



The Pythagorean angle lattice and the omni-metallic framework: arithmetic bridges and spectral relations

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Abstract: This paper establishes a synthesis between the Pythagorean angle lattice and the Omni-Metallic family, a four-parameter generalisation of the classical metallic means defined as the unique positive real root of a prescribed polynomial. The arithmetic kernel of the Omni-Metallic side is the counting function that records the number of Universal Metallic representations of a positive integer. The Mellin transform of this kernel is evaluated in closed form as a combination of the Riemann zeta function at shifted arguments, an identity here called the Bridge Identity, and its finite part at the double pole yields an alternative expression for the circle constant. The Borel transform of the same kernel is evaluated at unity and yields an alternative expression for the base of the natural logarithm. A regularised infinite product over the kernel is computed by zeta regularisation and is expressed in terms of the Glaisher and Kinkelin constant. The Generalized Integer Value Theorem is proved for all admissible parameters, characterising exactly when a member of the Omni-Metallic family is a positive integer. A Quarter-Angle Bridge is proved, giving in exact and unconditional form the value of the lattice under quarter-angle projection for every primitive Gaussian integer, and identifying precisely when that value is a classical metallic mean of integer index. The Crown Identity connecting the Universal Metallic family to primitive Pythagorean triples is established. For the cubic projection an exact irreducibility criterion is obtained: the associated cubic is reducible over the rationals precisely when the generating Gaussian integer is itself a cube, and explicit reducible instances are exhibited. The integer values attained by the lattice are determined completely: they occur only at even projection indices, and the associated multiplicities are computed. A spectral decomposition over odd projection indices is stated as a conjecture.

Key words: Pythagorean triples, metallic means, Gaussian integers, Dirichlet series, Mellin transform, spectral decomposition

1. Introduction

The classical metallic means, introduced by de Spinadel, are the positive roots of $x^2 - nx - 1 = 0$, $n \in \mathbb{N}$, with explicit form

$$M_n = \frac{n + \sqrt{n^2 + 4}}{2}.$$

These numbers include the Golden Ratio ($n = 1$), the Silver Ratio ($n = 2$), and their higher analogues [1–3].

In recent work [4], a geometric bijection was established between primitive Pythagorean triples and metallic means. For a primitive Pythagorean triple (a, b, c) with $c - b \in \{1, 2, 8\}$, the governing parameter

$n = 2a/(c - b)$ is an integer, and the relevant acute angle θ satisfies

$$\cot\left(\frac{\theta}{4}\right) = \frac{n + \sqrt{n^2 + 4}}{2} = M_n.$$

The metallic means family was subsequently generalized through the Universal Metallic family $\mathcal{U}_{2,1}(n, N) = n + \sqrt{n^2 + N}$, and ultimately the four-parameter Omni-Metallic family $\mathcal{U}_{m,r}(n, N)$, defined as the unique positive root of $x^m - 2nx^{m-1} - N^r = 0$. The Omni-Metallic framework [5] contains the classical metallic means as the special case $m = 2$, $r = 1$, $N = 4$:

$$\mathcal{U}_{2,1}(n, 4) = n + \sqrt{n^2 + 4} = 2M_n.$$

The Pythagorean angle lattice and the Omni-Metallic framework share a common arithmetic core. The Pythagorean angle lattice is generated by the angles of primitive Pythagorean triples under division by integers.

The key structural observation is that primitive Gaussian integers $W = u + vi$ with $u > v > 0$, $\gcd(u, v) = 1$, and $u \not\equiv v \pmod{2}$ induce a natural parity structure, and the parity of k likewise governs the counting function $r(k) = \lfloor (k - 1)/2 \rfloor$ through its $(-1)^k$ term, which in turn generates the relations to the constants π and e .

This paper is organized as follows. Section 2 presents the Omni-Metallic family and its fundamental properties. Section 3 presents the Pythagorean angle lattice. Section 4 establishes the exact quarter-angle bridge between the lattice and the classical metallic means, and determines the integer spectrum of the lattice for every D . Section 5 presents the exact summatory function. Section 6 presents the Mellin transform and a relation to π . Section 7 presents the Borel transform and a relation to e . Section 8 establishes the product formula. Section 9 presents the hierarchy of geometric bridges. Section 10 presents the cubic case. Section 11 presents a conjectural treatment of general odd D . Section 12 presents the general Dirichlet series. Section 13 presents the super-exponential approximation theorem. Section 14 presents a conjectural complete spectral decomposition. Section 15 lists open problems, and Section 16 concludes.

2. The Omni-Metallic family

2.1. Definition

For integers $m \geq 2$ and $n, N, r \in \mathbb{N}$, let $\mathcal{U}_{m,r}(n, N)$ denote the unique positive real root of

$$x^m - 2nx^{m-1} - N^r = 0. \tag{1}$$

2.2. The Mirror Identity

Theorem 2.1 (Mirror Identity). *For all parameters $m \geq 2$, $n, N, r \in \mathbb{N}$,*

$$\mathcal{U}_{m,r}(n, N) - \frac{N^r}{\mathcal{U}_{m,r}(n, N)^{m-1}} = 2n. \tag{2}$$

Proof. From the defining equation (1), dividing by $\mathcal{U}_{m,r}(n, N)^{m-1}$ gives $\mathcal{U}_{m,r}(n, N) = 2n + N^r / \mathcal{U}_{m,r}(n, N)^{m-1}$. Rearranging yields (2). $\square$

2.3. The Generalized Integer Value Theorem

Theorem 2.2 (Generalized Integer Value Theorem). *For $m \geq 2$, $n, N, r \in \mathbb{N}$,*

$$\mathfrak{U}_{m,r}(n, N) = k \iff N^r = k^{m-1}(k - 2n). \quad (3)$$

Proof. Substitute $x = k$ into (1): $k^m - 2nk^{m-1} - N^r = 0$, hence $N^r = k^{m-1}(k - 2n)$. Conversely, if $N^r = k^{m-1}(k - 2n)$, then k satisfies the defining equation. Uniqueness follows from Descartes' rule of signs: the polynomial $x^m - 2nx^{m-1} - N^r$ has exactly one sign change, hence exactly one positive real root. $\square$

Corollary 2.1. *For $m = 2$, $r = 1$, this reduces to the Integer Value Theorem:*

$$\mathfrak{U}_{2,1}(n, N) = k \iff N = k(k - 2n). \quad (4)$$

The number of representations with $n \geq 1$, $N \geq 1$ is

$$r(k) := \#\{(n, N) \in \mathbb{N}^2 : \mathfrak{U}_{2,1}(n, N) = k\} = \left\lfloor \frac{k-1}{2} \right\rfloor. \quad (5)$$

Proof. For fixed k , the condition $N = k(k - 2n)$ requires $N > 0$, so $k - 2n > 0$, i.e. $n < k/2$. Since $n \geq 1$, the possible values are $n = 1, 2, \dots, \lfloor (k-1)/2 \rfloor$. Each such n gives a unique N . Therefore the number of representations is $\lfloor (k-1)/2 \rfloor$. $\square$

3. The Pythagorean angle lattice

3.1. Primitive Pythagorean triples

A primitive Pythagorean triple is a triple $(a, b, c) \in \mathbb{N}^3$ satisfying $a^2 + b^2 = c^2$, $\gcd(a, b, c) = 1$. By Euclid's parametrization [6], there exist $u > v > 0$, $\gcd(u, v) = 1$, $u \not\equiv v \pmod{2}$ such that

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2.$$

3.2. Gaussian integer correspondence

Lemma 3.1. *Every primitive Pythagorean triple corresponds to a Gaussian integer*

$$a + bi = (u + vi)^2 = W^2, \quad W = u + vi \in \mathbb{Z}[i]. \quad (6)$$

3.3. The cotangent projection

For $W \in \mathbb{Z}[i]$ and $D \in \mathbb{N}$, define, using the principal branch $\theta = \arg(W^2) \in (0, \pi)$,

$$\phi_D(W) = \cot\left(\frac{\theta}{D}\right), \quad \theta = \arg(W^2). \quad (7)$$

Explicitly,

$$\phi_D(W) = i \frac{W^{2/D} + \overline{W}^{2/D}}{W^{2/D} - \overline{W}^{2/D}}, \quad (8)$$

where $W^{2/D}$ denotes the principal branch. Indeed, writing $Z = W^2 = re^{i\theta}$ with $\theta \in (0, \pi)$,

$$i \frac{Z^{1/D} + \overline{Z}^{1/D}}{Z^{1/D} - \overline{Z}^{1/D}} = i \frac{e^{i\theta/D} + e^{-i\theta/D}}{e^{i\theta/D} - e^{-i\theta/D}} = i \frac{2 \cos(\theta/D)}{2i \sin(\theta/D)} = \cot(\theta/D).$$

3.4. The lattice

Define the Pythagorean angle lattice as

$$\mathcal{L} = \{\phi_D(W) : W \in \mathbb{Z}[i]_{\text{prim}}, D \in \mathbb{N}\}. \quad (9)$$

3.5. Density theorem

Lemma 3.2. *The set $\{v/u : u > v > 0, \gcd(u, v) = 1, u \not\equiv v \pmod{2}\}$ is dense in $(0, 1)$.*

Proof. Let $t \in (0, 1)$ and $\varepsilon > 0$. By Euclid's theorem there are infinitely many primes, so choose an odd prime P with $P > 2/\varepsilon$, $P > 2/t$ and $P > 2/(1-t)$. Let v be an even integer nearest to tP , so that $|v - tP| \leq 1$, and set $u = P$. Then v is even and u is odd, so $u \not\equiv v \pmod{2}$. Since $0 < v < u$ and u is prime, $\gcd(u, v) = 1$. Finally

$$\left| \frac{v}{u} - t \right| = \frac{|v - tP|}{P} \leq \frac{1}{P} < \varepsilon.$$

Hence every $t \in (0, 1)$ is a limit of such fractions v/u , and the set is dense in $(0, 1)$. $\square$

Theorem 3.1 (Density). *The lattice $\mathcal{L}$ is dense in $\mathbb{R}^+$.*

Proof. The primitive Gaussian integers $W = u + vi$ with $u > v > 0$, $\gcd(u, v) = 1$, $u \not\equiv v \pmod{2}$, yield arguments $\arg(W^2) = 2 \arctan(v/u)$. By Lemma 3.2, the set of rational slopes v/u is dense in $(0, 1)$. Therefore $\{\arg(W^2)\}$ is dense in $(0, \pi/2)$.

Restricting to $D = 1$, we have $\phi_1(W) = \cot(\arg(W^2))$. The cotangent function is a continuous, strictly decreasing bijection from $(0, \pi/2)$ to $\mathbb{R}^+$. Continuous bijections carry dense sets to dense sets. Hence $\{\phi_1(W)\} \subset \mathcal{L}$ is dense in $\mathbb{R}^+$. Therefore $\mathcal{L}$ is dense in $\mathbb{R}^+$. $\square$

4. The quarter-angle bridge and the integer projection

We first establish, in exact form, the bridge between the lattice and the classical metallic means at $D = 4$.

Theorem 4.1 (Quarter-Angle Bridge). *Let $W = u + vi$ with $u > v > 0$, $\gcd(u, v) = 1$, $u \not\equiv v \pmod{2}$, and let $\theta = \arg(W^2)$. Then*

$$\phi_4(W) = \cot\left(\frac{\theta}{4}\right) = \frac{u + \sqrt{u^2 + v^2}}{v} = M_{2u/v}. \quad (10)$$

In particular $\phi_4(W)$ is a classical metallic mean M_n with $n \in \mathbb{N}$ if and only if $v \in \{1, 2\}$, in which case $n = 2u/v$ and $\phi_4(W) = M_n = \frac{1}{2}\mathcal{U}_{2,1}(n, 4)$.

Proof. Since $\theta = \arg(W^2) = 2 \arg(W) = 2\psi$ with $\psi = \arctan(v/u)$, we have $\theta/4 = \psi/2$, and the half-angle identity gives

$$\cot\left(\frac{\psi}{2}\right) = \frac{1 + \cos \psi}{\sin \psi} = \frac{1 + u/\sqrt{u^2 + v^2}}{v/\sqrt{u^2 + v^2}} = \frac{u + \sqrt{u^2 + v^2}}{v}.$$

On the other hand, with $n = 2u/v$,

$$M_n = \frac{n + \sqrt{n^2 + 4}}{2} = \frac{2u/v + (2/v)\sqrt{u^2 + v^2}}{2} = \frac{u + \sqrt{u^2 + v^2}}{v},$$

which proves (10). Finally, $n = 2u/v \in \mathbb{N}$ requires $v \mid 2u$; since $\gcd(u, v) = 1$ this holds precisely when $v \mid 2$, i.e. $v \in \{1, 2\}$. The identity $\mathcal{U}_{2,1}(n, 4) = n + \sqrt{n^2 + 4} = 2M_n$ is immediate from the definition. $\square$

Example 4.1. For $W = 2 + i$ we have $v = 1$, $n = 4$, and $\phi_4(W) = 2 + \sqrt{5} = M_4 = 4.2360679\dots$; the associated triple is $(3, 4, 5)$ with $c - b = 2$. For $W = 3 + 2i$ we have $v = 2$, $n = 3$, and $\phi_4(W) = (3 + \sqrt{13})/2 = M_3 = 3.3027756\dots$; the associated triple is $(5, 12, 13)$ with $c - b = 8$.

Remark 4.1. The condition $v \in \{1, 2\}$ corresponds exactly to $c - b \in \{2, 8\}$ in the associated triple, since $c - b = u^2 + v^2 - 2uv$ or $2v^2$ according to which leg is designated b . This recovers, in Gaussian-integer form, the one-two-eight rule of [4].

We next record the Crown Identity, which holds for every admissible triple and every N .

Theorem 4.2 (Crown Identity). *Let (a, b, c) be a primitive Pythagorean triple for which $n := 2a/(c - b) \in \mathbb{N}$. Then for every $N \in \mathbb{N}$,*

$$\mathcal{U}_{2,1}(n, N) = \frac{2a}{c - b} + \sqrt{\frac{8c}{c - b} + (N - 4)}. \quad (11)$$

Proof. Since $a^2 = c^2 - b^2 = (c - b)(c + b)$,

$$n^2 = \frac{4a^2}{(c - b)^2} = \frac{4(c + b)}{c - b} = \frac{8c - 4(c - b)}{c - b} = \frac{8c}{c - b} - 4.$$

Hence $n^2 + N = 8c/(c - b) + (N - 4)$, and $\mathcal{U}_{2,1}(n, N) = n + \sqrt{n^2 + N}$ gives (11). $\square$

4.1. The integer projection

By Corollary 2.1, the number of Universal Metallic representations of an integer k is exactly $r(k)$. Accordingly, let μ_{int} denote the counting measure on $\mathbb{N}_{\geq 2}$ defined by

$$\mu_{\text{int}}(\{k\}) := \#\{(n, N) \in \mathbb{N}^2 : \mathcal{U}_{2,1}(n, N) = k\} = r(k). \quad (12)$$

All spectral results of Sections 5–8 depend only on (12), which is established rigorously by Corollary 2.1.

The corresponding integer values on the lattice side are determined exactly by the following theorem.

Theorem 4.3 (Integer spectrum of the lattice). *Let $k \geq 2$ be an integer and $W \in \mathbb{Z}[i]_{\text{prim}}$, $D \in \mathbb{N}$. Then $\phi_D(W) = k$ forces*

$$\arg(W^2) = D \arccot(k), \quad D \text{ even}, \quad D \arccot(k) < \pi, \quad (13)$$

and for each such D there is at most one primitive W . In particular the set of integers attained by the lattice, together with their multiplicities, is completely determined.

Proof. Since $\arg(W^2) \in (0, \pi)$ and $D \geq 1$, we have $\arg(W^2)/D \in (0, \pi)$, where $\cot$ is injective; hence $\phi_D(W) = k$ is equivalent to $\arg(W^2) = D \arccot(k)$, which requires $D \arccot(k) < \pi$. As $\arg(k + i) = \arccot(k)$, the Gaussian integer W^2 lies in the direction of $(k + i)^D$; write $(k + i)^D = X + Yi$. From $W^2 = (u^2 - v^2) + 2uv i$ and the half-angle formula,

$$\frac{v}{u} = \frac{\sqrt{X^2 + Y^2} - X}{Y},$$

which is rational precisely when $X^2 + Y^2$ is a perfect square. Removing a common factor g divides $X^2 + Y^2$ by g^2 and so does not affect this condition, and $X^2 + Y^2 = (k^2 + 1)^D$. Since $k^2 + 1$ is a perfect square for no $k \geq 1$, the power $(k^2 + 1)^D$ is a perfect square if and only if D is even. The primitive W in a given direction is unique, which gives the last assertion. $\square$

Example 4.2. Applying Theorem 4.3 gives, for instance, $\phi_2(2+i) = 2$, $\phi_4(4+3i) = 3$, $\phi_2(4+i) = \phi_4(15+8i) = \phi_6(52+47i) = 4$ and $\phi_4(12+5i) = 5$. The resulting multiplicities are listed in Table 1.

Table 1. Multiplicity of the integer k in the lattice, compared with $r(k)$.

k	2	3	4	5	6	7	8	9
admissible D	2	4	2,4,6	4	2,4,6,8	4,8	2,...,12	4,8,12
lattice multiplicity	1	1	3	1	4	2	6	3
$r(k) = \lfloor (k-1)/2 \rfloor$	0	1	1	2	2	3	3	4

Remark 4.2. Table 1 shows that the multiplicity of k in the lattice is not $r(k)$; the same holds after restricting to the sub-lattice generated by triples with $c - b \in \{1, 2, 8\}$, where the multiplicities are 1, 1, 1, 0, 1, 0, 1, 0 for $k = 2, \dots, 9$. Accordingly the counting function $r(k)$ is an invariant of the Omni-Metallic side, established by Corollary 2.1, and the two frameworks are joined through Theorem 4.1 and Theorem 4.2 rather than through a common multiplicity. In particular the parametrization $N = 8c/(c-b) + (D-4)$ does not realise $\phi_D(W)$: by Theorem 4.2 it gives $\mathcal{U}_{2,1}(n, N) = n + \sqrt{2 \cdot 8c/(c-b) + D - 8}$, which for $(a, b, c) = (3, 4, 5)$ and $D = 4$ equals $6 + \sqrt{76} = 14.7178\dots$, whereas $\phi_4(2+i) = 4.2360\dots$ by Theorem 4.1.

The generating function of μ_{int} is

$$F(x) = \sum_{k=2}^{\infty} r(k)x^k = \frac{x^3}{(1-x)(1-x^2)}. \quad (14)$$

Lemma 4.1 (Exact decomposition of the generating function). *The generating function has the exact partial fraction decomposition*

$$F(x) = 1 - \frac{5}{4} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{(1-x)^2} - \frac{1}{4} \cdot \frac{1}{1+x}. \quad (15)$$

Consequently,

$$r(k) = \frac{k}{2} - \frac{3}{4} - \frac{1}{4}(-1)^k. \quad (16)$$

Proof. Direct computation of the partial fraction decomposition of $x^3/((1-x)(1-x^2))$ gives (15). Expanding (15) as a power series and comparing coefficients yields (16). $\square$

5. The exact summatory function

Theorem 5.1. *The summatory function of $r(k)$ is*

$$S(N) = \sum_{k=2}^N r(k) = \left\lfloor \frac{(N-1)^2}{4} \right\rfloor. \quad (17)$$

Proof. Using the closed form (16),

$$\begin{aligned} S(N) &= \frac{1}{2} \sum_{k=2}^N k - \frac{3}{4}(N-1) - \frac{1}{4} \sum_{k=2}^N (-1)^k \\ &= \frac{N^2}{4} - \frac{N}{2} + \frac{1}{4} - \frac{1}{4}(-1)^N = \left\lfloor \frac{(N-1)^2}{4} \right\rfloor. \end{aligned}$$

□

6. The Mellin transform and a relation to π

The Mellin transform of μ_{int} is

$$Z(s) = \int_1^\infty x^{-s} d\mu_{\text{int}}(x) = \sum_{k=2}^\infty \frac{r(k)}{k^s}. \quad (18)$$

Theorem 6.1 (Bridge Identity). *For $\Re(s) > 2$,*

$$Z(s) = \frac{1}{2}\zeta(s-1) - \frac{1}{2}(1+2^{-s})\zeta(s). \quad (19)$$

Proof. Substitute (16) into (18):

$$Z(s) = \frac{1}{2} \sum_{k=2}^\infty \frac{k}{k^s} - \frac{3}{4} \sum_{k=2}^\infty \frac{1}{k^s} - \frac{1}{4} \sum_{k=2}^\infty \frac{(-1)^k}{k^s}.$$

Using the standard evaluations [7, 10]

$$\sum_{k=2}^\infty \frac{k}{k^s} = \zeta(s-1) - 1, \quad \sum_{k=2}^\infty \frac{1}{k^s} = \zeta(s) - 1, \quad \sum_{k=2}^\infty \frac{(-1)^k}{k^s} = (2^{1-s} - 1)\zeta(s) + 1,$$

and combining yields (19). □

Corollary 6.1. *The pole at $s = 2$ has residue $1/2$. The finite part is*

$$\text{FP}_{s=2} Z(s) = \frac{\gamma}{2} - \frac{5\pi^2}{48}. \quad (20)$$

Proof. Expand $Z(2+\epsilon)$. Using $\zeta(1+\epsilon) = 1/\epsilon + \gamma + O(\epsilon)$ and $\zeta(2+\epsilon) = \pi^2/6 + \zeta'(2)\epsilon + O(\epsilon^2)$ [7], and $1+2^{-(2+\epsilon)} = 5/4 - (\log 2/4)\epsilon + O(\epsilon^2)$, we obtain

$$Z(2+\epsilon) = \frac{1}{2\epsilon} + \frac{\gamma}{2} - \frac{5\pi^2}{48} + O(\epsilon).$$

□

Corollary 6.2. *Using the standard identity $\zeta(2) = \pi^2/6$ [7], we obtain the relation*

$$\pi = \left[\frac{48}{5} \left(\frac{\gamma}{2} - \text{FP}_{s=2} Z(s) \right) \right]^{1/2}. \quad (21)$$

Remark 6.1. Equation (21) provides an alternative expression for π in terms of the lattice's spectral data. It is not an independent derivation of π ; it uses the standard identity $\zeta(2) = \pi^2/6$ [7].

6.1. The constant π and lattice spacing

Let $P(X)$ be the number of primitive Gaussian integers with $|W| \leq X$. Counting primitive Pythagorean triples with hypotenuse $\leq X$ gives

$$P(X) = \frac{1}{2\pi} X^2 + O(X \log X). \quad (22)$$

Corollary 6.3.

$$\pi = \lim_{X \rightarrow \infty} \frac{X^2}{2P(X)}. \quad (23)$$

Remark 6.2. The constant $1/(2\pi)$ in (22) follows from the standard lattice-point count: the density of coprime pairs is $6/\pi^2$, the opposite-parity fraction is $2/3$, the quarter-disc area gives $\pi/4$, and the restriction $u > v$ gives an additional factor of $1/2$, yielding $(6/\pi^2) \cdot (2/3) \cdot (\pi/4) \cdot (1/2) = 1/(2\pi)$.

7. The Borel transform and a relation to e

The Borel transform of the generating function F is

$$B(t) = \sum_{k=2}^{\infty} \frac{r(k)}{k!} t^k. \quad (24)$$

Theorem 7.1. At $t = 1$,

$$B(1) = \sum_{k=2}^{\infty} \frac{r(k)}{k!} = 1 - \frac{1}{2} \cosh(1). \quad (25)$$

Proof. Substitute (16) into the sum:

$$B(1) = \frac{1}{2} \sum_{k=2}^{\infty} \frac{k}{k!} - \frac{3}{4} \sum_{k=2}^{\infty} \frac{1}{k!} - \frac{1}{4} \sum_{k=2}^{\infty} \frac{(-1)^k}{k!}.$$

The standard exponential series evaluations give $\sum_{k \geq 2} k/k! = e - 1$, $\sum_{k \geq 2} 1/k! = e - 2$, and $\sum_{k \geq 2} (-1)^k/k! = e^{-1}$, whence $B(1) = 1 - (e + e^{-1})/4 = 1 - \frac{1}{2} \cosh(1)$. $\square$

Corollary 7.1. Using the identity $\cosh(1) = (e + e^{-1})/2$ [7], we obtain

$$e + e^{-1} = 4(1 - B(1)). \quad (26)$$

Let $a = 1 - B(1)$. Then

$$e = 2a + \sqrt{4a^2 - 1}. \quad (27)$$

Remark 7.1. Equation (27) provides an alternative expression for e in terms of the lattice's spectral data. It uses the standard identity $\cosh(1) = (e + e^{-1})/2$ [7].

8. The product formula and the Glaisher–Kinkelin constant

Define the regularized infinite product

$$P = \prod_{k=2}^{\infty} k^{r(k)}. \quad (28)$$

Theorem 8.1 (Product formula).

$$P = e^{-1/24} A^{1/2} 2^{-1/4} \pi^{-1/2}, \quad (29)$$

where A is the Glaisher–Kinkelin constant.

Proof. We have $\log P = -Z'(0)$, where $Z(s) = \frac{1}{2}\zeta(s-1) - \frac{1}{2}(1+2^{-s})\zeta(s)$. Differentiating,

$$Z'(s) = \frac{1}{2}\zeta'(s-1) - \frac{1}{2}(1+2^{-s})\zeta'(s) + \frac{1}{2}\ln 2 \cdot 2^{-s}\zeta(s).$$

At $s = 0$,

$$Z'(0) = \frac{1}{2}\zeta'(-1) - \zeta'(0) + \frac{1}{2}\ln 2 \zeta(0).$$

Using the standard values $\zeta(0) = -1/2$, $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$, and $\zeta'(-1) = \frac{1}{12} - \ln A$ [7, 10], we get

$$Z'(0) = \frac{1}{24} - \frac{1}{2}\ln A + \frac{1}{2}\ln(2\pi) - \frac{1}{4}\ln 2.$$

Thus $\log P = -\frac{1}{24} + \frac{1}{2}\ln A - \frac{1}{2}\ln(2\pi) + \frac{1}{4}\ln 2$. Exponentiating and simplifying yields (29). $\square$

Corollary 8.1.

$$\pi = e^{-1/12} A 2^{-1/2} P^{-2}. \quad (30)$$

Proof. From (29), raising to the power -2 gives $P^{-2} = e^{1/12} A^{-1} 2^{1/2} \pi$. Dividing both sides by $e^{1/12} A^{-1} 2^{1/2}$ yields $\pi = e^{-1/12} A 2^{-1/2} P^{-2}$. $\square$

Remark 8.1. Numerically, $P = 0.5153334\dots$ and (30) returns $3.1415926\dots$, confirming the identity. The product formula (29) expresses π in terms of the Omni-Metallic product P and the Glaisher–Kinkelin constant A . It relies on the standard values of the zeta function and its derivative at negative integers [7, 10].

9. The hierarchy of geometric bridges

The three geometric bridges of Table 2 apply to different levels of the framework.

Table 2. The hierarchy of geometric and algebraic bridges.

Level	Family	Geometric bridge	Algebraic bridge
1	Classical Metallic Means	$\cot(\theta/4) = M_n$	$N = 4$
2	Universal Metallic Family	Crown Identity (11)	$n^2 = 8c/(c-b) - 4$
3	Omni-Metallic Family	none (Remark 9.1)	$N^r = k^{m-1}(k-2n)$

Remark 9.1. Write $(y + i)^D = P_D(y) + iQ_D(y)$; the monic polynomial whose roots are $\cot((\theta + k\pi)/D)$ is $S_D(y) = P_D(y) - cQ_D(y)$. Since P_D contributes only in degrees $D - 2j$ and Q_D only in degrees $D - 1 - 2j$, the coefficient of y^{D-2} in S_D equals $-\binom{D}{2}$, independently of c . Every Omni-Metallic polynomial $x^m - 2nx^{m-1} - N^r$ has vanishing coefficients in all intermediate degrees. Hence for $D \geq 3$ the two polynomial families never coincide, while for $D = 2$ one has $S_2(y) = y^2 - 2cy - 1$, which is of Omni-Metallic shape with $m = 2$. The geometric bridge is therefore confined to the layer $m = 2$, where the quadratic radical carries the triple; this is exactly the range of the Crown Identity. Individual values may still coincide when S_D is reducible: for instance $\phi_4(u + i) = u + \sqrt{u^2 + 1} = \mathcal{U}_{2,1}(u, 1)$.

Levels 1 and 2 are established by Theorems 4.1 and 4.2 respectively. The integer projection $r(k) = \lfloor (k - 1)/2 \rfloor$ is common to all three levels on the Omni-Metallic side by Corollary 2.1; the lattice has its own integer spectrum, determined by Theorem 4.3.

10. The cubic case

For $D = 3$, let $c = \cot(\arg(W^2))$ and let $y = \phi_3(W) = \cot(\arg(W^2)/3)$.

Theorem 10.1. *The triple-angle identity gives the exact cubic*

$$y^3 - 3cy^2 - 3y + c = 0. \quad (31)$$

Proof. The triple-angle identity is $\cot(3\varphi) = (\cot^3 \varphi - 3 \cot \varphi) / (3 \cot^2 \varphi - 1)$. With $y = \cot \varphi$ and $c = \cot(3\varphi)$, we get $c(3y^2 - 1) = y^3 - 3y$, and rearranging gives (31). $\square$

The discriminant of this cubic is

$$\Delta_3 = 108(c^2 + 1)^2 = 3 \cdot [6(c^2 + 1)]^2. \quad (32)$$

We now determine exactly when (31) is irreducible. The answer is governed by cube structure in $\mathbb{Z}[i]$.

Theorem 10.2 (Irreducibility criterion). *Let $W = u + vi$ with $u > v > 0$, $\gcd(u, v) = 1$, $u \not\equiv v \pmod{2}$, and let $c = (u^2 - v^2)/(2uv)$. Then the cubic (31) has a rational root if and only if*

$$W = G^3 \quad \text{for some } G \in \mathbb{Z}[i], \quad (33)$$

that is, if and only if W is a cube in $\mathbb{Z}[i]$. In particular, (31) is irreducible over $\mathbb{Q}$ for every primitive W that is not a cube in $\mathbb{Z}[i]$.

Proof. Suppose $y = s/t \in \mathbb{Q}$ in lowest terms is a root. Substituting $c = (u^2 - v^2)/(2uv)$ into (31) and clearing denominators gives

$$(u^2 - v^2)t(3s^2 - t^2) = 2uv s(s^2 - 3t^2). \quad (34)$$

Setting $V = s + ti$, expansion of the cube yields $V^3 = s(s^2 - 3t^2) + t(3s^2 - t^2)i$.

We first check that s and t are of opposite parity. If both were odd, then $s^2 \equiv t^2 \equiv 1 \pmod{8}$, so $s^2 - 3t^2 \equiv -2$ and $3s^2 - t^2 \equiv 2 \pmod{8}$; both have 2-adic valuation exactly 1. Since $u \not\equiv v \pmod{2}$, the integer $u^2 - v^2$ is odd and $2uv$ has 2-adic valuation $\alpha \geq 1$. Comparing valuations in (34) then gives $1 = \alpha + 1$,

hence $\alpha = 0$, a contradiction. Therefore exactly one of s, t is even, and consequently $s(s^2 - 3t^2)$ is odd while $t(3s^2 - t^2)$ is even.

Next, $\gcd(s(s^2 - 3t^2), t(3s^2 - t^2)) = 1$. Indeed $\gcd(s, t) = 1$ gives $\gcd(s, 3s^2 - t^2) = \gcd(s, t^2) = 1$ and $\gcd(s^2 - 3t^2, t) = \gcd(s^2, t) = 1$; and since $3(s^2 - 3t^2) - (3s^2 - t^2) = -8t^2$, any common divisor of $s^2 - 3t^2$ and $3s^2 - t^2$ divides $8t^2$, hence divides 8, and is therefore 1 because $s^2 - 3t^2$ is odd. The same computation shows $\gcd(u^2 - v^2, 2uv) = 1$.

Both sides of (34) are therefore equalities of fractions in lowest terms, so comparing the real and imaginary parts of $W^2 = (u^2 - v^2) + 2uvi$ with those of V^3 gives $W^2 = \varepsilon V^3$ for a sign $\varepsilon \in \{\pm 1\}$. Since $(-V)^3 = -V^3$, we may absorb the sign and write $W^2 = V_0^3$ with $V_0 = \varepsilon V \in \mathbb{Z}[i]$.

Now $\mathbb{Z}[i]$ is a unique factorization domain, so comparing the exponent of each Gaussian prime ϖ in $W^2 = V_0^3$ gives $2a_\varpi = 3b_\varpi$, whence $3 \mid a_\varpi$ for every ϖ . Therefore $W = wG^3$ for some $G \in \mathbb{Z}[i]$ and some unit $w \in \{1, -1, i, -i\}$. Every unit of $\mathbb{Z}[i]$ is itself a cube, since $1 = 1^3$, $-1 = (-1)^3$, $-i = i^3$ and $i = (-i)^3$; writing $w = w_0^3$ gives $W = (w_0G)^3$, so W is a cube in $\mathbb{Z}[i]$.

Conversely, if $W = G^3$ then $W^2 = (G^2)^3$, and writing $G^2 = s + ti$ the number $y = s/t$ satisfies (31). A cubic over $\mathbb{Q}$ is reducible if and only if it has a rational root, which completes the proof. $\square$

Example 10.1 (An explicit reducible instance). Take $W = 11 + 2i$, which satisfies $u = 11 > v = 2 > 0$, $\gcd(11, 2) = 1$ and $11 \not\equiv 2 \pmod{2}$. Then $W = (-1 - 2i)^3$ is a cube in $\mathbb{Z}[i]$, and $G^2 = (-1 - 2i)^2 = -3 + 4i$, so the criterion predicts the rational root $y = -3/4$. Indeed $W^2 = 117 + 44i$ gives $c = 117/44$, and

$$\left(-\frac{3}{4}\right)^3 - 3 \cdot \frac{117}{44} \left(-\frac{3}{4}\right)^2 - 3 \left(-\frac{3}{4}\right) + \frac{117}{44} = 0.$$

Consequently the cubic is reducible for this W , and its Galois group is not the symmetric group on three letters. The next such W with $u < 200$ are $46 + 9i$, $52 + 47i$, $117 + 44i$, $142 + 65i$ and $198 + 107i$, with rational roots $5/12$, $15/8$, $7/24$, $-21/20$ and $35/12$ respectively.

Remark 10.1. Example 10.1 shows that irreducibility does not hold for every primitive W ; the exceptional set is exactly the set of cubes in $\mathbb{Z}[i]$, which is sparse but non-empty. The spectral analysis below is therefore stated for W outside this exceptional set.

Corollary 10.1. Let W be a primitive Gaussian integer that is not a cube in $\mathbb{Z}[i]$. Then (31) is irreducible, and since the discriminant $3 \cdot [6(c^2 + 1)]^2$ is not a square in $\mathbb{Q}$ (as 3 is not a square in $\mathbb{Q}$), the Galois group of (31) is the symmetric group S_3 [8].

The irreducible representations of S_3 are shown in Table 3.

Table 3. Irreducible representations and their spectral contributions.

Representation	Dimension	Contribution
Trivial	1	$\zeta(s-2), \zeta(s-1), \zeta(s)$
Sign	1	$L(s, \chi_{12})$
Standard	2	$L(s, \rho_2)$

Remark 10.2. The sign representation corresponds to the quadratic resolvent field $\mathbb{Q}(\sqrt{\Delta_3}) = \mathbb{Q}(\sqrt{3})$, whose Dirichlet character is the real character χ_{12} . This corrects an earlier use of the imaginary character χ_{-3} [8].

Thus the Mellin transform of $\mathcal{L}_3$ is

$$Z_3(s) = a_0\zeta(s-2) + a_1\zeta(s-1) + a_2\zeta(s) + a_3L(s, \chi_{12}) + a_4(s)L(s, \rho_2). \quad (35)$$

The coefficients a_0, a_1, a_2, a_3 are to be determined by direct computation of $Z_3(s)$ from the lattice of primitive Gaussian integers restricted to $D = 3$, excluding the exceptional set of Theorem 10.2. The coefficient $a_4(s)$ is given by

$$a_4(s) = \sum_{W \in \mathbb{Z}[i]_{\text{prim}}} \chi_2(\sigma_W) \cdot \phi_3(W)^{-s}, \quad (36)$$

where $\sigma_W \in S_3$ is the Frobenius element associated to $c = \cot(\arg(W^2))$.

Remark 10.3. *The computation of $a_4(s)$ is a Chebotarev-density problem and is left as an open problem. We do not assert that $a_4(s) = 0$ without proof. Consequently the resolved form of $Z_3(s)$ remains open.*

11. The general odd case

For odd D , the roots are

$$y_k = \cot\left(\frac{\theta + k\pi}{D}\right), \quad k = 0, 1, \dots, D-1.$$

Under the Möbius transformation $z = (y + i)/(y - i)$, the roots map to $z_k = \alpha\zeta_D^k$, and the action on the roots is $z_k \mapsto \alpha^a \zeta_D^{ak+b}$, which is the affine group

$$\text{Aff}(\mathbb{Z}/D\mathbb{Z}) = \{z \mapsto az + b : a \in (\mathbb{Z}/D\mathbb{Z})^\times, b \in \mathbb{Z}/D\mathbb{Z}\}.$$

Conjecture 11.1. *The character-weighted Dirichlet series*

$$c_{D,\rho}(s) = \sum_{W \in \mathbb{Z}[i]_{\text{prim}}} \chi_\rho(\sigma_W) \cdot \phi_D(W)^{-s} \quad (37)$$

satisfies $c_{D,\rho}(s) = 0$ for all $\rho \neq \mathbf{1}, \text{sgn}$.

Remark 11.1. *Since $\phi_D(W)$ depends only on $\arg(W^2)$, infinitely many primitive W produce values in any fixed interval, so the terms of $\sum_W \phi_D(W)^{-s}$ do not tend to zero. The series $Z_D(s)$ and the character sums (36) and (37) are therefore understood as formal Dirichlet series; establishing a regularisation under which they converge is part of Open Problem OP8.*

Remark 11.2. *This conjecture is plausible given the affine group structure and the square structure of $c^2 + 1$, but requires explicit character sum verification.*

11.1. A conjectural spectral decomposition

Assuming Conjecture 11.1, for odd D ,

$$Z_D(s) = c_{D,\mathbf{1}}(s)\zeta(s) + c_{D,\text{sgn}}(s)L(s, \chi_D). \quad (38)$$

The full lattice sum is $Z(s) = \sum_{D=1}^{\infty} Z_D(s)$, and the even- D sum is

$$\sum_{D \text{ even}} Z_D(s) = \frac{1}{2}\zeta(s-1) - \frac{1}{2}(1+2^{-s})\zeta(s). \quad (39)$$

Therefore, if the conjecture holds, $\sum_{D \text{ odd}} Z_D(s) = 0$, and substituting (38),

$$\sum_{D \text{ odd}} \sum_{\rho \neq 1, \text{sgn}} c_{D,\rho}(s) L(s, \rho) = 0. \quad (40)$$

Remark 11.3. *This cancellation identity is conjectural and requires verification.*

12. The general Dirichlet series

For arbitrary $m \geq 2$, $r \geq 1$, define

$$Z_{m,r}(s) := \sum_{k=2}^{\infty} \frac{r_{m,r}(k)}{k^s}, \quad (41)$$

the Dirichlet series of the integer multiplicities of the family, where

$$r_{m,r}(k) = \#\{n \in \mathbb{N} : 1 \leq n < k/2, k^{m-1}(k-2n) \text{ is a perfect } r\text{-th power}\}. \quad (42)$$

For $r = 1$, this reduces to the Bridge Identity:

$$Z_{m,1}(s) = Z_{2,1}(s) = \frac{1}{2}\zeta(s-1) - \frac{1}{2}(1+2^{-s})\zeta(s) \quad \text{for all } m \geq 2. \quad (43)$$

13. Super-exponential approximation

Theorem 13.1 (Super-exponential approximation). *For every $T > 0$, there exist sequences $W_k \in \mathbb{Z}[i]_{\text{prim}}$ and $D_k \in \mathbb{N}$ such that*

$$|\phi_{D_k}(W_k) - T| \leq \frac{C_T}{k^4 e^{k^2}}, \quad (44)$$

where $C_T = 8\pi e(1+T^2)$ is a constant depending only on T , with $|W_k| \leq k^2$ and $D_k = \lfloor e^{|W_k|} \rfloor$.

Proof. Let $\theta = \arctan T$. By Dirichlet's approximation theorem, for any N there exist coprime integers $u > v > 0$ with $u^2 + v^2 \leq 2N^2$ such that

$$\left| \frac{v}{u} - \tan(\theta/2) \right| \leq \frac{1}{uN} = O(1/N^2),$$

hence $|\theta_W - \theta| = O(1/N^2)$ for a suitable primitive W with $|W| = O(N)$, where θ_W is the corresponding acute angle. Using the cotangent difference formula

$$\phi_D(W) - T = -\frac{\sin(2\delta/D)}{\sin(\theta + 2\delta/D) \sin \theta}, \quad \delta = \theta_W - D\theta/2,$$

we get $|\phi_D(W) - T| \leq \frac{2(1+T^2)}{D}|\delta| + O(1/D^2)$. Setting $N = k^2$ and $D = \lfloor e^N \rfloor$, and choosing the representative of $D\theta/2$ modulo π nearest to θ_W , we obtain

$$|\phi_{D_k}(W_k) - T| \leq \frac{4\pi e(1+T^2)}{N^2 e^N} \leq \frac{8\pi e(1+T^2)}{k^4 e^{k^2}},$$

which is (44). $\square$

Remark 13.1. *The argument above controls δ through the choice of representative modulo π ; an unconditional and effective bound on δ uniform in k is Open Problem OP2.*

14. Complete spectral decomposition

For $D \geq 2$ with $4 \nmid D$, let $G_D = \text{Gal}(\mathbb{Q}(\zeta_D, i)/\mathbb{Q}) \cong (\mathbb{Z}/D\mathbb{Z})^\times \times C_2$, an abelian group. (For $4 \mid D$ one has $i \in \mathbb{Q}(\zeta_D)$ and $G_D \cong (\mathbb{Z}/D\mathbb{Z})^\times$.)

Conjecture 14.1 (Complete spectral decomposition). *For every $D \geq 2$,*

$$Z_D(s) = \sum_{W \in \mathbb{Z}[i]_{\text{prim}}} \phi_D(W)^{-s} = (-i)^s D \sum_{m=0}^{\infty} b_{D,mD}(s) \sum_{\chi \in \widehat{G_D}} c_{D,m,\chi} L(s, \chi), \quad (45)$$

where

$$b_{D,r}(s) = \sum_{j=0}^r \binom{s}{j} \binom{-s}{r-j} (-1)^j, \quad (46)$$

and

$$c_{D,m,\chi} = \frac{1}{\varphi(D)} \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \chi(a) \frac{1}{D} \sum_{k=0}^{D-1} \left(\frac{1 + e^{2\pi i k/D}}{1 - e^{2\pi i k/D}} \right)^m e^{-2\pi i a k/D}. \quad (47)$$

Remark 14.1. *For $c = \cot \theta$, the power sum of the D -section roots is*

$$S_{D,c}(s) = \sum_{k=0}^{D-1} \cot \left(\frac{\theta + k\pi}{D} \right)^{-s} = (-i)^s D \sum_{m=0}^{\infty} b_{D,mD}(s) \left(\frac{c-i}{c+i} \right)^m.$$

For $W = u + vi$ we have $c = (u^2 - v^2)/(2uv)$, so $(c-i)/(c+i) = \overline{W}^2/W^2$. The Grössencharacter $\psi(W) = \overline{W}^{2m}/W^{2m}$ has conductor dividing D . By Hecke theory over $\mathbb{Q}(i)$ [9, 11], it decomposes into Dirichlet L -functions over the character group $\widehat{G_D}$.

Corollary 14.1. *If Conjecture 14.1 holds, then for $D = 3$,*

$$Z_3(s) = \frac{1}{2}\zeta(s-1) - \frac{3}{4}\zeta(s) + a_4(s)L(s, \rho_2),$$

with $a_4(s)$ as in (36).

Remark 14.2. *Corollary 14.1 is consistent with the general form (35) precisely when $a_0 = 0$ and $a_3 = 0$, which Conjecture 14.1 entails. Unconditionally the coefficients $a_0, \dots, a_3$ remain undetermined, as does the Chebotarev term $a_4(s)$; see Section 10.*

Corollary 14.2. *If Conjecture 14.1 holds, for $D = 5$,*

$$Z_5(s) = \frac{1}{2}\zeta(s-1) - \frac{5}{8}\zeta(s) + \frac{3}{8}L(s, \chi_5) + \frac{1}{8}L(s, \chi_{-5}).$$

15. Future directions

The following open problems arise naturally from the framework.

- OP1 Prove the complete spectral decomposition conjecture (45) in full rigor.
- OP2 Determine the exact error term in the super-exponential approximation theorem, with an effective bound uniform in k .
- OP3 Find a closed form for the general Dirichlet series $Z_{m,r}(s)$ for $r > 1$.
- OP4 Investigate the connection between the parity structure and the Langlands program.
- OP5 Explore the modular properties of the generating function $F(x)$.
- OP6 Compute the Chebotarev-density coefficient $a_4(s)$ in (36).
- OP7 Extend the quarter-angle bridge of Theorem 4.1 to arbitrary D , and find a closed form for the lattice multiplicities of Theorem 4.3.
- OP8 Determine the full spectral decomposition for odd D , and a regularisation making $Z_D(s)$ convergent (Remark 11.1).
- OP9 Determine the density of primitive Gaussian integers that are cubes in $\mathbb{Z}[i]$, that is, the exceptional set of Theorem 10.2.

16. Conclusion

The Pythagorean angle lattice $\mathcal{L}$ and the Omni-Metallic family $\mathfrak{U}_{m,r}(n, N)$ are joined by the Quarter-Angle Bridge and the Crown Identity. For the Universal Metallic family ($m = 2$, $r = 1$) the arithmetic kernel is $\mu_{\text{int}}(\{k\}) = r(k) = \lfloor (k-1)/2 \rfloor$, with generating function $F(x) = x^3/((1-x)(1-x^2))$, and every spectral result below rests on it.

The Mellin transform is the Bridge Identity, whose finite part at $s = 2$ gives an alternative expression for π using the identity $\zeta(2) = \pi^2/6$. The Borel transform gives an alternative expression for e using the identity $\cosh(1) = (e + e^{-1})/2$. The product formula $\prod_{k \geq 2} k^{r(k)} = e^{-1/24} A^{1/2} 2^{-1/4} \pi^{-1/2}$ connects the Omni-Metallic counting function to the fundamental constants.

The Generalized Integer Value Theorem is proven for all $m \geq 2$, $r \geq 1$. The Quarter-Angle Bridge (Theorem 4.1) gives the exact identity $\phi_4(W) = (u + \sqrt{u^2 + v^2})/v = M_{2u/v}$, an unconditional link between the lattice and the classical metallic means, and the Crown Identity connects the Universal Metallic family to primitive Pythagorean triples.

For the cubic case we have determined the exact irreducibility criterion: the associated cubic is reducible over $\mathbb{Q}$ precisely when W is a cube in $\mathbb{Z}[i]$, with $W = 11 + 2i = (-1 - 2i)^3$ an explicit instance. The spectral decomposition coefficients require further computation, and the decomposition over odd D remains conjectural. The integer spectrum of the lattice is determined by Theorem 4.3; a closed form for its multiplicities remains open.

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