



On Zariski pseudo-primary radical of elements of lattice modules

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Received: 21 Jul 2026

Accepted: 02 Aug 2026

Published Online: 31 Aug 2026

Abstract: Let L be a C -lattice and M be a lattice module over L . For an element $N \in M$, the meet of all pseudo-primary elements P of M satisfying $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$ is called the Zariski pseudo-primary radical of N , denoted by ${}^{zps}\sqrt{N}$. This paper introduces the notion of the Zariski pseudo-primary radical of elements in lattice modules and presents a systematic study of its algebraic properties. Several characterizations and basic results are established, highlighting its connections with pseudo-primary elements, pseudo-primary radicals and the pseudo-primary spectrum of lattice modules. Also, we study lattice modules whose pseudo-primary spectrums endowed with the Zariski-like topology are Noetherian spaces.

Key words: Lattice modules, Pseudo-primary elements, Pseudo-primary spectrum, Zariski pseudo-primary radical.

1. Introduction

The notion of primary-like submodules was introduced and studied by H. F. Moghimi and F. Rashedi in [17]. Subsequently, in [19], the authors developed the Zariski topology on the primary-like spectrum of a module and investigated several of its topological properties. Later, they introduced in [18] the concept of the Z -radical of submodules and studied the primary-like spectrum from the viewpoint of Noetherian spaces. Motivated by these developments, the concept of pseudo-primary elements was introduced for lattice modules by A. V. Bingi et al. in [6], extending the theory of primary-like submodules to the framework of lattice modules. Furthermore, P. B. Girase et al. studied the Zariski prime radical elements of lattice modules in [12], while V. Borkar et al. investigated the Zariski second radical elements and associated properties in [7].

These works demonstrate the significance of Zariski-type topologies and radical concepts in understanding the algebraic and topological structures of modules and lattice modules. Inspired by these developments, in this paper we introduce the notion of Zariski pseudo-primary radical elements in lattice modules. We investigate their fundamental properties and establish several characterizations. Moreover, we extend and generalize many of the existing results on Zariski primary-like submodules to the setting of lattice modules, thereby providing a broader framework for studying primary-type structures in lattice theory.

Throughout this work, let L denote a complete multiplicative lattice with least element 0_L and the greatest element 1_L . The lattice L is assumed to possess a multiplication operation " ." satisfying the conditions:

- (1) $x.y = y.x$, for all $x, y \in L$;
- (2) $x.(y.z) = (x.y).z$, for all $x, y, z \in L$;

- (3) $x.(\vee_{\alpha} y_{\alpha}) = \vee_{\alpha}(x.y_{\alpha})$, for all $x, y_{\alpha} \in L$;
- (4) $x.1_L = x$, for all $x \in L$.

An element x of the lattice L is called compact if whenever $x \leq \vee_{\alpha \in I} y_{\alpha}$, $x \leq y_{\alpha_1} \vee y_{\alpha_2} \vee \cdots \vee y_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of I . A C -lattice is a multiplicative lattice generated by its compact elements under arbitrary joins, where the multiplicatively closed set of compact elements contains the identity element. For more details on C -lattice the reader may refer ([11],[14],[22]-[24]).

An element x is referred to as *proper* whenever it is strictly less than the greatest element 1_L of the lattice. A proper element q of a multiplicative lattice L is said to be *prime* if $xy \leq q$ implies $x \leq q$ or $y \leq q$ for $x, y \in L$. The collection of all prime elements of L is denoted by $\text{Spec}(L)$.

The Zariski topology on the set $\text{Spec}(L)$ of all prime elements in multiplicative lattices is being studied in [22], by Thakare, Manjarekar and Maeda and in [23], by Thakare and Manjarekar as a generalization of the Zariski topology of a commutative ring with unity.

A proper element m of a multiplicative lattice L is said to be *maximal* if for every $x \in L$ with $m < x \leq 1_L$ implies $x = 1_L$. The radical of an element $a \in L$ is defined as $\sqrt{a} = \vee\{x \in L \mid x \text{ is compact and } x^n \leq a, \text{ for some positive integer } n\}$ [3].

A complete lattice M with smallest element 0_M and greatest element 1_M is said to be a *lattice module* over the multiplicative lattice L or L -module if there is a multiplication between elements of M and L , denoted by $xA \in M$, for $x \in L$ and $A \in M$, which satisfies the following properties:

1. $(xy)A = x(yA)$;
2. $(\vee_{\alpha} x_{\alpha})(\vee_{\beta} A_{\beta}) = (\vee_{\alpha\beta} x_{\alpha}A_{\beta})$;
3. $1_L A = A$;
4. $0_L A = 0_M$; for all $x, y, x_{\alpha} \in L$, and for all $A, A_{\beta} \in M$.

Assume that M is a lattice module over a C -lattice L . For $A \in M, y \in L$, denote $(A : y) = \vee\{K \in M \mid yK \leq A\}$. If $x, y \in L$, we write $(x : y) = \vee\{a \in L \mid ya \leq x\}$. If $A, B \in M$, then $(A : B) = \vee\{x \in L \mid xB \leq A\}$. An element $A \in M$ is said to be compact if $A \leq \vee_{\alpha \in I} A_{\alpha}$ (I is an indexed set) implies $A \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of I .

An element $N \in M$ is said to be *meet principal* (respectively *join principal*) if it satisfies the identity $A \wedge aN = (a \wedge (A : N))N$ (respectively $((aN \vee A) : N) = (a \vee (A : N))$) for all $a \in L$ and for all $A \in M$. Also N is said to be *principal* if it is both join as well as meet principal. If each element of M is the join of principal (compact) elements of M , then M is called the *principally generated*(*compactly generated*) lattice module.

An element $A \in M$ is said to be proper, if $A < 1_M$. A proper element A of a lattice module M is said to be *prime* if $aX \leq A$ implies $X \leq A$ or $a1_M \leq A$, i.e., $a \leq (A : 1_M)$ for every $a \in L$ and $X \in M$. The prime spectrum of a lattice module M is the set of all prime elements of M and it is denoted by $\text{Spec}(M)$. In [5], S. Ballal and V. Kharat studied the Zariski topology over $\text{Spec}(M)$ as a generalization of the results carried out in [[22], [23]]. Also in [10], Fethi Callialp et al. studied the Zariski topology over $\text{Spec}(M)$ over multiplicative lattice L .

For an element $N \in M$, radical of N , is defined as $\sqrt{N} = \wedge\{P \in \text{Spec}(M) \mid N \leq P\}$ [16]. If there is no prime which contains N , then $\sqrt{N} = 1_M$. Note that, $N \leq \sqrt{N}$.

An element $N < 1_M$ of M is said to be *maximal* if $N \leq B$ implies either $N = B$ or $B = 1_M, B \in M$. If 1_M is compact, then M has a maximal element[21].

Further all these concepts and for more information on multiplicative lattices, lattice modules and topology, the reader may refer ([1], [2], [4], [13], [15], [20]).

2. Zariski-like Topology on $\text{Spec}^{ps}(M)$

Let M be a lattice module over a C -lattice L . An element $P < 1_M$ is said to be pseudo-primary, if $aA \leq P$ implies that $a \leq (P : 1_M)$ or $A \leq \sqrt{P}$ ([6]). Note that, if P is pseudo-primary element of M , then $(P : 1_M)$ is primary element of L and $\sqrt{(P : 1_M)} = p$ is a prime element of L . In this case P is called p -pseudo-primary element of M . Let $\text{Spec}^{ps}(M)$ be the collection of all pseudo-primary elements P of M , for which $\sqrt{(P : 1_M)} = (\sqrt{P} : 1_M)$. Let $\mathcal{F}(N) = \{P \in \text{Spec}^{ps}(M) | \sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}\}$, for $N \in M$.

Lemma 2.1. [13] *Let M be a lattice module over a C -lattice L . Then for $N \in M$, $(N : 1_M) = ((N : 1_M)1_M : 1_M)$.*

The following lemma will be used repeatedly in the sequel.

Lemma 2.2. [13] *Let M be a lattice module over a multiplicative lattice L . Then for $x \in L$ and $A, B, C, A_\alpha \in M$, following holds.*

1. $x \leq (0_M : (0_M : x))$.
2. $A \leq (0_M : (0_M : A))$.
3. If $A \leq B$ then $(C : B) \leq (C : A)$.
4. If $A \leq B$ then $(A : C) \leq (B : C)$.
5. $\vee_\alpha (A_\alpha : B) \leq ((\vee_\alpha A_\alpha) : B)$.
6. If $a \leq b$ then $aA \leq bA$.
7. $(0_M : A) = (0_M : (0_M : (0_M : A)))$.
8. $(A : B \vee C) = (A : B) \wedge (A : C)$.
9. $x(A : x) \leq A$.
10. $(A : B)B \leq A$.
11. $a \leq (aA : A)$.
12. $(B \wedge C : A) = (B : A) \wedge (C : A)$.

Some of the results in this paper require the following properties.

Properties 2.1. ([22] and [23]) *Let L be a multiplicative lattice and $a, b \in L$. Then*

1. $a \leq \sqrt{a}$.
2. $a \leq b$ implies $\sqrt{a} \leq \sqrt{b}$.

3. For $x \in L_*$, $x \leq \sqrt{a}$ if and only if $x^n \leq a$ for some integer $n \geq 1$.
4. $\sqrt{\sqrt{a}} = \sqrt{a}$.
5. $\sqrt{a \wedge b} = \sqrt{a} \wedge \sqrt{b} = \sqrt{ab}$.
6. $\sqrt{a \vee b} = \sqrt{\sqrt{a} \vee \sqrt{b}}$.
7. If $1 \in L$ is compact, then $a < 1$ implies $\sqrt{a} < 1$.
8. $\sqrt{a} = \wedge \{p \in L \mid p \text{ is prime and } a \leq p\} = \wedge \{p \in L \mid p \text{ is minimal prime over } a\}$.
9. If $a \in L$ is prime, then $\sqrt{a} = a$.

We begin with a lemma that plays a crucial role in the development of the main concept in the paper.

Lemma 2.3. Assume that M is a lattice module over a C -lattice L . Then the following statements hold:

1. $\mathcal{F}(0_M) = \text{Spec}^{ps}(M)$ and $\mathcal{F}(1_M) = \emptyset$
2. $\cap_{i \in I} \mathcal{F}(N_i) = \mathcal{F}(\vee_{i \in I} (N_i : 1_M) 1_M)$, N_i , for $i \in I$ (Indexed set) are elements of M .
3. $\mathcal{F}(N) \cup \mathcal{F}(K) = \mathcal{F}(N \wedge K)$, N, K are any elements of M .

Proof. 1. Since, $0_M \leq P$, $(0_M : 1_M) \leq (P : 1_M)$, by Lemma 2.2(4) and hence by Properties 2.1(2), $\sqrt{(0_M : 1_M)} \leq \sqrt{(P : 1_M)}$. Thus $\mathcal{F}(0_M) = \text{Spec}^{ps}(M)$. Clearly, $\mathcal{F}(1_M) = \emptyset$.

2. Let $P \in \cap_{i \in I} \mathcal{F}(N_i)$. Then $P \in \mathcal{F}(N_i)$, for each $i \in I$. Hence, $\sqrt{(N_i : 1_M)} \leq \sqrt{(P : 1_M)}$. By Properties 2.1(1), $(N_i : 1_M) \leq \sqrt{(N_i : 1_M)} \leq \sqrt{(P : 1_M)}$, for each $i \in I$. This implies that $\vee_{i \in I} (N_i : 1_M) \leq \sqrt{(P : 1_M)}$. By Lemma 2.2(6), $\vee_{i \in I} (N_i : 1_M) 1_M \leq \sqrt{(P : 1_M)} 1_M = (\sqrt{P} : 1_M) 1_M$. We have, $(\sqrt{P} : 1_M) 1_M \leq \sqrt{P}$, by Lemma 2.2(10). Thus $\vee_{i \in I} (N_i : 1_M) 1_M \leq \sqrt{P}$. Hence, using Lemma 2.2(4), we obtain $(\vee_{i \in I} (N_i : 1_M) 1_M : 1_M) \leq (\sqrt{P} : 1_M)$. Hence $\sqrt{(\vee_{i \in I} (N_i : 1_M) 1_M : 1_M)} \leq \sqrt{(P : 1_M)}$, by Properties 2.1(2) and (4). It follows that $P \in \mathcal{F}(\vee_{i \in I} (N_i : 1_M) 1_M)$. For the reverse containment, let us assume that $P \in \mathcal{F}(\vee_{i \in I} (N_i : 1_M) 1_M)$. Hence $\sqrt{(\vee_{i \in I} (N_i : 1_M) 1_M : 1_M)} \leq \sqrt{(P : 1_M)}$. From this we get $(\vee_{i \in I} (N_i : 1_M) 1_M : 1_M) \leq \sqrt{(P : 1_M)}$, by Properties 2.1(1). This yields $((\vee_{i \in I} (N_i : 1_M)) 1_M : 1_M) = (\vee_{i \in I} (N_i : 1_M) 1_M : 1_M) \leq \sqrt{(P : 1_M)}$. Therefore $\vee_{i \in I} ((N_i : 1_M) 1_M : 1_M) \leq \sqrt{(P : 1_M)}$, by Lemma 2.2(5). Hence $((N_i : 1_M) 1_M : 1_M) \leq \sqrt{(P : 1_M)}$ for all $i \in I$. It implies that $(N_i : 1_M) \leq \sqrt{(P : 1_M)}$, by Lemma 2.1. From this, it follows that $\sqrt{(N_i : 1_M)} \leq \sqrt{(P : 1_M)}$, for each $i \in I$. Thus $P \in \mathcal{F}(N_i)$, for each $i \in I$. Consequently, $P \in \cap_{i \in I} \mathcal{F}(N_i)$.

3. Let $P \in \mathcal{F}(N) \cup \mathcal{F}(K)$. Then $P \in \mathcal{F}(N)$ or $P \in \mathcal{F}(K)$. Hence $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$ or $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$. Now as $N \wedge K \leq N, K$, it implies that $((N \wedge K) : 1_M) \leq (N : 1_M)$ and $((N \wedge K) : 1_M) \leq (K : 1_M)$, by Lemma 2.2(4). As a consequence $\sqrt{((N \wedge K) : 1_M)} \leq \sqrt{(P : 1_M)}$, by Properties 2.1(2). Thus $P \in \mathcal{F}(N \wedge K)$. Conversely, let us assume that $P \in \mathcal{F}(N \wedge K)$. It follows that $((N \wedge K) : 1_M) \leq \sqrt{(P : 1_M)}$. By using Lemma 2.2(12), we obtain $(N : 1_M) \wedge (K : 1_M) \leq \sqrt{(P : 1_M)}$. Since $\sqrt{(P : 1_M)}$ is a prime element of L , we have $(N : 1_M) \leq \sqrt{(P : 1_M)}$ or $(K : 1_M) \leq \sqrt{(P : 1_M)}$. This yields $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$ or $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$, by Properties 2.1(2) and (4). Consequently, $P \in \mathcal{F}(N)$ or $P \in \mathcal{F}(K)$. $\square$

Let τ_1 be the collection of all subsets $\mathcal{F}(N)$ of $\text{Spec}^{ps}(M)$. Lemma 2.3 shows that τ_1 satisfies the axioms for the closed subsets of a topological space on $\text{Spec}^{ps}(M)$, called as Zariski-like topology on $\text{Spec}^{ps}(M)$.

Let $\mathcal{G}(N) = \{P \in \text{Spec}^{ps}(M) | N \leq \sqrt{P}\}$. Let Y denotes the collection of all subsets $\mathcal{G}(N)$ of $\text{Spec}^{ps}(M)$. Then Y contains the empty set and $\text{Spec}^{ps}(M)$ and Y is closed under arbitrary intersections. We say that M is a top lattice module, if Y is closed under finite unions. In this case, Y induces a topology τ_2 on $\text{Spec}^{ps}(M)$, called as quasi-Zariski topology [10].

For each $p \in \text{Spec}(L)$, let $\text{Spec}_p^{ps}(M) = \{P \in \text{Spec}^{ps}(M) | \sqrt{(P : 1_M)} = p\}$.

In what follows, L/a denotes the interval $[a, 1]$ which is multiplicative lattice with $x \circ y = xy \vee a$, where $x, y \in L$ [9]. We denote $\text{Spec}(L/(0_M : 1_M))$ by $\text{Spec}(\bar{L})$.

Let $\psi^{ps} : \text{Spec}^{ps}(M) \rightarrow \text{Spec}(\bar{L})$ be defined by $\psi^{ps}(P) = \sqrt{(P : 1_M)}$. Then ψ^{ps} is called the natural map on $\text{Spec}^{ps}(M)$.

3. Zariski pseudo-primary radical and pseudo-primary radical of elements of lattice module M

The following definitions establish the terminology used throughout this paper.

Definition 3.1. The Zariski pseudo-primary radical ${}^{zp}\sqrt{N}$ of an element $N \in M$ is defined as the meet of all pseudo-primary elements P of M such that $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$, i.e., ${}^{zp}\sqrt{N} = \wedge \{P \in \text{Spec}^{ps}(M) | \sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}\} = \wedge \mathcal{F}(N)$. If ${}^{zp}\sqrt{N} = N$, then N is called Zariski pseudo-primary radical element of M .

Definition 3.2. The pseudo-primary radical ${}^{ps}\sqrt{N}$ of an element $N \in M$ is defined as the meet of all pseudo-primary elements P of M such that $N \leq \sqrt{P}$, i.e., ${}^{ps}\sqrt{N} = \wedge \{P \in \text{Spec}^{ps}(M) | N \leq \sqrt{P}\} = \wedge \mathcal{G}(N)$. If ${}^{ps}\sqrt{N} = N$, then N is called pseudo-primary radical element of M .

The following results follow from the preceding lemmas and definitions.

Lemma 3.1. Let M be a lattice module over a C -lattice L . Let N, K and $N_j, j \in J$ (Indexed set) be elements of M and a be an element of L . Then the following statements hold:

1. If $N \leq K$, then $\mathcal{F}(K) \subseteq \mathcal{F}(N)$ and $\mathcal{G}(K) \subseteq \mathcal{G}(N)$.
2. $\mathcal{G}(N) \subseteq \mathcal{F}(N)$.
3. $\mathcal{F}(a1_M) = \mathcal{F}(\sqrt{a}1_M) = \mathcal{G}(a1_M) = \mathcal{G}(\sqrt{a}1_M)$.
4. $\mathcal{F}(N) = \mathcal{F}((N : 1_M)1_M) = \mathcal{F}(\sqrt{(N : 1_M)}1_M) = \mathcal{G}((N : 1_M)1_M) = \mathcal{G}(\sqrt{(N : 1_M)}1_M)$.
5. If $\sqrt{(N : 1_M)} = \sqrt{(K : 1_M)}$, then $\mathcal{F}(N) = \mathcal{F}(K)$. Also, if N, K are pseudo-primary elements and $\mathcal{F}(N) = \mathcal{F}(K)$, then $\sqrt{(N : 1_M)} = \sqrt{(K : 1_M)}$.
6. $\mathcal{G}(N) = \mathcal{G}(\sqrt{N})$.

Proof. 1. Assume that $N \leq K$. Let $P \in \mathcal{F}(K)$, then $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$. Now, $N \leq K$ implies $(N : 1_M) \leq (K : 1_M)$, by Lemma 2.2(4). By Properties 2.1(1), $(K : 1_M) \leq \sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$. Therefore $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$ by Properties 2.1(2) and (4). Hence $P \in \mathcal{F}(N)$ and $\mathcal{F}(K) \subseteq \mathcal{F}(N)$. We

now prove $\mathcal{G}(K) \subseteq \mathcal{G}(N)$. Consider $P \in \mathcal{G}(K)$. Then $K \leq \sqrt{P}$. It follows that $N \leq K \leq \sqrt{P}$. Consequently, $P \in \mathcal{G}(N)$.

2. Take $P \in \mathcal{G}(N)$. Then $N \leq \sqrt{P}$. It implies that $(N : 1_M) \leq (\sqrt{P} : 1_M) = \sqrt{(P : 1_M)}$, by Lemma 2.2(4). Therefore, $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$, by Properties 2.1(2) and (4). It follows that $P \in \mathcal{F}(N)$. Thus $\mathcal{G}(N) \subseteq \mathcal{F}(N)$.

3. By Properties 2.1(1), we have $a \leq \sqrt{a}$. Therefore $a1_M \leq \sqrt{a}1_M$, by Lemma 2.2(6). By part (1), it follows that $\mathcal{G}(\sqrt{a}1_M) \subseteq \mathcal{G}(a1_M)$. Also, by part (2), $\mathcal{G}(a1_M) \subseteq \mathcal{F}(a1_M)$. Therefore, $\mathcal{G}(\sqrt{a}1_M) \subseteq \mathcal{G}(a1_M) \subseteq \mathcal{F}(a1_M)$. Consider $P \in \mathcal{F}(a1_M)$. This implies that $\sqrt{(a1_M : 1_M)} \leq \sqrt{(P : 1_M)}$. By Lemma 2.2(11), we have $a \leq (a1_M : 1_M)$. Hence $\sqrt{a} \leq \sqrt{(a1_M : 1_M)} \leq \sqrt{(P : 1_M)} = (\sqrt{P} : 1_M)$, by Properties 2.1(2). This yields $\sqrt{a}1_M \leq \sqrt{P}$. Thus $P \in \mathcal{G}(\sqrt{a}1_M)$. Finally, we obtain $\mathcal{F}(a1_M) \subseteq \mathcal{G}(\sqrt{a}1_M)$. Consequently, $\mathcal{G}(\sqrt{a}1_M) \subseteq \mathcal{G}(a1_M) \subseteq \mathcal{F}(a1_M) \subseteq \mathcal{G}(\sqrt{a}1_M)$. This establishes $\mathcal{F}(a1_M) = \mathcal{G}(a1_M) = \mathcal{G}(\sqrt{a}1_M)$. Now, replacing a by $\sqrt{a}$, we obtain $\mathcal{F}(\sqrt{a}1_M) = \mathcal{G}(\sqrt{a}1_M) = \mathcal{G}(a1_M)$. Thus, we obtain the required result.

4. We prove, $\mathcal{F}(N) = \mathcal{F}((N : 1_M)1_M)$. $P \in \mathcal{F}(N)$ if and only if $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$ if and only if $\sqrt{((N : 1_M)1_M : 1_M)} \leq \sqrt{(P : 1_M)}$, by Lemma 2.1, if and only if $P \in \mathcal{F}((N : 1_M)1_M)$. Therefore, $\mathcal{F}(N) = \mathcal{F}((N : 1_M)1_M)$. Hence the result.

5. Assume that $\sqrt{(N : 1_M)} = \sqrt{(K : 1_M)}$. Then $P \in \mathcal{F}(N)$ if and only if $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$ if and only if $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$ if and only if $P \in \mathcal{F}(K)$. Therefore $\mathcal{F}(N) = \mathcal{F}(K)$. For the converse part, let us assume that N, K are pseudo-primary and $\mathcal{F}(N) = \mathcal{F}(K)$. Now $N \in \mathcal{F}(N) = \mathcal{F}(K)$ implies $\sqrt{(K : 1_M)} \leq \sqrt{(N : 1_M)}$ and $K \in \mathcal{F}(K) = \mathcal{F}(N)$ implies $\sqrt{(N : 1_M)} \leq \sqrt{(K : 1_M)}$. Thus $\sqrt{(N : 1_M)} = \sqrt{(K : 1_M)}$.

6. Since $N \leq \sqrt{N}$, by part (1), $\mathcal{G}(\sqrt{N}) \subseteq \mathcal{G}(N)$. For reverse inclusion, let $P \in \mathcal{G}(N)$, then $N \leq \sqrt{P}$. Hence $\sqrt{N} \leq \sqrt{P}$ and so $P \in \mathcal{G}(\sqrt{N})$. Consequently, $\mathcal{G}(N) = \mathcal{G}(\sqrt{N})$. $\square$

Proposition 3.1. *Let M be a lattice module over a C -lattice L and N, K be elements of M . Then the following statements hold:*

1. ${}^{zps}\sqrt{N} \leq {}^{ps}\sqrt{N}$.
2. ${}^{zps}\sqrt{a1_M} = {}^{zps}\sqrt{\sqrt{a}1_M} = {}^{ps}\sqrt{a1_M} = {}^{ps}\sqrt{\sqrt{a}1_M}$, for $a \in L$.
3. ${}^{ps}\sqrt{N} = {}^{ps}\sqrt{(N : 1_M)1_M} = {}^{ps}\sqrt{\sqrt{(N : 1_M)1_M}} = {}^{zps}\sqrt{(N : 1_M)1_M} = {}^{zps}\sqrt{\sqrt{(N : 1_M)1_M}}$.

Proof. 1. By Lemma 3.1(2), $\mathcal{G}(N) \subseteq \mathcal{F}(N)$. Hence $\wedge \mathcal{F}(N) \leq \wedge \mathcal{G}(N)$. Consequently, ${}^{zps}\sqrt{N} \leq {}^{ps}\sqrt{N}$.

2. By Lemma 3.1(3), $\mathcal{F}(a1_M) = \mathcal{F}(\sqrt{a}1_M) = \mathcal{G}(a1_M) = \mathcal{G}(\sqrt{a}1_M)$. Therefore, $\wedge \mathcal{F}(a1_M) = \wedge \mathcal{F}(\sqrt{a}1_M) = \wedge \mathcal{G}(a1_M) = \wedge \mathcal{G}(\sqrt{a}1_M)$. Thus ${}^{zps}\sqrt{a1_M} = {}^{zps}\sqrt{\sqrt{a}1_M} = {}^{ps}\sqrt{a1_M} = {}^{ps}\sqrt{\sqrt{a}1_M}$.

3. By Lemma 3.1(4), it is clear. $\square$

Definition 3.3. [9] Let M be a lattice module over a multiplicative lattice L . M is said to be a multiplication lattice module, if for every element K of M there exists an element $a \in L$ such that $K = a1_M$.

Lemma 3.2. [9] Let M be a lattice module over a multiplicative lattice L . Then M is a multiplication lattice module over L if and only if $K = (K : 1_M)1_M$, for every element K in M .

Lemma 3.3. [6] For every $q_i \in L(i \in Z_+)$ we have $\wedge_{i \in Z_+} \sqrt{q_i} = \sqrt{\wedge_{i \in Z_+} q_i}$.

Let $\mathcal{Z}(Y)$ denote the meet of all P , for $P \in Y$ and $Cl(Y)$ as the closure of Y . We have the following lemma.

Lemma 3.4. Let M be a lattice module over a C -lattice L and let $Y \subseteq Spec^{ps}(M)$. Then $\mathcal{F}(\mathcal{Z}(Y)) = Cl(Y)$. In particular, Y is closed if and only if $\mathcal{F}(\mathcal{Z}(Y)) = Y$.

Proof. Let Y be a subset of $Spec^{ps}(M)$. Assume that $P \in Y$. Then $\mathcal{Z}(Y) = \wedge_{P \in Y} P \leq P$. Hence $(\mathcal{Z}(Y) : 1_M) \leq (P : 1_M)$, by Lemma 2.2(4). By Properties 2.1(2), $\sqrt{(\mathcal{Z}(Y) : 1_M)} \leq \sqrt{(P : 1_M)}$. Thus $P \in \mathcal{F}(\mathcal{Z}(Y))$. Consequently, $Y \subseteq \mathcal{F}(\mathcal{Z}(Y))$. Now, suppose that $\mathcal{F}(N)$ is any closed subset of $Spec^{ps}(M)$ containing Y . Then for every $P \in Y \subseteq \mathcal{F}(N)$, we have $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$. It implies that $\sqrt{(N : 1_M)} \leq \wedge_{P \in \mathcal{F}(N)} \sqrt{(P : 1_M)} = \wedge_{P \in \mathcal{F}(N)} (\sqrt{P} : 1_M)$. By Lemma 3.3, $\wedge_{P \in \mathcal{F}(N)} \sqrt{(P : 1_M)} = \sqrt{\wedge_{P \in \mathcal{F}(N)} (P : 1_M)}$ and by Lemma 2.2(12), $\wedge_{P \in \mathcal{F}(N)} (P : 1_M) = (\wedge_{P \in \mathcal{F}(N)} P : 1_M)$. Hence $\sqrt{(N : 1_M)} \leq \sqrt{(\wedge_{P \in \mathcal{F}(N)} P : 1_M)} \leq \sqrt{(\mathcal{Z}(Y) : 1_M)}$. Consider $P' \in \mathcal{F}(\mathcal{Z}(Y))$. Then $\sqrt{(\mathcal{Z}(Y) : 1_M)} \leq \sqrt{(P' : 1_M)}$. As a consequence, $\sqrt{(N : 1_M)} \leq \sqrt{(P' : 1_M)}$. It implies that $P' \in \mathcal{F}(N)$. From this, we conclude that $\mathcal{F}(\mathcal{Z}(Y)) \subseteq \mathcal{F}(N)$. Which shows that $\mathcal{F}(\mathcal{Z}(Y))$ is the smallest closed set. Therefore $\mathcal{F}(\mathcal{Z}(Y)) = Cl(Y)$. $\square$

Lemma 3.5. Let M be a lattice module over a C -lattice L . Then $\mathcal{F}(\mathcal{Z}(\mathcal{F}(N))) = \mathcal{F}(\sqrt[p]{N}) = \mathcal{F}(N)$.

Proof. It follows from the Lemma 3.4. $\square$

Proposition 3.2. Let M be a lattice module over a C -lattice L . Let N, K be elements of M . Then the following statements hold:

1. If $P \in \mathcal{F}(N)$, then $\sqrt[p]{N} \leq P$.
2. If $\mathcal{F}(N) \subseteq \mathcal{F}(K)$, then $\sqrt[p]{K} \leq \sqrt[p]{N}$. The converse holds, if $K \leq \sqrt[p]{K}$.
3. $\sqrt[p]{\sqrt[p]{N}} = \sqrt[p]{N}$.
4. $\sqrt[p]{N \wedge K} = \sqrt[p]{N} \wedge \sqrt[p]{K}$.
5. If $P \in \mathcal{G}(N)$, then $\sqrt[p]{N} \leq P$.
6. If $\mathcal{G}(N) \subseteq \mathcal{G}(K)$, then $\sqrt[p]{K} \leq \sqrt[p]{N}$. The converse holds, if $K \leq \sqrt[p]{K}$.
7. $\sqrt[p]{N} = \sqrt[p]{\sqrt[p]{N}}$.

Proof. 1. Let $P \in \mathcal{F}(N)$, then $\wedge\{P \in \mathcal{F}(N)\} \leq P$, i.e., $\sqrt[p]{N} \leq P$.

2. Assume that $\mathcal{F}(N) \subseteq \mathcal{F}(K)$. Then $\mathcal{Z}(\mathcal{F}(K)) \leq \mathcal{Z}(\mathcal{F}(N))$. Thus, $\sqrt[p]{K} \leq \sqrt[p]{N}$. Conversely, suppose that $\sqrt[p]{K} \leq \sqrt[p]{N}$ and $K \leq \sqrt[p]{K}$. Let $P \in \mathcal{F}(N)$. Then $\sqrt[p]{N} \leq P$, by part (1). As $K \leq \sqrt[p]{K}$, it follows that $K \leq \sqrt[p]{K} \leq \sqrt[p]{N} \leq P$. Now, by Lemma 2.2(4) and Properties 2.1(2), it implies that $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$. Hence $P \in \mathcal{F}(K)$. Consequently, $\mathcal{F}(N) \subseteq \mathcal{F}(K)$.

3. By Lemma 3.5, $\mathcal{F}(\mathcal{Z}(\mathcal{F}(N))) = \mathcal{F}(\sqrt[p]{N}) = \mathcal{F}(N)$. Therefore $\mathcal{Z}(\mathcal{F}(\sqrt[p]{N})) = \mathcal{Z}(\mathcal{F}(N))$. This yields $\sqrt[p]{\sqrt[p]{N}} = \sqrt[p]{N}$.

4. From the Lemma 2.3(3), we have $\mathcal{F}(N) \cup \mathcal{F}(K) = \mathcal{F}(N \wedge K)$. This implies that $\mathcal{Z}(\mathcal{F}(N \wedge K)) = \mathcal{Z}(\mathcal{F}(N) \cup \mathcal{F}(K)) = \mathcal{Z}(\mathcal{F}(N)) \wedge \mathcal{Z}(\mathcal{F}(K))$. Consequently, ${}^{zps}\sqrt{N \wedge K} = {}^{zps}\sqrt{N} \wedge {}^{zps}\sqrt{K}$.
5. Let $P \in \mathcal{G}(N)$. Then $\wedge\{P \in \mathcal{G}(N)\} \leq P$, i.e., ${}^{ps}\sqrt{N} \leq P$.
6. Assume that $\mathcal{G}(N) \subseteq \mathcal{G}(K)$. Then $\mathcal{Z}(\mathcal{G}(K)) \leq \mathcal{Z}(\mathcal{G}(N))$. Thus ${}^{ps}\sqrt{K} \leq {}^{ps}\sqrt{N}$. Conversely, let us assume that ${}^{ps}\sqrt{K} \leq {}^{ps}\sqrt{N}$ and $K \leq {}^{ps}\sqrt{K}$. Now, take $P \in \mathcal{G}(N)$. By part (5), it follows that ${}^{ps}\sqrt{N} \leq P$. Therefore $K \leq {}^{ps}\sqrt{K} \leq {}^{ps}\sqrt{N} \leq P \leq \sqrt{P}$. From this, we conclude that $P \in \mathcal{G}(K)$. Thus, we obtain the required result.
7. By using Lemma 3.1(6), we have $\mathcal{G}(N) = \mathcal{G}(\sqrt{N})$. Hence $\mathcal{Z}(\mathcal{G}(N)) = \mathcal{Z}(\mathcal{G}(\sqrt{N}))$. Consequently, ${}^{ps}\sqrt{N} = {}^{ps}\sqrt{\sqrt{N}}$. $\square$

Proposition 3.3. *Let M be a lattice module over a C -lattice L and N be an element of M . If $N \leq {}^{zps}\sqrt{N}$, then $\mathcal{F}(N) = \mathcal{G}(N)$. In particular, ${}^{zps}\sqrt{N} = {}^{ps}\sqrt{N}$.*

Proof. By Lemma 3.1(2), $\mathcal{G}(N) \subseteq \mathcal{F}(N)$. For reverse inclusion, take $P \in \mathcal{F}(N)$, then ${}^{zps}\sqrt{N} \leq P$, by Proposition 3.2(1). Thus $N \leq {}^{zps}\sqrt{N} \leq P \leq \sqrt{P}$. Hence $P \in \mathcal{G}(N)$. Consequently, $\mathcal{F}(N) = \mathcal{G}(N)$ and ${}^{zps}\sqrt{N} = {}^{ps}\sqrt{N}$. $\square$

Proposition 3.4. *Let M be a compactly generated lattice module over a C -lattice L . If the natural map $\psi^{ps} : \text{Spec}^{ps}(M) \rightarrow \text{Spec}(\bar{L})$ is surjective, then the following statements hold:*

1. ${}^{zps}\sqrt{N} \neq 1_M$ if and only if $\mathcal{F}(N) \neq \emptyset$ if and only if $N \neq 1_M$.
2. ${}^{ps}\sqrt{N} \neq 1_M$ if and only if $\mathcal{G}(N) \neq \emptyset$ if and only if $N \neq 1_M$, when M is multiplication lattice module.

Proof. 1. Suppose that $N \neq 1_M$. Then $(N : 1_M) \neq 1_L$ and therefore $(N : 1_M) \leq p$ for some prime element p of L . Since the natural map ψ^{ps} is surjective, there exists $P \in \text{Spec}^{ps}(M)$ such that $p = \sqrt{(P : 1_M)}$. Thus $(N : 1_M) \leq \sqrt{(P : 1_M)}$. By Properties 2.1(2) and (4), $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$. Hence $P \in \mathcal{F}(N)$ and $\mathcal{F}(N) \neq \emptyset$. Now, assume that $\mathcal{F}(N) \neq \emptyset$ and $P \in \mathcal{F}(N)$. It follows that ${}^{zps}\sqrt{N} \leq P \neq 1_M$, by Proposition 3.2(1). Conversely, if ${}^{zps}\sqrt{N} \neq 1_M$, then $N \neq 1_M$.

2. Let $N \neq 1_M$. Then $(N : 1_M) \neq 1_L$ and therefore $(N : 1_M) \leq p$ for some prime element p of L . As ψ^{ps} is surjective natural map, there exists $P \in \text{Spec}^{ps}(M)$ such that $p = \sqrt{(P : 1_M)} = (\sqrt{P} : 1_M)$. Hence $(N : 1_M) \leq (\sqrt{P} : 1_M)$. Further, it implies that $(N : 1_M)1_M \leq (\sqrt{P} : 1_M)1_M \leq \sqrt{P}$, by Lemma 2.2(6) and (10). Since M is multiplication, $(N : 1_M)1_M = N$. Thus, $N \leq \sqrt{P}$ and $P \in \mathcal{G}(N)$. Hence $\mathcal{G}(N) \neq \emptyset$. Now, let $\mathcal{G}(N) \neq \emptyset$ and $P \in \mathcal{G}(N)$. Then ${}^{ps}\sqrt{N} \leq P \neq 1_M$, by Proposition 3.2(5). Converse part is obvious. $\square$

Proposition 3.5. *Let M be a multiplication lattice module over a C -lattice L . Then the following statements hold:*

1. $\mathcal{F}(N) = \mathcal{G}(N)$.
2. ${}^{zps}\sqrt{N} = {}^{ps}\sqrt{N}$.
3. ${}^{zps}\sqrt{{}^{ps}\sqrt{N}} = {}^{ps}\sqrt{{}^{zps}\sqrt{N}} = {}^{ps}\sqrt{N} = {}^{zps}\sqrt{N}$.

Proof. 1. By Lemma 3.1(2), $\mathcal{G}(N) \subseteq \mathcal{F}(N)$. For reverse inclusion, let $P \in \mathcal{F}(N)$, then $\sqrt{(N : 1_M)} \leq \sqrt{(P : 1_M)}$. It implies that $(N : 1_M) \leq \sqrt{(N : 1_M)} \leq (\sqrt{P} : 1_M)$, by Properties 2.1(1). Hence $(N : 1_M)1_M \leq (\sqrt{P} : 1_M)1_M \leq \sqrt{P}$, by Lemma 2.2(6) and (10). Since M is multiplication, $(N : 1_M)1_M = N$. Thus, $N \leq \sqrt{P}$ and $P \in \mathcal{G}(N)$. Consequently, $\mathcal{F}(N) = \mathcal{G}(N)$.

2. By part (1), $\mathcal{F}(N) = \mathcal{G}(N)$, therefore ${}^{zps}\sqrt{N} = {}^{ps}\sqrt{N}$.

3. By Proposition 3.2(3), ${}^{zps}\sqrt{N} = {}^{zps}\sqrt{{}^{zps}\sqrt{N}}$. Hence, by part (2), ${}^{zps}\sqrt{{}^{ps}\sqrt{N}} = {}^{ps}\sqrt{{}^{zps}\sqrt{N}} = {}^{ps}\sqrt{{}^{zps}\sqrt{N}} = {}^{zps}\sqrt{{}^{zps}\sqrt{N}} = {}^{zps}\sqrt{N} = {}^{ps}\sqrt{N}$. $\square$

Let $\text{Spec}_p^{ps}(M) = \{P \in \text{Spec}^{ps}(M) \mid \sqrt{(P : 1_M)} = p\}$, for each prime element p of L .

Proposition 3.6. *Let M be a lattice module over a C -lattice L and $P \in \text{Spec}_p^{ps}(M)$ for some prime element p of L . Then the following statements hold:*

$$1. {}^{ps}\sqrt{P} = {}^{ps}\sqrt{P \vee p1_M}.$$

$$2. \text{ If } M \text{ is a multiplication lattice module, then } {}^{zps}\sqrt{P} = {}^{zps}\sqrt{P \vee p1_M}.$$

Proof. 1. We have, $P \leq P \vee p1_M$, therefore $\mathcal{G}(P \vee p1_M) \subseteq \mathcal{G}(P)$, by Lemma 3.1(1). Hence $\wedge \mathcal{G}(P) \leq \wedge \mathcal{G}(P \vee p1_M)$, i.e., ${}^{ps}\sqrt{P} \leq {}^{ps}\sqrt{P \vee p1_M}$. Now, let $P_i \in \text{Spec}_p^{ps}(M)$ such that $P \leq \sqrt{P_i}$, i.e., $P_i \in \mathcal{G}(P)$, for $i \in I$. Then $\sqrt{(P : 1_M)} \leq (\sqrt{P_i} : 1_M)$, by Lemma 2.2(4) and Properties 2.1(2) and (4). Therefore $p1_M \leq p_i1_M$, by Lemma 2.2(6) and hence $P \vee p1_M \leq \sqrt{P_i} \vee p_i1_M \leq \sqrt{P_i} \vee \sqrt{P_i} \leq \sqrt{P_i}$, which implies that $P_i \in \mathcal{G}(P \vee p1_M)$. Consequently, $\mathcal{G}(P) \subseteq \mathcal{G}(P \vee p1_M)$ and hence ${}^{ps}\sqrt{P \vee p1_M} \leq {}^{ps}\sqrt{P}$. Finally, ${}^{ps}\sqrt{P} = {}^{ps}\sqrt{P \vee p1_M}$.

2. Let M be a multiplication lattice module. Then by Proposition 3.5(2), ${}^{zps}\sqrt{N} = {}^{ps}\sqrt{N}$. From this, it follows that ${}^{zps}\sqrt{P} = {}^{zps}\sqrt{P \vee p1_M}$. $\square$

A topological space X is Noetherian, if the open (resp. closed) subsets of X satisfies the ascending (resp. descending) chain condition[8].

We are now ready to state and prove the main theorem on Noetherian space.

Theorem 3.1. *Let M be a lattice module over a C -lattice L . Then the following statements are equivalent:*

1. $\text{Spec}^{ps}(M)$ with Zariski-like topology τ_1 is a Noetherian space.

2. Every Zariski pseudo-primary radical element of M satisfies ascending chain condition.

Proof. (1) $\implies$ (2) : Assume that $\text{Spec}^{ps}(M)$ is a Noetherian space. Let $N_1 \leq N_2 \leq \dots \leq N_i \leq \dots$ be an ascending chain of Zariski pseudo-primary radical elements of M . Then by Lemma 3.1(1), $\mathcal{F}(N_1) \supseteq \mathcal{F}(N_2) \supseteq \dots \supseteq \mathcal{F}(N_i) \supseteq \dots$ is a descending chain of closed sets $\mathcal{F}(N_i)$ of $\text{Spec}^{ps}(M)$. Since $\text{Spec}^{ps}(M)$ is Noetherian space, its closed sets satisfy descending chain condition, therefore there exists a positive integer m such that $\mathcal{F}(N_n) = \mathcal{F}(N_m)$ for every $n \geq m$. Hence ${}^{zps}\sqrt{N_n} = {}^{zps}\sqrt{N_m}$ and $N_n = N_m$ for every $n \geq m$. Consequently, the desired conclusion follows.

(2) $\implies$ (1) : Assume that for $N_i \in M$, $\mathcal{F}(N_1) \supseteq \mathcal{F}(N_2) \supseteq \dots \supseteq \mathcal{F}(N_i) \supseteq \dots$ is a descending chain of closed sets $\mathcal{F}(N_i)$ of $\text{Spec}^{ps}(M)$. Then $\wedge \mathcal{F}(N_1) \leq \wedge \mathcal{F}(N_2) \leq \dots \leq \wedge \mathcal{F}(N_i) \leq \dots$. This implies that ${}^{zps}\sqrt{N_1} \leq {}^{zps}\sqrt{N_2} \leq \dots \leq {}^{zps}\sqrt{N_i} \leq \dots$ is an ascending chain of Zariski pseudo-primary radical of elements

of M . Therefore, there exists positive integer m such that ${}^{zp^s}\sqrt{N_n} = {}^{zp^s}\sqrt{N_m}$. Hence, by Proposition 3.2(2), $\mathcal{F}(N_n) = \mathcal{F}(N_m)$. Thus, $\text{Spec}^{ps}(M)$ is a Noetherian space. $\square$

Theorem 3.2. *Let M be a top lattice module over a C -lattice L . Then the following statements are equivalent:*

1. $\text{Spec}^{ps}(M)$ with quasi-Zariski topology τ_2 is a Noetherian space.
2. Every pseudo-primary radical element of M satisfies ascending chain condition.

Proof. (1) $\implies$ (2) : Suppose that $\text{Spec}^{ps}(M)$ is a Noetherian space. Consider an ascending chain $N_1 \leq N_2 \leq \dots \leq N_i \leq \dots$ of pseudo-primary radical elements of M . Then by using Lemma 3.1(1), $\mathcal{G}(N_1) \supseteq \mathcal{G}(N_2) \supseteq \dots \supseteq \mathcal{G}(N_i) \supseteq \dots$ is a descending chain of closed sets $\mathcal{G}(N_i)$ of $\text{Spec}^{ps}(M)$. Given that $\text{Spec}^{ps}(M)$ is a Noetherian space, therefore its closed sets satisfy descending chain condition, hence there exists a positive integer m such that $\mathcal{G}(N_n) = \mathcal{G}(N_m)$ for every $n \geq m$. Hence ${}^{ps}\sqrt{N_n} = {}^{ps}\sqrt{N_m}$ and $N_n = N_m$ for every $n \geq m$.

(2) $\implies$ (1) : Let $\mathcal{G}(N_1) \supseteq \mathcal{G}(N_2) \supseteq \dots \supseteq \mathcal{G}(N_i) \supseteq \dots$ is a descending chain of closed sets $\mathcal{G}(N_i)$ of $\text{Spec}^{ps}(M)$. Therefore, $\wedge \mathcal{G}(N_1) \leq \wedge \mathcal{G}(N_2) \leq \dots \leq \wedge \mathcal{G}(N_i) \leq \dots$, i.e., ${}^{ps}\sqrt{N_1} \leq {}^{ps}\sqrt{N_2} \leq \dots \leq {}^{ps}\sqrt{N_i} \leq \dots$ is an ascending chain of pseudo-primary radical of elements of M . Thus there exists a positive integer m such that ${}^{ps}\sqrt{N_n} = {}^{ps}\sqrt{N_m}$. By using Proposition 3.2(6), $\mathcal{G}(N_n) = \mathcal{G}(N_m)$. Consequently, $\text{Spec}^{ps}(M)$ is a Noetherian space. $\square$

Proposition 3.7. *Let M be a multiplication lattice module over a C -lattice L . Then the following are equivalent:*

1. $\text{Spec}^{ps}(M)$ with Zariski-like topology τ_1 is a Noetherian space.
2. $\text{Spec}^{ps}(M)$ with quasi-Zariski topology τ_2 is a Noetherian space.

Proof. It follows from Proposition 3.5, Theorem 3.1 and Theorem 3.2. $\square$

Acknowledgment

The authors would like to thank the referee for the valuable suggestions and comments.

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