defining condition of a $\mathcal{IV}\Lambda$ -rg-closed set (Definition 3.3). Hence $[\mathcal{M}^-, \mathcal{M}^+]^c$ is $\mathcal{IV}\Lambda$ -rg-closed, and therefore, by definition, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open.

Since $[\mathcal{M}^-, \mathcal{M}^+]$ was an arbitrary $\mathcal{IV}$ -open set, it follows that every $\mathcal{IV}$ -open set is $\mathcal{IV}\Lambda$ -rg-open. $\square$

Remark 3.1. *The converse of the statement in Lemma 3.1 is not necessarily true, as seen from the following example.*

Example 3.3. *Consider the interval-valued topological space $\mathcal{T}^X$ from Example 3.1. We show that $[\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ is $\mathcal{IV}\Lambda$ -rg-open but not $\mathcal{IV}$ -open.*

1. $[\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ **is not $\mathcal{IV}$ -open.**

The members of $\mathcal{T}^X$ are precisely

$$\tilde{\emptyset}, \quad [\emptyset, \{\mathfrak{r}^2\}], \quad [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}], \quad [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], \quad \tilde{X}.$$

Since $[\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ does not coincide with any of these five sets, it does not belong to $\mathcal{T}^X$. Hence $[\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ is not $\mathcal{IV}$ -open.

2. $[\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ **is $\mathcal{IV}\Lambda$ -rg-open.**

By definition, $[\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ is $\mathcal{IV}\Lambda$ -rg-open if and only if its complement

$$([\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}])^c = [\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]$$

is $\mathcal{IV}\Lambda$ -rg-closed.

Compute the closure. The $\mathcal{IV}$ -closed sets in $\mathcal{T}^X$ are

$$\tilde{\emptyset}, \quad [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], \quad [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], \quad [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], \quad \tilde{X}.$$

The smallest of these containing $[\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]$ is $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$, since we need $\{\mathfrak{r}^1, \mathfrak{r}^4\} \subseteq$ (minus-part) and $\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\} \subseteq$ (plus-part); checking each candidate, $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$ is the only one that fits both conditions minimally. Hence

$$\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X].$$

List the relevant $\mathcal{IV}\lambda_r$ -open sets. Recall from Example 3.1 that the $\mathcal{IV}\lambda_r$ -open sets are

$$\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \tilde{X}.$$

We need those containing $[\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]$, i.e. satisfying $\{\mathfrak{r}^1, \mathfrak{r}^4\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\} \subseteq \mathcal{K}^+$. Checking each set in the list against these two containments, only

$$[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X] \quad \text{and} \quad \tilde{X}$$

qualify (every other listed set fails to contain $\mathfrak{r}^4$ in its minus-part).

Check the closure condition for each.

- For $[\mathcal{K}^-, \mathcal{K}^+] = [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$: we need $\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X] \subseteq [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$, which holds.
- For $[\mathcal{K}^-, \mathcal{K}^+] = \tilde{X}$: we need $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X] \subseteq \tilde{X} = \tilde{X}$, which holds trivially.

Since $\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$ holds for every $\mathcal{IV}\lambda_r$ -open set $[\mathcal{K}^-, \mathcal{K}^+]$ containing $[\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]$, it follows by Definition 3.2 that $[\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Since $([\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}])^c = [\{\mathfrak{r}^1, \mathfrak{r}^4\}, \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed, it follows that $[\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ is $\mathcal{IV}\Lambda$ -rg-open.

Combining (1) and (2): $[\{\mathfrak{r}^2\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ is $\mathcal{IV}\Lambda$ -rg-open but not $\mathcal{IV}$ -open, so the converse of Lemma 3.1 fails, i.e. the class of $\mathcal{IV}\Lambda$ -rg-open sets is strictly larger than the class of $\mathcal{IV}$ -open sets.

Remark 3.2. The example below demonstrates that there is no inclusion relation between $\mathcal{IV}\Lambda$ -rg-closed sets and $\mathcal{IV}\lambda_r$ -closed sets.

Example 3.4. Consider the interval-valued topological space $\mathcal{T}^X$ from Example 3.1. The $\mathcal{IV}r$ -open sets (equivalently the $\mathcal{IV}\Lambda_r$ -sets) are:

$$\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], \tilde{X}.$$

By Definition 2.7(2), the $\mathcal{IV}\lambda_r$ -closed sets are obtained as intersections of $\mathcal{IV}\Lambda_r$ -set with a $\mathcal{IV}$ -closed set. Carrying this out gives the full list of $\mathcal{IV}\lambda_r$ -closed sets:

$$\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], [\emptyset, \{\mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], \tilde{X}.$$

Taking complements gives the $\mathcal{IV}\lambda_r$ -open sets:

$$\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \tilde{X}.$$

1. $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed, but $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ is not $\mathcal{IV}\lambda_r$ -closed.

2. $[\emptyset, \{\mathfrak{r}^2\}]$ is $\mathcal{IV}\lambda_r$ -closed, but $[\emptyset, \{\mathfrak{r}^2\}]$ is not $\mathcal{IV}\Lambda$ -rg-closed.

1. $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ **is not $\mathcal{IV}\lambda_r$ -closed** : Comparing $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ against the list of $\mathcal{IV}\lambda_r$ -closed sets above, it does not appear; hence it is not $\mathcal{IV}\lambda_r$ -closed.

$[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed. First, since $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is the smallest $\mathcal{IV}$ -closed set containing $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$,

$$\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}].$$

Next, find every $\mathcal{IV}\lambda_r$ -open set $[\mathcal{K}^-, \mathcal{K}^+]$ with $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, i.e. $\{\mathfrak{r}^1\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^4\} \subseteq \mathcal{K}^+$. Checking the list of $\mathcal{IV}\lambda_r$ -open sets, the qualifying sets are

$$[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], \quad [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], \quad [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \quad \tilde{X}.$$

For each, we verify $\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$:

- $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$: $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$.
- $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$: $\{\mathfrak{r}^1\} \subseteq \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq X$.
- $[\{\mathfrak{r}^1, \mathfrak{r}^3\}, X]$: $\{\mathfrak{r}^1\} \subseteq \{\mathfrak{r}^1, \mathfrak{r}^3\}$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq X$.
- $\tilde{X}$: trivially holds.

The condition holds for every $\mathcal{IV}\lambda_r$ -open superset, so by Definition 3.2, $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

2. $[\emptyset, \{\mathfrak{r}^2\}]$ **is $\mathcal{IV}\lambda_r$ -closed** : This set appears directly in the list of $\mathcal{IV}\lambda_r$ -closed sets derived above (it arises by intersecting the $\mathcal{IV}\Lambda_r$ -set $[\emptyset, \{\mathfrak{r}^2\}]$ with the $\mathcal{IV}$ -closed set $\tilde{X}$).

$[\emptyset, \{\mathfrak{r}^2\}]$ is not $\mathcal{IV}\Lambda$ -rg-closed. First,

$$\mathcal{IVC}([\emptyset, \{\mathfrak{r}^2\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}],$$

since $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is the smallest $\mathcal{IV}$ -closed set containing $[\emptyset, \{\mathfrak{r}^2\}]$.

Take the witness $\mathcal{IV}\lambda_r$ -open set $[\mathcal{K}^-, \mathcal{K}^+] = [\emptyset, \{\mathfrak{r}^2\}]$ itself, noting that $[\emptyset, \{\mathfrak{r}^2\}]$ is indeed $\mathcal{IV}\lambda_r$ -open (it lies in τ , hence is $\mathcal{IV}$ -open, hence $\mathcal{IV}\lambda_r$ -open by Lemma 3.1). Clearly

$$[\emptyset, \{\mathfrak{r}^2\}] \subseteq [\emptyset, \{\mathfrak{r}^2\}],$$

so this is an admissible test case in Definition 3.2. However,

$$\mathcal{IVC}([\emptyset, \{\mathfrak{r}^2\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \not\subseteq [\emptyset, \{\mathfrak{r}^2\}],$$

since $\{\mathfrak{r}^1\} \not\subseteq \emptyset$. Thus the required containment fails for this particular $\mathcal{IV}\lambda_r$ -open superset. Hence $[\emptyset, \{\mathfrak{r}^2\}]$ is not $\mathcal{IV}\Lambda$ -rg-closed.

Part (1) exhibits a set that is $\mathcal{IV}\Lambda$ -rg-closed but not $\mathcal{IV}\lambda_r$ -closed, while part (2) exhibits a set that is $\mathcal{IV}\lambda_r$ -closed but not $\mathcal{IV}\Lambda$ -rg-closed. Therefore neither class of sets is contained in the other.

Theorem 3.1. *In an interval-valued topological space $\mathcal{T}^X$, the union of any two $\mathcal{IV}\Lambda$ -rg-closed sets is again $\mathcal{IV}\Lambda$ -rg-closed set.*

Proof.

Let $[\mathcal{M}^-, \mathcal{M}^+] \cup [\mathcal{B}^-, \mathcal{B}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, where $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\lambda_r$ -open set.

Then $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$ and $[\mathcal{B}^-, \mathcal{B}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$.

Since $[\mathcal{M}^-, \mathcal{M}^+]$ and $[\mathcal{B}^-, \mathcal{B}^+]$ are $\mathcal{IV}\Lambda$ -rg-closed sets,

we have $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$ and $\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$.

Therefore,

$$\begin{aligned} \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+] \cup [\mathcal{B}^-, \mathcal{B}^+]) &= \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cup \mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) \\ &\subseteq [\mathcal{K}^-, \mathcal{K}^+]. \end{aligned}$$

Hence $[\mathcal{M}^-, \mathcal{M}^+] \cup [\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed. $\square$

Example 3.5. *Consider the interval-valued topological space $\mathcal{T}^X$ from Example 3.1. The $\mathcal{IV}\lambda_r$ -open sets are:*

$$\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \tilde{X}.$$

Let

$$[\mathcal{M}^-, \mathcal{M}^+] = [\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2\}] \quad \text{and} \quad [\mathcal{B}^-, \mathcal{B}^+] = [\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^4\}].$$

Step 1: $[\mathcal{M}^-, \mathcal{M}^+] = [\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Closure: We seek the smallest $\mathcal{IV}$ -closed set $[\mathcal{K}^-, \mathcal{K}^+]$ with $\emptyset \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2\} \subseteq \mathcal{K}^+$. Among the closed sets, $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ qualifies, and it is contained in both $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ and $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$, which also qualify. Hence it is the smallest, so

$$\mathcal{IVC}([\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}].$$

Relevant λ_r -open supersets. We list every $\mathcal{IV}\lambda_r$ -open set $[\mathcal{K}^-, \mathcal{K}^+]$ with $\emptyset \subseteq \mathcal{K}^-$ (automatic) and $\{\mathfrak{r}^1, \mathfrak{r}^2\} \subseteq \mathcal{K}^+$. Scanning the λ_r -open list, the qualifying sets are

$$[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \tilde{X}.$$

Check the containment $\mathcal{IVC}(M) \subseteq K$ for each:

- $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$: $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$, so the containment holds.
- $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$: $\{\mathfrak{r}^1\} \subseteq \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq X$, so the containment holds.
- $[\{\mathfrak{r}^1, \mathfrak{r}^3\}, X]$: $\{\mathfrak{r}^1\} \subseteq \{\mathfrak{r}^1, \mathfrak{r}^3\}$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq X$, so the containment holds.
- $\tilde{X}$: the containment holds trivially.

The condition holds in every case, so $[\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Step 2: $[\mathcal{B}^-, \mathcal{B}^+] = [\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Closure. We seek the smallest $\mathcal{IV}$ -closed set with $\emptyset \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^4\} \subseteq \mathcal{K}^+$. Both $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ and $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ qualify, with $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$, and $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$ as well. So

$$\mathcal{IVC}([\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}].$$

Relevant λ_r -open supersets. We need $\{\mathfrak{r}^1, \mathfrak{r}^4\} \subseteq \mathcal{K}^+$. Checking the λ_r -open list, the qualifying sets are

$$[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], \quad [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], \quad [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \quad \tilde{X}$$

(sets such as $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}]$ are excluded since $\{\mathfrak{r}^1, \mathfrak{r}^4\} \not\subseteq \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}$).

Check the containment $\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$ for each: this uses the identical closure $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ and the identical candidate list as in Step 1, so the containment holds in all four cases by the same reasoning.

Hence $[\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Step 3: The union $[\mathcal{M}^-, \mathcal{M}^+] \cup [\mathcal{B}^-, \mathcal{B}^+]$.

Using the union formula for interval-valued sets,

$$[\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2\}] \cup [\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^4\}] = [\emptyset \cup \emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2\} \cup \{\mathfrak{r}^1, \mathfrak{r}^4\}] = [\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}].$$

Step 4: $[\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Closure. We seek the smallest $\mathcal{IV}$ -closed set with $\emptyset \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq \mathcal{K}^+$. Checking the closed sets, $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ qualifies and is contained in both $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ and $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$, which also qualify. So

$$\mathcal{IVC}([\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}].$$

Relevant λ_r -open supersets. We need $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq \mathcal{K}^+$. Checking the λ_r -open list, the qualifying sets are

$$[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], \quad [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], \quad [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \quad \tilde{X}$$

(sets such as $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}]$ are excluded since $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \not\subseteq \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}$).

Check the containment: As in Steps 1–2, $\mathcal{IVC}([\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$ holds for all four candidates.

Hence $[\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed. Both $[\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$ and $[\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^4\}]$ are $\mathcal{IV}\Lambda$ -rg-closed, and their union $[\emptyset, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is also $\mathcal{IV}\Lambda$ -rg-closed, illustrating Theorem 3.1 (that the union of two $\mathcal{IV}\Lambda$ -rg-closed sets is again $\mathcal{IV}\Lambda$ -rg-closed) with a concrete instance.

Theorem 3.2. In an interval-valued topological space $\mathcal{T}^X$, the intersection of two $\mathcal{IV}\Lambda$ -rg-open sets is $\mathcal{IV}\Lambda$ -rg-open.

Proof.

Let $[\mathcal{M}^-, \mathcal{M}^+]$ and $[\mathcal{B}^-, \mathcal{B}^+]$ be two $\mathcal{IV}\Lambda$ -rg-open sets in $\mathcal{T}^X$. Then their complements $([\mathcal{M}^-, \mathcal{M}^+])^c$ and $([\mathcal{B}^-, \mathcal{B}^+])^c$ are $\mathcal{IV}\Lambda$ -rg-closed sets. By Theorem 3.1, the union of two $\mathcal{IV}\Lambda$ -rg-closed sets is again $\mathcal{IV}\Lambda$ -rg-closed.

Hence $([\mathcal{M}^-, \mathcal{M}^+])^c \cup ([\mathcal{B}^-, \mathcal{B}^+])^c$ is $\mathcal{IV}\Lambda$ -rg-closed.

Using De Morgan's law, $([\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{B}^-, \mathcal{B}^+])^c = ([\mathcal{M}^-, \mathcal{M}^+])^c \cup ([\mathcal{B}^-, \mathcal{B}^+])^c$. Therefore, $([\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{B}^-, \mathcal{B}^+])^c$ is $\mathcal{IV}\Lambda$ -rg-closed. Hence $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{IV}\Lambda$ -rg-open. $\square$

Example 3.6. Consider the interval-valued topological space $\mathcal{T}^X$ from Example 3.1. The $\mathcal{IV}\lambda_r$ -open sets are:

$$\tilde{\emptyset}, [\emptyset, \{\mathfrak{r}^2\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3, \mathfrak{r}^4\}], [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}], [\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X], [\{\mathfrak{r}^1, \mathfrak{r}^3\}, X], \tilde{X}.$$

Let

$$[\mathcal{M}^-, \mathcal{M}^+] = [\{\mathfrak{r}^4\}, \{\mathfrak{r}^4\}] \quad \text{and} \quad [\mathcal{B}^-, \mathcal{B}^+] = [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}].$$

Step 1: $[\mathcal{M}^-, \mathcal{M}^+] = [\{\mathfrak{r}^4\}, \{\mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open.

By definition this holds if and only if the complement

$$([\{\mathfrak{r}^4\}, \{\mathfrak{r}^4\}])^c = [(\{\mathfrak{r}^4\})^c, (\{\mathfrak{r}^4\})^c] = [\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}]$$

is $\mathcal{IV}\Lambda$ -rg-closed.

Closure. We seek the smallest $\mathcal{IV}$ -closed set $[\mathcal{K}^-, \mathcal{K}^+]$ with $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\} \subseteq \mathcal{K}^+$. Checking the closed sets: $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$ fails since $\mathfrak{r}^2 \notin \{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}$; $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ fails since $\mathfrak{r}^3 \notin \{\mathfrak{r}^1, \mathfrak{r}^2\}$; $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ fails since $\mathfrak{r}^2, \mathfrak{r}^3 \notin \{\mathfrak{r}^1\}$. Only $\tilde{X}$ satisfies both containments. Hence

$$\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}]) = \tilde{X}.$$

Relevant λ_r -open supersets. We need $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\} \subseteq \mathcal{K}^+$. Scanning the λ_r -open list, every minus-part other than X itself omits at least one of $\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3$ (e.g. $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ has minus-part $\{\mathfrak{r}^1\}$, missing $\mathfrak{r}^2, \mathfrak{r}^3$; $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$ has minus-part missing $\mathfrak{r}^2$; $[\{\mathfrak{r}^1, \mathfrak{r}^3\}, X]$ has minus-part missing $\mathfrak{r}^2$). So the only qualifying set is $\tilde{X}$.

Check the containment. $\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}]) = \tilde{X} \subseteq \tilde{X}$, which holds trivially.

Hence $[\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}]$ is $\mathcal{IV}\Lambda$ -rg-closed, so $[\{\mathfrak{r}^4\}, \{\mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open.

Step 2: $[\mathcal{B}^-, \mathcal{B}^+] = [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open.

This holds if and only if the complement

$$([\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}])^c = [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$$

is $\mathcal{IV}\Lambda$ -rg-closed.

Closure. Since $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is itself $\mathcal{IV}$ -closed set, its closure is itself:

$$\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}].$$

Relevant λ_r -open supersets. We need $\{\mathfrak{r}^1, \mathfrak{r}^2\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \subseteq \mathcal{K}^+$. Scanning the λ_r -open list, every minus-part except X itself omits $\mathfrak{r}^1$ or $\mathfrak{r}^2$ (e.g. $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ has minus-part $\{\mathfrak{r}^1\}$, missing $\mathfrak{r}^2$; $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$ and $[\{\mathfrak{r}^1, \mathfrak{r}^3\}, X]$ both miss $\mathfrak{r}^2$). So the only qualifying set is $\tilde{X}$.

Check the containment. $\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]) = [\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}] \subseteq \tilde{X}$, which holds trivially.

Hence $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-closed, so $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open.

Step 3: The intersection $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{B}^-, \mathcal{B}^+]$.

Using the intersection formula for interval-valued sets,

$$[\{\mathfrak{r}^4\}, \{\mathfrak{r}^4\}] \cap [\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}] = [\{\mathfrak{r}^4\} \cap \{\mathfrak{r}^3\}, \{\mathfrak{r}^4\} \cap \{\mathfrak{r}^3, \mathfrak{r}^4\}] = [\emptyset, \{\mathfrak{r}^4\}].$$

Step 4: $[\emptyset, \{\mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open.

This holds if and only if the complement

$$([\emptyset, \{\mathfrak{r}^4\}])^c = [(\{\mathfrak{r}^4\})^c, (\emptyset)^c] = [\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}, X]$$

is $\mathcal{IV}\Lambda$ -rg-closed.

Closure. We seek the smallest $\mathcal{IV}$ -closed set with $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\} \subseteq \mathcal{K}^-$ and $X \subseteq \mathcal{K}^+$, i.e. $\mathcal{K}^+ = X$. Among the closed sets, $[\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}, X]$ has plus-part X but minus-part $\{\mathfrak{r}^1, \mathfrak{r}^3, \mathfrak{r}^4\}$, which does not contain $\mathfrak{r}^2$, so it fails. $[\{\mathfrak{r}^1, \mathfrak{r}^2\}, \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\}]$ has plus-part $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^4\} \neq X$, so it fails. Only $\tilde{X}$ satisfies both. Hence

$$\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}, X]) = \tilde{X}.$$

Relevant λ_r -open supersets. We need $\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\} \subseteq \mathcal{K}^-$ and $X \subseteq \mathcal{K}^+$. As in Step 1, the only λ_r -open set with minus-part containing all of $\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3$ is $\tilde{X}$.

Check the containment. $\mathcal{IVC}([\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}, X]) = \tilde{X} \subseteq \tilde{X}$, which holds trivially.

Hence $[\{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}, X]$ is $\mathcal{IV}\Lambda$ -rg-closed, so $[\emptyset, \{\mathfrak{r}^4\}]$ is $\mathcal{IV}\Lambda$ -rg-open. Both $[\{\mathfrak{r}^4\}, \{\mathfrak{r}^4\}]$ and $[\{\mathfrak{r}^3\}, \{\mathfrak{r}^3, \mathfrak{r}^4\}]$ are $\mathcal{IV}\Lambda$ -rg-open, and their intersection $[\emptyset, \{\mathfrak{r}^4\}]$ is also $\mathcal{IV}\Lambda$ -rg-open, illustrating the theorem that the intersection of two $\mathcal{IV}\Lambda$ -rg-open sets is again $\mathcal{IV}\Lambda$ -rg-open.

Remark 3.3. In an interval-valued topological space $\mathcal{T}^X$, the intersection of two $\mathcal{IV}\Lambda$ -rg-closed sets need not be $\mathcal{IV}\Lambda$ -rg-closed.

Example 3.7. Let $X = \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\}$ and consider the interval-valued topology

$$\tau = \{\tilde{\emptyset}, [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}], \tilde{X}\}.$$

$\mathcal{IV}$ -closed sets: Taking complements of the members of τ via $([\mathcal{M}^-, \mathcal{M}^+])^c = [(\mathcal{M}^+)^c, (\mathcal{M}^-)^c]$:

$$\tilde{\emptyset}^c = \tilde{X}, \quad [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]^c = [\{\mathfrak{r}^2, \mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}], \quad \tilde{X}^c = \tilde{\emptyset}.$$

So the $\mathcal{IV}$ -closed sets are $\tilde{\emptyset}$, $[\{\mathfrak{r}^2, \mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$, $\tilde{X}$.

$\mathcal{IV}r$ -open sets: We check each member of τ against $M = \mathcal{IVI}(\mathcal{IVC}(M))$.

- $[\mathcal{M}^-, \mathcal{M}^+] = \tilde{\emptyset}$: $\mathcal{IVC}(\tilde{\emptyset}) = \tilde{\emptyset}$, $\mathcal{IVI}(\tilde{\emptyset}) = \tilde{\emptyset}$. So $\tilde{\emptyset}$ is $\mathcal{IV}r$ -open.
- $[\mathcal{M}^-, \mathcal{M}^+] = \tilde{X}$: $\mathcal{IVC}(\tilde{X}) = \tilde{X}$, $\mathcal{IVI}(\tilde{X}) = \tilde{X}$. So $\tilde{X}$ is $\mathcal{IV}r$ -open.
- $[\mathcal{M}^-, \mathcal{M}^+] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$: among the closed sets, only $\tilde{X}$ contains $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$ (since $[\{\mathfrak{r}^2, \mathfrak{r}^3\}, \{\mathfrak{r}^2, \mathfrak{r}^3\}]$ does not contain $\mathfrak{r}^1$ in its minus-part). So $\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]) = \tilde{X}$, and $\mathcal{IVI}(\tilde{X}) = \tilde{X} \neq [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$. So $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$ is not $\mathcal{IV}r$ -open.

Hence the $\mathcal{IV}r$ -open sets are exactly $\tilde{\emptyset}$ and $\tilde{X}$.

$\mathcal{IV}\Lambda_r$ -sets. For any nonempty $[\mathcal{M}^-, \mathcal{M}^+] \neq \tilde{X}$, the only $\mathcal{IV}r$ -open superset of $[\mathcal{M}^-, \mathcal{M}^+]$ is $\tilde{X}$ (since $\tilde{\emptyset}$ cannot contain a nonempty set), so $\mathcal{IV}r\text{-ker}([\mathcal{M}^-, \mathcal{M}^+]) = \tilde{X} \neq [\mathcal{M}^-, \mathcal{M}^+]$. For $[\mathcal{M}^-, \mathcal{M}^+] = \tilde{\emptyset}$, $\mathcal{IV}r\text{-ker}(\tilde{\emptyset}) = \tilde{\emptyset} \cap \tilde{X} = \tilde{\emptyset}$. For $[\mathcal{M}^-, \mathcal{M}^+] = \tilde{X}$, $\mathcal{IV}r\text{-ker}(\tilde{X}) = \tilde{X}$. Hence the only $\mathcal{IV}\Lambda_r$ -sets are $\tilde{\emptyset}$ and $\tilde{X}$.

$\mathcal{IV}\lambda_r$ -closed sets: By Definition 2.7(2), these are intersections $[S^-, S^+] \cap [\mathcal{K}^-, \mathcal{K}^+]$ of an $\mathcal{IV}\Lambda_r$ -set $[S^-, S^+] \in \{\tilde{\mathcal{O}}, \tilde{X}\}$ with an $\mathcal{IV}$ -closed set $[\mathcal{K}^-, \mathcal{K}^+] \in \{\tilde{\mathcal{O}}, [\{\mathfrak{r}^2, \mathfrak{r}^3\}\{\mathfrak{r}^2, \mathfrak{r}^3\}], \tilde{X}\}$:

$$\tilde{\mathcal{O}} \cap [\mathcal{K}^-, \mathcal{K}^+] = \tilde{\mathcal{O}} \text{ (for any } [\mathcal{K}^-, \mathcal{K}^+]), \quad \tilde{X} \cap \tilde{\mathcal{O}} = \tilde{\mathcal{O}}, \tilde{X} \cap [\{\mathfrak{r}^2, \mathfrak{r}^3\}\{\mathfrak{r}^2, \mathfrak{r}^3\}] = [\{\mathfrak{r}^2, \mathfrak{r}^3\}\{\mathfrak{r}^2, \mathfrak{r}^3\}],$$

$$\tilde{X} \cap \tilde{X} = \tilde{X}.$$

So the $\mathcal{IV}\lambda_r$ -closed sets are $\tilde{\mathcal{O}}, [\{\mathfrak{r}^2, \mathfrak{r}^3\}\{\mathfrak{r}^2, \mathfrak{r}^3\}], \tilde{X}$.

$\mathcal{IV}\lambda_r$ -open sets. Taking complements: $\tilde{\mathcal{O}}^c = \tilde{X}, [\{\mathfrak{r}^2, \mathfrak{r}^3\}\{\mathfrak{r}^2, \mathfrak{r}^3\}]^c = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}],$

$\tilde{X}^c = \tilde{\mathcal{O}}$. So the $\mathcal{IV}\lambda_r$ -open sets are $\tilde{\mathcal{O}}, [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}], \tilde{X}$.

Now take

$$[\mathcal{M}^-, \mathcal{M}^+] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}] \quad \text{and} \quad [\mathcal{B}^-, \mathcal{B}^+] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^3\}].$$

Step 1: $[\mathcal{M}^-, \mathcal{M}^+] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Closure. We seek the smallest $\mathcal{IV}$ -closed set with $\{\mathfrak{r}^1\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2\} \subseteq \mathcal{K}^+$. Checking the closed sets: $[\{\mathfrak{r}^2, \mathfrak{r}^3\}\{\mathfrak{r}^2, \mathfrak{r}^3\}]$ fails since $\mathfrak{r}^1 \notin \{\mathfrak{r}^2, \mathfrak{r}^3\}$. Only $\tilde{X}$ satisfies both containments. So

$$\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]) = \tilde{X}.$$

Relevant λ_r -open supersets. We need $\{\mathfrak{r}^1\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^2\} \subseteq \mathcal{K}^+$. Checking the λ_r -open list: $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$ fails since $\{\mathfrak{r}^1, \mathfrak{r}^2\} \not\subseteq \{\mathfrak{r}^1\}$. Only $\tilde{X}$ qualifies.

Check the containment. $\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]) = \tilde{X} \subseteq \tilde{X}$, which holds trivially.

Hence $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Step 2: $[\mathcal{B}^-, \mathcal{B}^+] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^3\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Closure. We seek the smallest $\mathcal{IV}$ -closed set with $\{\mathfrak{r}^1\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^3\} \subseteq \mathcal{K}^+$. Again $[\{\mathfrak{r}^2, \mathfrak{r}^3\}\{\mathfrak{r}^2, \mathfrak{r}^3\}]$ fails since $\mathfrak{r}^1 \notin \{\mathfrak{r}^2, \mathfrak{r}^3\}$, so only $\tilde{X}$ qualifies. So

$$\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^3\}]) = \tilde{X}.$$

Relevant λ_r -open supersets. We need $\{\mathfrak{r}^1\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1, \mathfrak{r}^3\} \subseteq \mathcal{K}^+$.

Checking the λ_r -open list: $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$ fails since $\{\mathfrak{r}^1, \mathfrak{r}^3\} \not\subseteq \{\mathfrak{r}^1\}$. Only $\tilde{X}$ qualifies.

Check the containment. $\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^3\}]) = \tilde{X} \subseteq \tilde{X}$, which holds trivially.

Hence $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^3\}]$ is $\mathcal{IV}\Lambda$ -rg-closed.

Step 3: The intersection $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{B}^-, \mathcal{B}^+]$.

Using the intersection formula for interval-valued sets,

$$[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}] \cap [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^3\}] = [\{\mathfrak{r}^1\} \cap \{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\} \cap \{\mathfrak{r}^1, \mathfrak{r}^3\}] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}].$$

Step 4: $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$ is not $\mathcal{IV}\Lambda$ -rg-closed.

Closure. We seek the smallest $\mathcal{IV}$ -closed set with $\{\mathfrak{r}^1\} \subseteq \mathcal{K}^-$ and $\{\mathfrak{r}^1\} \subseteq \mathcal{K}^+$. Again $[\{\mathfrak{r}^2, \mathfrak{r}^3\}\{\mathfrak{r}^2, \mathfrak{r}^3\}]$ fails since $\mathfrak{r}^1 \notin \{\mathfrak{r}^2, \mathfrak{r}^3\}$, so only $\tilde{X}$ qualifies. So

$$\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]) = \tilde{X}.$$

A witness λ_r -open superset that breaks the condition. Consider $[\mathcal{K}^-, \mathcal{K}^+] = [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$, which is itself an $\mathcal{IV}\lambda_r$ -open set (it appears in the λ_r -open list above), and it satisfies

$$[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}] \subseteq [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}].$$

For $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$ to be $\mathcal{IV}\Lambda$ -rg-closed, we would need

$$\mathcal{IVC}([\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]) = \tilde{X} \subseteq [\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}],$$

but this fails, since $\tilde{X} = [X, X]$ and $X = \{\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3\} \not\subseteq \{\mathfrak{r}^1\}$.

Since the defining condition of Definition 3.2 fails for this particular $\mathcal{IV}\lambda_r$ -open superset, $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$ is not $\mathcal{IV}\Lambda$ -rg-closed. Both $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^2\}]$ and $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1, \mathfrak{r}^3\}]$ are $\mathcal{IV}\Lambda$ -rg-closed, yet their intersection $[\{\mathfrak{r}^1\}, \{\mathfrak{r}^1\}]$ is not. Hence the intersection of two $\mathcal{IV}\Lambda$ -rg-closed sets need not be $\mathcal{IV}\Lambda$ -rg-closed.

Theorem 3.3. If a subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $\mathcal{T}^X$ is $\mathcal{IV}\Lambda$ -rg-closed, then $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ contains no nonempty $\mathcal{IV}$ -closed set.

Proof.

Let $[\mathcal{B}^-, \mathcal{B}^+]$ be a $\mathcal{IV}$ -closed subset contained in $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$. Clearly $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{B}^-, \mathcal{B}^+]^c$, where $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed and $[\mathcal{B}^-, \mathcal{B}^+]^c$ is a $\mathcal{IV}$ -open subset of $\tilde{X}$.

Thus $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{B}^-, \mathcal{B}^+]^c$, or equivalently $[\mathcal{B}^-, \mathcal{B}^+] \subseteq (\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))^c$. Then $[\mathcal{B}^-, \mathcal{B}^+] \subseteq (\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))^c \cap (\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]) \subseteq (\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))^c \cap \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = \tilde{\emptyset}$. This shows that $[\mathcal{B}^-, \mathcal{B}^+] = \tilde{\emptyset}$. $\square$

Theorem 3.4. A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $\mathcal{T}^X$ is $\mathcal{IV}\Lambda$ -rg-closed if and only if $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ contains no nonempty $\mathcal{IV}\lambda_r$ -closed set.

Proof.

Necessity: Assume that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed. Let $[\mathcal{B}^-, \mathcal{B}^+]$ be a nonempty $\mathcal{IV}\lambda_r$ -closed subset of $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$. Then $[\mathcal{M}^-, \mathcal{M}^+] \subseteq ([\mathcal{B}^-, \mathcal{B}^+])^c$. Since $[\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{IV}\lambda_r$ -closed, $([\mathcal{B}^-, \mathcal{B}^+])^c$ is $\mathcal{IV}\lambda_r$ -open. As $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed, we obtain $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq ([\mathcal{B}^-, \mathcal{B}^+])^c$.

Therefore, $[\mathcal{B}^-, \mathcal{B}^+] \subseteq (\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))^c$. Hence, $[\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap (\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))^c = \tilde{\emptyset}$. Thus $[\mathcal{B}^-, \mathcal{B}^+] = \tilde{\emptyset}$.

Therefore, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ contains no nonempty $\mathcal{IV}\lambda_r$ -closed set.

Sufficiency: Assume that $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ contains no nonempty $\mathcal{IV}\lambda_r$ -closed set. Let $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, where $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\lambda_r$ -open. Suppose that $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \not\subseteq [\mathcal{K}^-, \mathcal{K}^+]$. Then $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{K}^-, \mathcal{K}^+])^c \neq \tilde{\emptyset}$. Since $([\mathcal{K}^-, \mathcal{K}^+])^c$ is $\mathcal{IV}\lambda_r$ -closed, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{K}^-, \mathcal{K}^+])^c$ is a nonempty $\mathcal{IV}\lambda_r$ -closed subset of $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, which is a contradiction. Hence, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$. Therefore, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed. $\square$

Theorem 3.5. In an interval-valued topological space $\mathcal{T}^X$, if $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed and $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$, then $[\mathcal{B}^-, \mathcal{B}^+]$ is also $\mathcal{IV}\Lambda$ -rg-closed.

Proof.

Assume that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$. Then $\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$.

Consequently, $\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) - [\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ - rg -closed, Theorem 3.4 implies that $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ contains no nonempty $\mathcal{IV}\lambda_r$ -closed set. Hence, $\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) - [\mathcal{B}^-, \mathcal{B}^+]$ also contains no nonempty $\mathcal{IV}\lambda_r$ -closed set. Again by Theorem 3.4, $[\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{IV}\Lambda$ - rg -closed. $\square$

Theorem 3.6. *In an interval-valued topological space $\mathcal{T}^X$, if $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\lambda_r$ -open and $\mathcal{IV}\Lambda$ - rg -closed, then $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}$ -closed.*

Proof.

Let $[\mathcal{M}^-, \mathcal{M}^+]$ be both $\mathcal{IV}\lambda_r$ -open and $\mathcal{IV}\Lambda$ - rg -closed. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\lambda_r$ -open, we may take $[\mathcal{K}^-, \mathcal{K}^+] = [\mathcal{M}^-, \mathcal{M}^+]$. As $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ - rg -closed and $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, where $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\lambda_r$ -open, it follows that $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$. Since every set is contained in its closure, $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$. Therefore, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = [\mathcal{M}^-, \mathcal{M}^+]$. Hence $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}$ -closed. $\square$

Theorem 3.7. *For each $p \in \tilde{X}$, either $\{p\}$ is $\mathcal{IV}\lambda_r$ -closed or $\{p\}^c$ is $\mathcal{IV}\Lambda$ - rg -closed.*

Proof.

Assume that $\{p\}$ is not $\mathcal{IV}\lambda_r$ -closed. Then its complement $\{p\}^c$ is not $\mathcal{IV}\lambda_r$ -open. Let $\{p\}^c \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, where $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\lambda_r$ -open. Since $\{p\}^c$ is not $\mathcal{IV}\lambda_r$ -open, the only $\mathcal{IV}\lambda_r$ -open set containing $\{p\}^c$ is $\tilde{X}$ itself. Hence $[\mathcal{K}^-, \mathcal{K}^+] = \tilde{X}$. Also, $\mathcal{IVC}(\{p\}^c) \subseteq \tilde{X}$. Therefore, by the definition of $\mathcal{IV}\Lambda$ - rg -closed sets, $\{p\}^c$ is $\mathcal{IV}\Lambda$ - rg -closed. Hence, for each $p \in \tilde{X}$, either $\{p\}$ is $\mathcal{IV}\lambda_r$ -closed or $\{p\}^c$ is $\mathcal{IV}\Lambda$ - rg -closed. $\square$

Theorem 3.8. *In an interval-valued topological space $\mathcal{T}^X$, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ - rg -open $\iff [\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$ whenever $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\lambda_r$ -closed and $[\mathcal{K}^-, \mathcal{K}^+] \subseteq [\mathcal{M}^-, \mathcal{M}^+]$.*

Proof.

Assume that $[\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$ whenever $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\lambda_r$ -closed and $[\mathcal{K}^-, \mathcal{K}^+] \subseteq [\mathcal{M}^-, \mathcal{M}^+]$. Let $[\mathcal{M}^-, \mathcal{M}^+]^c \subseteq [\mathcal{B}^-, \mathcal{B}^+]$, where $[\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{IV}\lambda_r$ -open.

Hence $[\mathcal{B}^-, \mathcal{B}^+]^c \subseteq [\mathcal{M}^-, \mathcal{M}^+]$. By assumption $[\mathcal{B}^-, \mathcal{B}^+]^c \subseteq \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$, which implies that $(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))^c \subseteq [\mathcal{B}^-, \mathcal{B}^+]$, so $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c) \subseteq [\mathcal{B}^-, \mathcal{B}^+]$.

Hence $[\mathcal{M}^-, \mathcal{M}^+]^c$ is $\mathcal{IV}\Lambda$ - rg -closed, that is, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ - rg -open.

Conversely, let $[\mathcal{M}^-, \mathcal{M}^+]$ be $\mathcal{IV}\Lambda$ - rg -open. Then $[\mathcal{M}^-, \mathcal{M}^+]^c$ is $\mathcal{IV}\Lambda$ - rg -closed. Also let $[\mathcal{K}^-, \mathcal{K}^+]$ be an $\mathcal{IV}\lambda_r$ -closed set contained in $[\mathcal{M}^-, \mathcal{M}^+]$. Then $[\mathcal{K}^-, \mathcal{K}^+]^c$ is $\mathcal{IV}\lambda_r$ -open. Therefore, whenever $[\mathcal{M}^-, \mathcal{M}^+]^c \subseteq [\mathcal{K}^-, \mathcal{K}^+]^c$, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c) \subseteq [\mathcal{K}^-, \mathcal{K}^+]^c$. This implies that $[\mathcal{K}^-, \mathcal{K}^+] \subseteq (\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c))^c = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$. Thus $[\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$. $\square$

Theorem 3.9. *In an interval-valued topological space $\mathcal{T}^X$, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ - rg -open $\iff [\mathcal{B}^-, \mathcal{B}^+] = \tilde{X}$ whenever $[\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{IV}\lambda_r$ -open and $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \cup [\mathcal{M}^-, \mathcal{M}^+]^c \subseteq [\mathcal{B}^-, \mathcal{B}^+]$.*

Proof.

Let $[\mathcal{M}^-, \mathcal{M}^+]$ be $\mathcal{IV}\Lambda$ -rg-open, let $[\mathcal{B}^-, \mathcal{B}^+]$ be $\mathcal{IV}\lambda_r$ -open, and suppose $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \cup [\mathcal{M}^-, \mathcal{M}^+]^c \subseteq [\mathcal{B}^-, \mathcal{B}^+]$. Then $[\mathcal{B}^-, \mathcal{B}^+]^c \subseteq (\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))^c \cap ([\mathcal{M}^-, \mathcal{M}^+]^c)^c = (\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))^c - [\mathcal{M}^-, \mathcal{M}^+]^c = \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c) - [\mathcal{M}^-, \mathcal{M}^+]^c$. Since $[\mathcal{M}^-, \mathcal{M}^+]^c$ is $\mathcal{IV}\Lambda$ -rg-closed and $[\mathcal{B}^-, \mathcal{B}^+]^c$ is $\mathcal{IV}\lambda_r$ -closed, by Theorem 3.4 it follows that $[\mathcal{B}^-, \mathcal{B}^+]^c = \emptyset$. Therefore $[\mathcal{B}^-, \mathcal{B}^+] = \tilde{X}$.

Conversely, suppose that $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\lambda_r$ -closed and $[\mathcal{K}^-, \mathcal{K}^+] \subseteq [\mathcal{M}^-, \mathcal{M}^+]$. Then $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \cup [\mathcal{M}^-, \mathcal{M}^+]^c \subseteq \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \cup [\mathcal{K}^-, \mathcal{K}^+]^c$. It follows that $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \cup [\mathcal{K}^-, \mathcal{K}^+]^c = \tilde{X}$, and hence $[\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$. Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open. $\square$

Theorem 3.10. *In an interval-valued topological space $\mathcal{T}^X$, if $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{B}^-, \mathcal{B}^+] \subseteq [\mathcal{M}^-, \mathcal{M}^+]$ and $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open, then $[\mathcal{B}^-, \mathcal{B}^+]$ is also $\mathcal{IV}\Lambda$ -rg-open.*

Proof.

Assume that $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{B}^-, \mathcal{B}^+] \subseteq [\mathcal{M}^-, \mathcal{M}^+]$ and that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open.

Taking complements, we obtain $([\mathcal{M}^-, \mathcal{M}^+]^c) \subseteq ([\mathcal{B}^-, \mathcal{B}^+]^c) \subseteq (\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))^c$. Since $(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))^c = \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c)$, we have $([\mathcal{M}^-, \mathcal{M}^+]^c) \subseteq ([\mathcal{B}^-, \mathcal{B}^+]^c) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]^c)$. Also, $([\mathcal{M}^-, \mathcal{M}^+]^c)$ is $\mathcal{IV}\Lambda$ -rg-closed, since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open. By Theorem 3.5, $([\mathcal{B}^-, \mathcal{B}^+]^c)$ is $\mathcal{IV}\Lambda$ -rg-closed. Therefore, $[\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{IV}\Lambda$ -rg-open. $\square$

Theorem 3.11. *In a interval-valued topological space $\mathcal{T}^X$, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed if and only if $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open.*

Proof.

Necessity. Assume that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed. Let $[\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, where $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\lambda_r$ -closed. By Theorem 3.4, $[\mathcal{K}^-, \mathcal{K}^+] = \emptyset$. Hence, $[\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+])$.

Therefore, by Theorem 3.8, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open.

Sufficiency. Assume that $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open. Let $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{B}^-, \mathcal{B}^+]$, where $[\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{IV}\lambda_r$ -open. Then $([\mathcal{B}^-, \mathcal{B}^+]^c)$ is $\mathcal{IV}\lambda_r$ -closed. Now, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+]^c) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{M}^-, \mathcal{M}^+]^c)$. Hence, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+]^c) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$.

Since $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+]^c)$ is $\mathcal{IV}\lambda_r$ -closed and $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open, by Theorem 3.8,

$$\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+]^c) \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]) = \emptyset.$$

Therefore, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+]^c) = \emptyset$. This implies that $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{B}^-, \mathcal{B}^+]$. Hence $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed. $\square$

Theorem 3.12. *In an interval-valued topological space $\mathcal{T}^X$, the following statements are equivalent:*

1. $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-closed.
2. $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ contains no nonempty $\mathcal{IV}\lambda_r$ -closed set.
3. $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ -rg-open.

Proof.

(1) $\Rightarrow$ (2).

Assume $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\Lambda$ - rg -closed. Let $[\mathcal{B}^-, \mathcal{B}^+]$ be a nonempty $\mathcal{TV}\lambda_r$ -closed subset of $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$. Then $[\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])$ and $[\mathcal{B}^-, \mathcal{B}^+] \cap [\mathcal{M}^-, \mathcal{M}^+] = \tilde{\emptyset}$, so $[\mathcal{M}^-, \mathcal{M}^+] \subseteq ([\mathcal{B}^-, \mathcal{B}^+])^c$. Since $[\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{TV}\lambda_r$ -closed, its complement $([\mathcal{B}^-, \mathcal{B}^+])^c$ is $\mathcal{TV}\lambda_r$ -open. As $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\Lambda$ - rg -closed, applying Definition 3.3 to this $\mathcal{TV}\lambda_r$ -open superset gives $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq ([\mathcal{B}^-, \mathcal{B}^+])^c$.

Consequently,

$$[\mathcal{B}^-, \mathcal{B}^+] \subseteq (\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]))^c, \text{ and therefore } [\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \cap (\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]))^c = \tilde{\emptyset}.$$

Thus $[\mathcal{B}^-, \mathcal{B}^+] = \tilde{\emptyset}$, contradicting the assumption that it is nonempty. Hence no nonempty $\mathcal{TV}\lambda_r$ -closed set is contained in $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$.

(2) $\Rightarrow$ (1).

Assume $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ contains no nonempty $\mathcal{TV}\lambda_r$ -closed set. Let $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, where $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{TV}\lambda_r$ -open. Suppose, for contradiction, that $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \not\subseteq [\mathcal{K}^-, \mathcal{K}^+]$. Then $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{K}^-, \mathcal{K}^+])^c \neq \tilde{\emptyset}$. Since $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{TV}\lambda_r$ -open, its complement $([\mathcal{K}^-, \mathcal{K}^+])^c$ is $\mathcal{TV}\lambda_r$ -closed, and hence so is the intersection $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{K}^-, \mathcal{K}^+])^c$. Moreover, since $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, we have $[\mathcal{M}^-, \mathcal{M}^+] \cap ([\mathcal{K}^-, \mathcal{K}^+])^c = \tilde{\emptyset}$, so this nonempty $\mathcal{TV}\lambda_r$ -closed set is disjoint from $[\mathcal{M}^-, \mathcal{M}^+]$ and contained in $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])$; that is, it is a nonempty $\mathcal{TV}\lambda_r$ -closed subset of $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, contradicting the hypothesis. Hence $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$. Since $[\mathcal{K}^-, \mathcal{K}^+]$ was an arbitrary $\mathcal{TV}\lambda_r$ -open superset of $[\mathcal{M}^-, \mathcal{M}^+]$, it follows that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\Lambda$ - rg -closed.

This establishes $(1) \Leftrightarrow (2)$.

(1) $\Rightarrow$ (3).

Assume $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\Lambda$ - rg -closed. Let $[\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, where $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{TV}\lambda_r$ -closed. By the already-established equivalence $(1) \Leftrightarrow (2)$, we have $[\mathcal{K}^-, \mathcal{K}^+] = \tilde{\emptyset}$. In particular, $[\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{TV}I(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+])$, since $\tilde{\emptyset}$ is contained in every set, including the interior of any set. As $[\mathcal{K}^-, \mathcal{K}^+]$ was an arbitrary $\mathcal{TV}\lambda_r$ -closed set contained in $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, it follows by Theorem 3.8 (applied to the set $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ in place of $[\mathcal{M}^-, \mathcal{M}^+]$) that $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\Lambda$ - rg -open.

(3) $\Rightarrow$ (1).

Assume $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\Lambda$ - rg -open. Let $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{B}^-, \mathcal{B}^+]$, where $[\mathcal{B}^-, \mathcal{B}^+]$ is $\mathcal{TV}\lambda_r$ -open. Then $([\mathcal{B}^-, \mathcal{B}^+])^c$ is $\mathcal{TV}\lambda_r$ -closed.

Observe that $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+])^c \subseteq \mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{M}^-, \mathcal{M}^+])^c = \mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, using $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{B}^-, \mathcal{B}^+]$, hence $([\mathcal{B}^-, \mathcal{B}^+])^c \subseteq ([\mathcal{M}^-, \mathcal{M}^+])^c$. Since $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+])^c$ is $\mathcal{TV}\lambda_r$ -closed, and it is contained in $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, which is $\mathcal{TV}\Lambda$ - rg -open by hypothesis, Theorem 3.8 gives $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+])^c \subseteq \mathcal{TV}I(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+])$. Since $\mathcal{TV}I(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, and $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+])^c$ is itself $\mathcal{TV}\lambda_r$ -closed and contained in $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, the equivalence $(1) \Leftrightarrow (2)$ established

above forces this set to equal $\tilde{\mathcal{O}}$.

Hence $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \cap ([\mathcal{B}^-, \mathcal{B}^+])^c = \tilde{\mathcal{O}}$, which gives $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{B}^-, \mathcal{B}^+]$. Since $[\mathcal{B}^-, \mathcal{B}^+]$ was an arbitrary $\mathcal{IV}\lambda_r$ -open superset of $[\mathcal{M}^-, \mathcal{M}^+]$, it follows that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\Lambda$ - rg -closed.

This establishes $(1) \Leftrightarrow (3)$.

Combining both equivalences, statements (1), (2), and (3) are pairwise equivalent. $\square$

Conclusion

In this paper, we introduced and studied the concept of interval-valued Λ - rg -closed sets and their complements, interval-valued Λ - rg -open sets, in interval-valued topological spaces. We established that every interval-valued closed (resp. open) set is interval-valued Λ - rg -closed (resp. Λ - rg -open), and we constructed examples showing that the converse fails in general, confirming that these new classes are strictly weaker than the classical notions. We also showed that no inclusion relation holds between interval-valued Λ - rg -closed sets and interval-valued λ_r -closed sets. Furthermore, we proved that the family of interval-valued Λ - rg -closed sets is closed under finite unions, while the corresponding Λ - rg -open sets are closed under finite intersections; an example was given to show that the intersection of two interval-valued Λ - rg -closed sets need not be interval-valued Λ - rg -closed. Several characterizations of interval-valued Λ - rg -closed and interval-valued Λ - rg -open sets were obtained in terms of the interval-valued closure and interior operators and the set $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) - [\mathcal{M}^-, \mathcal{M}^+]$, culminating in an equivalence theorem unifying these characterizations. These results extend the theory of generalized closed and open sets to the setting of interval-valued topology and lay the groundwork for further investigation, such as the study of interval-valued Λ - rg -continuous functions, interval-valued Λ - rg -separation axioms, and related covering and connectedness properties in future work.

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