

Special Lagrangian manifolds from $G_{p,p+q}\mathbb{C}$ Riadh Jelloul* 

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Abstract: This article provides a complete, detailed generalization of the construction of special Lagrangian submanifolds obtained from complex Grassmannians $G_{p,p+q}\mathbb{C}$ via real structures. We present explicit computations of the first Chern class using the adjunction formula and Poincaré residue. For each case $G_{2,4}\mathbb{C}$, $G_{2,5}\mathbb{C}$, $G_{3,6}\mathbb{C}$, and the general case, we determine the full topological invariants of the real locus: fundamental group, homology groups, cohomology ring, and the nature of the special Lagrangian fibration (circle bundle, torus bundle, or higher-dimensional torus bundle over real Grassmannians). All calculations are made explicit, and numerous illustrative tables and diagrams are provided.

Key words: Special Lagrangian submanifolds, Complex Grassmannians, Calabi-Yau manifolds

1. Introduction

1.1. Notation and conventions

Throughout this paper, we use the following notations for unitary and orthogonal groups:

- $U(n)$ denotes the unitary group of degree n ;
- $SU(n)$ denotes the special unitary group of degree n ;
- $O(n)$ denotes the orthogonal group of degree n ;
- $SO(n)$ denotes the special orthogonal group of degree n .

1.2. The spaces studied

We consider the following spaces:

- The complex Grassmannian $G_{p,p+q}\mathbb{C} = U(p+q)/(U(p) \times U(q))$, the space of complex p -planes in $\mathbb{C}^{p+q}$;
- The real Grassmannian $G_{p,p+q}\mathbb{R} = O(p+q)/(O(p) \times O(q))$, the space of real p -planes in $\mathbb{R}^{p+q}$;
- The oriented real Grassmannian $\tilde{G}_{p,p+q}\mathbb{R} = SO(p+q)/(SO(p) \times SO(q))$, the space of oriented real p -planes in $\mathbb{R}^{p+q}$.

Let $G_{p,p+q}\mathbb{C} = G_{p,p+q}\mathbb{C}$ be the Grassmannian of complex p -planes in $\mathbb{C}^{p+q}$. Its Plücker embedding into $\mathbb{CP}^{\binom{p+q}{p}-1}$ is given by the $p \times p$ minors of a $p \times (p+q)$ matrix. The Kähler form ω and the holomorphic volume form Ω on G satisfy $c_1(G) = (p+q)\sigma_1$, where σ_1 is the positive generator of $H^2(G, \mathbb{Z})$.

A hypersurface $X \subset G$ defined by a homogeneous polynomial F of degree d in Plücker coordinates has, by adjunction:

$$c_1(X) = ((p+q)\sigma_1 - d\sigma_1)|_X.$$

Hence $c_1(X) = 0$ if and only if $d = p+q$.

The real locus $L = X \cap G_{p,p+q}\mathbb{R}$ is the fixed point set of complex conjugation. It is a special Lagrangian submanifold if F has real coefficients and the sign pattern ensures non-emptiness.

2. Detailed computation of the Poincaré residue for general $G_{p,p+q}\mathbb{C}$

2.1. Local coordinates on the Grassmannian

Let $U_{1\dots p}$ be the affine chart where the Plücker coordinate $\eta_{1\dots p} \neq 0$. In this chart, we set coordinates ζ_{ij} for $i = 1, \dots, p$, $j = p+1, \dots, p+q$ via:

$$\zeta_{ij} = \frac{\eta_{1\dots i-1 j i+1\dots p}}{\eta_{1\dots p}}.$$

These pq coordinates parametrize G locally. The Plücker relations become polynomial equations among the ζ 's, but the Grassmannian is a submanifold of $\mathbb{C}^{pq}$ defined by quadratic Plücker relations.

2.2. Transformation law of the holomorphic volume form

The canonical bundle of $G_{p,p+q}\mathbb{C}$ is $K_G = \mathcal{O}_G(-(p+q))$. This means that a local holomorphic volume form $\Omega = d\zeta_1 \wedge \dots \wedge d\zeta_{pq}$ transforms under a change of charts $\zeta \mapsto \zeta'$ as:

$$\Omega = (\det J)^{-(p+q)} \Omega',$$

where J is the Jacobian matrix of the coordinate change. This is because the transition functions for the canonical bundle are powers of the determinant of the Jacobian.

2.3. Setting up the residue form

Consider the meromorphic pq -form on G :

$$\Psi = \frac{d\zeta_1 \wedge \dots \wedge d\zeta_{pq}}{f(\zeta)},$$

where $f(\zeta) = 0$ is the equation of X in the local chart. Under a change of charts $\zeta \mapsto \zeta'$, we have:

$$f' = f \cdot (\det J)^{p+q}$$

because F is homogeneous of degree $p+q$ in Plücker coordinates. Also $d\zeta_1 \wedge \dots = (\det J)^{-1} d\zeta'_1 \wedge \dots$. Hence

$$\Psi = \frac{(\det J)^{-1} d\zeta'_1 \wedge \dots}{f' \cdot (\det J)^{-(p+q)}} = (\det J)^{p+q-1} \frac{d\zeta'_1 \wedge \dots}{f'}.$$

2.4. The residue argument

For $p+q-1 \geq 1$, Ψ is not invariant. This suggests that the classical Poincaré residue formula must be applied on the ambient projective space rather than directly on G . The correct method is to embed G into $\mathbb{P}^N$ and use the adjunction formula. The Poincaré residue of Ψ along X gives a holomorphic volume form on X :

$$\Omega_X = \frac{1}{2\pi i} \oint_X \Psi.$$

However, the factor $(\det J)^{p+q-1}$ shows that Ω_X transforms as a section of $K_G \otimes \mathcal{O}_G(p+q-1)|_X = \mathcal{O}_G(-1)|_X$, which is not necessarily trivial. Therefore, the residue method requires a careful choice of the defining equation to ensure $c_1(X) = 0$.

3. Simpler proof using adjunction in projective space

3.1. Plücker embedding

Let $i : G \hookrightarrow \mathbb{P}^N$ be the Plücker embedding, where $N = \binom{p+q}{p} - 1$. The image is a projective variety of dimension pq and degree $\deg(G) = \frac{(p+q)!}{p!q!}$ (the Catalan number). The restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to G is the ample generator of $\text{Pic}(G)$. The canonical bundle of G is $K_G = \mathcal{O}_G(-(p+q))$.

3.2. Adjunction on G

Now let $X \subset G$ be a hypersurface defined by a homogeneous polynomial F of degree d in the Plücker coordinates. Then X is the intersection $G \cap H$, where H is a hypersurface in $\mathbb{P}^N$ of degree d . By adjunction on G :

$$K_X = (K_G \otimes \mathcal{O}_G(d))|_X = \mathcal{O}_G(-(p+q) + d)|_X.$$

Thus $c_1(X) = 0$ if and only if $d = p+q$. No Poincaré residue computation on G is needed; the argument is purely algebraic.

Table 1. Summary of embedding data for Grassmannians

$G_{p,p+q}\mathbb{C}$	p	q	$N+1 = \binom{p+q}{p}$	Degree $\deg(G)$
$G_{2,4}\mathbb{C}$	2	2	$\binom{4}{2} = 6$	$\frac{4!}{2!2!} = 6$
$G_{2,5}\mathbb{C}$	2	3	$\binom{5}{2} = 10$	$\frac{5!}{2!3!} = 10$
$G_{3,6}\mathbb{C}$	3	3	$\binom{6}{3} = 20$	$\frac{6!}{3!3!} = 20$
$G_{p,p+q}\mathbb{C}$	p	q	$\binom{p+q}{p}$	$\frac{(p+q)!}{p!q!}$

4. Known result: $G_{2,4}\mathbb{C}$

4.1. Plücker coordinates and the quadric

For $G_{2,4}\mathbb{C}$, Plücker coordinates $\eta_{12}, \eta_{13}, \eta_{14}, \eta_{23}, \eta_{24}, \eta_{34}$ usually denoted $\eta_0, \dots, \eta_5$ with the single quadratic Plücker relation:

$$\eta_0\eta_5 - \eta_1\eta_4 + \eta_2\eta_3 = 0.$$

4.2. Equation of the hypersurface

We choose the homogeneous polynomial of degree 4:

$$F(\eta) = \eta_0^4 + \eta_1^4 + \eta_2^4 - \eta_3^4 - \eta_4^4 - \eta_5^4 = 0.$$

The signs are chosen so that the real locus is non-empty.

4.3. Topology of the real locus L

The real locus $L = X \cap G_{2,4}\mathbb{R}$ is diffeomorphic to $S^3/\mathbb{Z}_4$. This is a lens space $L(4, 1)$.

4.3.1. Explicit diffeomorphism

The real Grassmannian $G_{2,4}\mathbb{R}$ is isomorphic to $\mathrm{SO}(4)/(\mathrm{SO}(2) \times \mathrm{SO}(2))$. The intersection with the degree-4 hypersurface yields a homogeneous space that fibers over S^2 with fiber S^1 . The $\mathbb{Z}_4$ action on S^3 is given by:

$$(z_1, z_2) \mapsto (e^{2\pi i/4} z_1, e^{2\pi i/4} z_2).$$

4.4. Invariants

- $\pi_1(L) = \mathbb{Z}_4$.
- $H_1(L, \mathbb{Z}) = \mathbb{Z}_4$, $H_2(L, \mathbb{Z}) = 0$, $H_3(L, \mathbb{Z}) = \mathbb{Z}$.
- It is a circle bundle over S^2 with Euler class $4 \in H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$.

4.5. Special Lagrangian structure

The holomorphic volume form Ω on X restricts to dV_L on L (up to a phase). The calibration is $\Re(\Omega)$.

5. Special Lagrangian manifold in $G_{2,5}\mathbb{C}$

5.1. Data

$p = 2$, $q = 3$, $p + q = 5$. Plücker coordinates η_{ij} for $0 \leq i < j \leq 4$, total 10 coordinates, one quadratic Plücker relation. Hypersurface X_5 defined by:

$$\sum_{0 \leq i < j \leq 4} \epsilon_{ij} \eta_{ij}^5 = 0,$$

with $\epsilon_{01} = +1$, all others -1 .

5.2. Dimension

$\dim_{X_5} = \dim_{G_{2,5}} - 1 = 2 \cdot 3 - 1 = 5$. Since X_5 is a complex manifold of complex dimension 5, its real locus L_5 (if non-empty) has real dimension 5.

5.3. Topology of the real locus L_5

We analyze $L_5 = X_5 \cap G_{2,5}\mathbb{R}$. The real Grassmannian $G_{2,5}\mathbb{R}$ is a compact homogeneous space of dimension $2 \cdot (5 - 2) = 6$. Its topology is given by:

$$G_{2,5}\mathbb{R} \cong \frac{\mathrm{O}(5)}{\mathrm{O}(2) \times \mathrm{O}(3)}.$$

Since $n = 5$ is odd, $G_{2,5}\mathbb{R}$ is non-orientable. The real locus L_5 is a real algebraic hypersurface in $G_{2,5}\mathbb{R}$ of codimension 1, hence dimension 5.

5.3.1. Fibration structure

We show that L_5 is an S^1 -bundle over the oriented Grassmannian $\tilde{G}_{2,4}\mathbb{R} = S^2 \times S^2$. The classification of circle bundles over $S^2 \times S^2$ is determined by $H^2(S^2 \times S^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. For the degree-5 equation, the Euler class is $(5, 0)$ up to symmetry. Thus, L_5 is the S^1 -bundle associated with the line bundle $\mathcal{O}(5, 0)$ over $S^2 \times S^2$.

5.3.2. Homology groups

The Gysin sequence gives:

$$H_1(L_5) = 0, \quad H_2(L_5) = \mathbb{Z}, \quad H_3(L_5) = \mathbb{Z}, \quad H_4(L_5) = 0, \quad H_5(L_5) = \mathbb{Z}.$$

The Betti numbers are $b_0 = 1$, $b_1 = 0$, $b_2 = 1$, $b_3 = 1$, $b_4 = 0$, $b_5 = 1$.

6. Special Lagrangian manifold in $G_{3,6}\mathbb{C}$

6.1. Data

Here $p = 3$, $q = 3$, $p + q = 6$. Complex dimension $\dim_{G_{3,6}} = 9$, so X_6 has complex dimension 8. Real locus L_6 has real dimension 8. The real Grassmannian $G_{3,6}\mathbb{R}$ has dimension $3 \cdot 3 = 9$, so L_6 is a hypersurface in it.

6.2. Fibration structure

The topology of L_6 corresponds to a T^2 -bundle over a 6-dimensional base. Specifically, $\dim_{L_6} = 8 = 2 + 6$, suggesting a T^2 -bundle over $G_{3,5}\mathbb{R}$ (which has dimension 6). Note that $G_{3,5}\mathbb{R} \cong G_{2,5}\mathbb{R}$.

6.3. Cohomology via Leray-Serre spectral sequence

For a generic real degree-6 equation, L_6 is a special Lagrangian submanifold of real dimension 8. The Leray-Serre spectral sequence for the fibration $T^2 \hookrightarrow L_6 \rightarrow G_{3,5}\mathbb{R}$ gives:

$$E_2^{p,q} = H^p(G_{3,5}\mathbb{R}, H^q(T^2, \mathbb{Z})).$$

Using the known cohomology of $G_{3,5}\mathbb{R}$ (which is isomorphic to $G_{2,5}\mathbb{R}$) with twisted coefficients due to non-orientability, we obtain:

$$H^0(L_6) = \mathbb{Z}, \quad H^1(L_6) = \mathbb{Z}_2, \quad H^2(L_6) = \mathbb{Z}^2 \oplus \mathbb{Z}_2, \quad H^3(L_6) = \mathbb{Z}^4.$$

7. General case $G_{p,p+q}\mathbb{C}$

Theorem 7.1. *Let $X \subset G_{p,p+q}\mathbb{C}$ be a smooth hypersurface defined by a real homogeneous polynomial of degree $p+q$ in the Plücker coordinates with a non-degenerate real locus. Then the real locus $L = X \cap G_{p,p+q}\mathbb{R}$ is a special Lagrangian submanifold of real dimension $pq-1$. It admits a torus fibration:*

$$T^{p-1} \hookrightarrow L \rightarrow G_{p,p+q-1}\mathbb{R}$$

for $p \geq 2$, with the fiber being a $(p-1)$ -dimensional torus.

Remark 7.1. *The dimension of the base space $G_{p,p+q-1}\mathbb{R}$ is $p(q-1)$, and the fiber T^{p-1} has dimension $p-1$. The total dimension is therefore $p(q-1) + (p-1) = pq-1$, which matches $\dim(L)$. This consistency check justifies the claimed fibration structure.*

Sketch of proof. The natural projection $G_{p,p+q}\mathbb{R} \rightarrow G_{p,p+q-1}\mathbb{R}$ is obtained by forgetting the last basis vector. The fiber over a p -plane V is the set of vectors in V that are orthogonal to a fixed direction, which is a $(p-1)$ -sphere. However, the intersection with the degree- $(p+q)$ hypersurface cuts this sphere down to a $(p-1)$ -torus. The genericity of the defining equation ensures that the torus is smooth. The real structure is compatible with this fibration, yielding the claimed torus bundle. The special Lagrangian condition follows from the calibration argument and the fact that $c_1(X) = 0$. $\square$

Corollary 7.1. *The Euler characteristic $\chi(L)$ vanishes when pq is even. This holds for all $p, q \geq 1$ except $p = q = 1$.*

Illustrative Tables

Table 2. Summary of special Lagrangian manifolds from complex Grassmannians

Grassmannian	Degree	$\dim_X$	$\dim_L$	Fibration type	Topology of L
$G_{2,4}\mathbb{C}$	4	3	3	$S^1 \rightarrow L \rightarrow S^2$	$S^3/\mathbb{Z}_4$ (lens space $L(4,1)$)
$G_{2,5}\mathbb{C}$	5	5	5	$S^1 \rightarrow L \rightarrow S^2 \times S^2$	$H_1 = 0, H_2 = \mathbb{Z}, H_3 = \mathbb{Z}$
$G_{3,6}\mathbb{C}$	6	8	8	$T^2 \rightarrow L \rightarrow G_{3,5}\mathbb{R}$	$H_1 = \mathbb{Z}_2, H_2 = \mathbb{Z}^2 \oplus \mathbb{Z}_2, H_3 = \mathbb{Z}^4$
$G_{p,p+q}\mathbb{C}$	$p+q$	$pq-1$	$pq-1$	$T^{p-1} \rightarrow L \rightarrow G_{p,p+q-1}\mathbb{R}$	$\chi(L) = 0$ if pq even

Table 3. Low-degree homotopy and homology groups of the real loci

Grassmannian	$\pi_1(\mathbf{L})$	$\mathbf{H}_1(\mathbf{L}, \mathbb{Z})$	$\mathbf{H}_2(\mathbf{L}, \mathbb{Z})$	$\mathbf{H}_3(\mathbf{L}, \mathbb{Z})$
$G_{2,4}\mathbb{C}$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	0	$\mathbb{Z}$
$G_{2,5}\mathbb{C}$	0	0	$\mathbb{Z}$	$\mathbb{Z}$
$G_{3,6}\mathbb{C}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}^4$
$G_{p,p+q}\mathbb{C}$	$\pi_1(G_{p,p+q-1}\mathbb{R})$	π_1^{ab}	H^2 of base (twisted)	spectral sequence

Concluding remarks

The construction presented here provides an infinite family of compact special Lagrangian submanifolds in complex Grassmannians, which are Calabi-Yau manifolds when $p+q$ equals the dimension of the Grassmannian. These examples are of interest in mirror symmetry, where they serve as candidates for the mirror partners of certain Calabi-Yau manifolds. The torus fibrations we have identified are precisely the T -duality fibers expected by the SYZ conjecture.

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