



CCM-spaces and fixed point results of R-contractive mappings

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Abstract: In this paper, we obtain existence results in CCM (Complete Cone Metric) – Spaces and fixed point theorem for R-contractive mappings. These results generalize and extend several well-known results existing in the literature.

Key words: Cone metric space, contractive condition, fixed point, normal cone, R-contraction.

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1. Introduction and Preliminaries

Banach contraction principle is the first and foremost theorem in the fixed point theory. Huang and Zhang [3] generalized the metric space into cone metric space, where the set of real numbers is replaced by an ordered Banach space and obtained some fixed point results in CMS (Cone Metric Space). Later on many Mathematicians have been working in this area of results, they extended the results in different ways with different types of contractive conditions (see for e.g. [1,2, 5-17]). Beiranvand, Moradi., et.al.[2] introduced the class of T-contraction and T-contractive functions, extending the Banach contraction principle and Edelstein's fixed point theorems. S.Moradi [6] introduced the T-Kannan contractive mapping. The main aim of this paper is to analyze the existence of fixed point theorems of R-contractive condition. Our result is a generalization results of [5].

The following are needed to obtain our main result which are due to [3,5].

1.1. Sections and subsections

Definition 1.1. Let S be a real Banach space and Q a subset of S . Q is called a cone if and only if:

- (a). Q is closed, non empty, and $Q \neq 0$.
- (b). $\alpha, \beta \in R, \alpha, \beta \geq 0, x, y \in Q$ implies $\alpha x + \beta y \in Q$.
- (c). $x \in Q$ and $-x \in Q$ implies $x = 0$.

Definition 1.2. Given a cone $Q \subset S$, we define a partial ordering $\leq$ with respect to Q by: $x \leq y$ iff $y - x \in Q$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{interior of } Q$. The cone Q is called normal if there is a number $L > 0$ such that for all $x, y \in S, 0 \leq x \leq y$ implies $\|x\| \leq L\|y\|$. In this case the least positive number L is called the normal constant of Q .

Definition 1.3. Let X be a non-empty set. Suppose the $\rho : X \longrightarrow X$ satisfies the following: (a). $\rho(x, y) > 0$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.

- (b). $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$.

(c). $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, for all $x, y, z \in X$. Then ρ is called a cone metric on X , and (X, ρ) is called a CMS(Cone Metric Space).

Definition 1.4. Let (X, ρ) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $d \in S$, $d \gg 0$ there is N such that for all $n > N$, $\rho(x_n, x) \ll d$, then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. We denote this $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 1.5. Let (X, ρ) be a CMS(Cone Metric Space). Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $d \in S$, $d \gg 0$ there is N such that for all $n, m > N$, $\rho(x_n, x_m) \ll d$, then $\{x_n\}$ is said to be Cauchy sequence in X .

Definition 1.6. Let (X, ρ) be a CMS(Cone Metric Space). If every Cauchy sequence is convergent, then X is called a CCMS(Complete Cone Metric Space).

Definition 1.7. Let (X, ρ) be a CM (Cone Metric)- space. Q is normal cone with normal constant L , and a mapping $A : X \rightarrow X$ is said to be continuous if $\lim_{n \rightarrow \infty} x_n = x$ implies that $\lim_{n \rightarrow \infty} T(x_n) = T(x)$, for all x_n in X .

Definition 1.8. Let (X, ρ) be a CM (Cone Metric)- space, Q is normal cone with normal constant L , and a mapping $A : X \rightarrow X$ is said to be sub sequentially convergent if we have for every sequence x_n , if $\{Ax_n\}$ is convergence then $\{x_n\}$ is also convergence sub sequence.

Proposition 1.1. Let (X, ρ) be a CM (Cone Metric)- space, then every function $A : X \rightarrow X$ is sub sequentially convergent and every continuous function $A : X \rightarrow X$ is sub sequentially convergent.

We introduced R -(Reich)-Contraction.

Definition 1.9. Let (X, ρ) be a CCM(Complete Cone Metric)- space.. Let “ Q ” be a normal cone with normal constant “ L ” and $A, B : X \rightarrow X$ be mapping. A mapping B is said to be R -Contraction if there exists non negative numbers a_1, a_2, a_3 satisfying $a_1 + a_2 + a_3 < 1$ such that for each $x, y \in X$, $\rho(ABx, ABx) \leq a_1\rho(Ax, ABx) + a_2\rho(Ay, ABx) + a_3\rho(Ax, Ay)$.

2. Main Result

Theorem 2.1. Let (X, ρ) be a CCM-space and Q be a normal cone with normal constant L . Suppose that a mapping $A : X \rightarrow X$ be one -to -one continuous function and $B : X \rightarrow X$ a R -Contraction. Then

$$\rho(BAx, ABx) \leq a\rho(Ax, Ay) + b[\rho(Ax, ABx) + \rho(Ay, ABx)].$$

Then (i). For every $x_0 \in X$, $\lim_{n \rightarrow \infty} \rho(AB^n x_0, AB^{n+1} x_0) = 0$.

(ii). There is $q \in X$ such that $\lim_{n \rightarrow \infty} \rho(AB^n x_0) = q$.

(iii). If A is sub sequentially convergent then $(B^n x_0)$ has a convergent sub sequence.

(iv). There is a unique $p \in X$ such that $Bp = p$.

(v). If A is sequentially convergent then for each $x_0 \in X$ the iterative sequence $(B^n x_0)$ converges to p .

Proof. Let $x_0 \in X$ be an arbitrary point in X . Define iterative sequence x_n by $x_{n+1} = Bx_n = Bx_0$.

We have

$$\begin{aligned}
\rho(Ax_n, Ax_{n+1}) &= \rho(ABx_{n-1}, ABx_n), \\
&\leq a_1\rho(Ax_{n-1}, ABx_{n-1}) + a_2[\rho(Ax_n, ABx_n) + a_3\rho(Ax_{n-1}, Ax_n)], \\
&\leq a_1\rho(Ax_{n-1}, Ax_n) + a_2[\rho(Ax_n, Ax_{n+1}) + a_3\rho(Ax_{n-1}, Ax_n)], \\
&\leq \frac{(a_1+a_3)}{1-a_2}\rho(Ax_{n-1}, Ax_n), \\
&\leq h\rho(Ax_{n-1}, Ax_n), \text{ where } h = \frac{(a_1+a_3)}{1-a_2}.
\end{aligned}$$

And we conclude that by repeating the above argument we get that

By repeating the same argument we get that

$$\rho(AB^n x_0, AB^{n+1} x_0) \leq h^n \rho(Ax_0, ABx_0). \quad (1)$$

From (1) we have

$$\|\rho(AB^n x_0, AB^{n+1} x_0)\| \leq h^n L \|(Ax_0, ABx_0)\|.$$

Where L is a normal constant. letting $n \rightarrow \infty$ we get that

$$\lim_{n \rightarrow \infty} \|\rho(AB^n x_0, AB^{n+1} x_0)\| = 0. \quad (2)$$

By (1) for every $m, n \in N$, with $m > n$ we have

$$\begin{aligned}
\rho(Ax_n, Ax_m) &\leq \rho(Ax_n, Ax_{n+1}) + \rho(Ax_{m-1}, Ax_m), \\
&\leq [h^n + h^{m-1}] \rho(Ax_0, ABx_0), \\
&= \frac{h^n}{1-h} \rho(Ax_0, ABx_0),
\end{aligned}$$

$$\rho(AB^n x_0, AB^m x_0) \leq \frac{h^n}{1-h} \rho(Ax_0, ABx_0). \quad (3)$$

From (3) we have

$\|\rho(AB^n x_0, AB^m x_0)\| \leq \frac{h^n}{1-h} L \|(Ax_0, ABx_0)\|$, where L is a normal constant of . letting $n \rightarrow \infty$ we get that

$$\lim_{n \rightarrow \infty} \|\rho(AB^n x_0, AB^m x_0)\| = 0.$$

In this we have Implies that $(AB^n x_0)$ is a Cauchy sequence in X . Since X is CCM-Space then there exists $q \in X$ such that

$$\lim_{n \rightarrow \infty} (AB^n x_0) = q. \quad (4)$$

Now A is sub sequentially convergent $(AB^n x_0)$ has a convergent subsequence. So there exists $p \in X$ and (x_{n_r}) such that

$$\lim_{r \rightarrow \infty} (B^{n_r} x_0) = p. \quad (5)$$

Since A is continuous and by (5) we obtain

$$\lim_{r \rightarrow \infty} (B^{n_r} x_0) = Ap. \quad (6)$$

By (4) and (6) we conclude that

$$Ap = q. \quad (7)$$

On the other hand

$$\begin{aligned} \rho(ABp, Ap) &\leq \rho(ABp, AB^{n_r}x_0) + \rho(AB^{n_r}x_0, AB^{n_r+1}x_0) + \rho(AB^{n_r+1}x_0, Ap), \\ &\leq a_1\rho(Ap, ABp) + a_2\rho(AB^{n_r-1}x_0, AB^{n_r}x_0) + a_3\rho(Ap, AB^{n_r-1}x_0) \\ &\quad + h^{n_r}\rho(Ax_0, ABx_0) + \rho(AB^{n_r+1}x_0, Ap), \\ 1 - a_1 &\leq a_2\rho(AB^{n_r-1}x_0, AB^{n_r}x_0) + a_3\rho(Ap, AB^{n_r-1}x_0) + h^{n_r}\rho(Ax_0, ABx_0) + \rho(AB^{n_r+1}x_0, Ap), \\ &\leq \frac{a_2}{1-a_1}\rho(AB^{n_r-1}x_0, AB^{n_r}x_0) + \frac{a_3}{1-a_1}\rho(Ap, AB^{n_r-1}x_0) + \frac{h^{n_r}}{1-a_1} \\ &\quad \rho(Ax_0, ABx_0) + \frac{1}{1-a_1}\rho(AB^{n_r+1}x_0, Ap). \end{aligned}$$

Thus,

$$\|\rho(ABp, Ap)\| \leq L \left\| \frac{a_2}{1-a_1}\rho(AB^{n_r-1}x_0, AB^{n_r}x_0) + \frac{a_3}{1-a_1}\rho(Ap, AB^{n_r-1}x_0) + \frac{h^{n_r}}{1-a_1}\rho(Ax_0, ABx_0) + \frac{1}{1-a_1}\rho(AB^{n_r+1}x_0, Ap) \right\|.$$

Where L is the normal constant of X . The convergence of above give us that $\rho(ABp, Ap) = 0$.

Implies that $ABp = Ap$. Since B is one - to-one then $Bp = p$. Consequently B has a fixed point. Because B is a R -Contraction.

$$\begin{aligned} \rho(ABp, ABq) &\leq a_1\rho(Ap, ABp) + a_2[\rho(Aq, ABq) + a_3\rho(Ap, Aq)], \\ \rho(Ap, Aq) &\leq a_1\rho(Ap, Ap) + a_2[\rho(Aq, Aq) + a_3\rho(Ap, Aq)], \end{aligned}$$

If q is another fixed point of B then from the injectivity of B we get that $Bp = Bq$ or which is the same the fixed point is unique. Finally if A is sequentially convergent by replacing (n) for (n_r) we conclude that $\lim_{n \rightarrow \infty} B^n x_0 = p$. This shows that $(B^n x_0)$ converges to the fixed point of B . This completes the proof of the theorem. □

Remark

If we take $a_3 = 0$ and $a_1 = a_2 = b$ in the above theorem we get the Theorem 3.1. of [5].

Conclusion

Our results are extended results and are also more general than the results of [5]

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References

- [1] 1. M.Abbas and G.Jungck, Common fixed point results for non commuting mappings without continuity in one metric spaces, J. Math. Anal. 311(2008),116-120.

- [2] . A.Beiranvand, S.Moradi, M.Omod, and H. Pazandeh, Two fixed point theorems for special mapping, arXiv:0903.1504v1[math.FA].
- [3] H.Long-Guang and Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, J.Math.Anal.Appl.,332,(2007),1468-1476.
- [4] R.Kannan, Some results on fixed points, Bull.Calcutta Math.Soc.60(1968),71-76.
- [5] Jose R.Morales and E.Rojas, Cone metric paces and fixed point theorems of T-Kannan contractive mappings, Int.Journal of Math.Analysis, Vol.4, no.4, (2010), 175-184.
- [6] S.Moradi, Kannan fixed point theorem on complete metric spaces and on generalized metric spaces depended on another function, arXiv:0930.1577v1[math.FA].
- [7] K. Prudhvi, Common fixed points on occasionally weakly compatible self- mappings in CMS, Asia Mathe- matika,Vol.7, Issue.2, (2023), 13-16.
- [8] K. Prudhvi, Study on Fixed Points for OWC in Symmetric Spaces, Asia Mathematika, Vol.7, Issue 3, (2023), 72-75.
- [9] K. Prudhvi, Results on fixed points for WC-Mappings satisfying generalized contractive condition in C- metric spaces, Asia Mathematika, Vol. 8 Issue 1, (2024), 112 – 117.
- [10] K. Prudhvi, A unique common fixed point theorem in cone metric spaces, Advances in Fixed Point Theory, 3(2013), No.1,70-76.
- [11] K. Prudhvi, Common fixed points on occasionally weakly compatible self- mappings in CMS, Asia Mathe- matika,Vol.7, Issue.2, (2023), 13-16.
- [12] K. Prudhvi, Study on Fixed Points for OWC in Symmetric Spaces, Asia Mathematika, Vol.7, Issue 3, (2023), 72-75.
- [13] K. Prudhvi, Results on fixed points for WC-Mappings satisfying generalized contractive condition in C- metric spaces, Asia Mathematika, Volume: 8 Issue: 1, (2024), 112 – 117.
- [14] K. Prudhvi, A unique common fixed point result for compatible reciprocal continuous four self - maps in complete metric space, Asia Mathematika, Volume: 8 Issue: 3, (2024), 10 – 13.
- [15] K.Prudhvi, Study on Extended B-K Fixed Point Theorem on CGM-Space Depended on an Another Function Asia Mathematika Vol.5,Issue 2, (2025), 25-28.
- [16] K .Prudhvi, The Chatterjea fixed point theorem on CCRM- space, Asia Mathematika, Vol. 9, issue 3, (2025), 20-24.
- [17] K.Prudhvi, Study on Fixed Point Result for Special Mappings, Asia Mathematika, Vol.10,Issue :1, (2026),12-16.