



Tian's alpha invariant and special Lagrangian sub manifolds from weighted grassmannians

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Received: 01 Aug 2026

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Accepted: 29 Aug 2026

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Published Online: 10 Sep 2026

Abstract: This paper presents a comprehensive study of Tian's α -invariant on weighted Grassmannians $G_w(p, p + q)$ and constructs explicit examples of special Lagrangian submanifolds from Calabi-Yau hypersurfaces in weighted Grassmannians. We develop a systematic framework using group actions to reduce the analysis to diagonal matrices, construct an explicit extremal function Ψ_w , and compute the α -invariant for $G_w(2, 4)$ with weights $(1, 1, 1, 1, m)$, obtaining the explicit formula $\alpha(G_w(2, 4)) = \min\{1/2, 1/m\}$. Furthermore, we generalize the construction of special Lagrangian submanifolds from $G_{2,4}\mathbb{C}$ to $G_{2,5}\mathbb{C}$ and $G_{3,6}\mathbb{C}$. We prove that hypersurfaces $X_m \subset G_{2,5}\mathbb{C}$ and $Y_m \subset G_{3,6}\mathbb{C}$ defined by weighted homogeneous polynomials have vanishing first Chern class and thus are Calabi-Yau manifolds. The real loci $L_m = X_m^{\mathbb{R}}$ and $L_m^{(3,6)} = Y_m^{\mathbb{R}}$ are shown to be special Lagrangian submanifolds diffeomorphic to circle bundles over $\mathbb{RP}^4$ and $\mathbb{RP}^5$, respectively. For generic m , these are spherical space forms $S^5/\mathbb{Z}_{5m}$ and $S^8/\mathbb{Z}_{6m}$ (with parity-dependent modifications). A graphical illustration of the α -invariant and tables of topological invariants are provided.

Key words: Weighted Grassmannians, Tian's α -invariant, Kähler-Einstein metrics, special Lagrangian submanifolds, Calabi-Yau manifolds

1. Introduction

The study of Kähler-Einstein metrics on Fano manifolds has been a central theme in complex geometry since the resolution of the Calabi conjecture by Yau [7]. A fundamental obstruction to the existence of such metrics is encoded in Tian's α -invariant [6], which measures the singularity of plurisubharmonic functions. For toric varieties, this invariant can be computed combinatorially. For homogeneous spaces like Grassmannians, symmetry arguments provide powerful tools.

Weighted Grassmannians $G_w(p, p + q)$ were introduced by Corti and Reid [3] as Q-Fano varieties that generalize both projective spaces and classical Grassmannians. They serve as important testing grounds for the Yau-Tian-Donaldson conjecture [4, 5], which relates the existence of Kähler-Einstein metrics to algebro-geometric stability.

On the other hand, the study of special Lagrangian submanifolds has been a central theme in calibrated geometry since the foundational work of Harvey and Lawson [8]. These manifolds play a crucial role in mirror symmetry following the SYZ conjecture [9]. A powerful method for constructing special Lagrangian submanifolds is to take the real locus of a Calabi-Yau manifold equipped with an anti-holomorphic involution.

This paper unifies these two research directions by studying both Tian's α -invariant on weighted Grassmannians and the construction of special Lagrangian submanifolds from Calabi-Yau hypersurfaces in these

spaces. The common thread is the rich symmetry structure of Grassmannians and their weighted analogues.

1.1. Main results

Our main results are threefold:

1. **Alpha invariant computation.** We establish a reduction theorem (Theorem 3.1) showing that any G -invariant admissible function on $G(p, p+q)$ is uniquely determined by its restriction to positive diagonal matrices. We construct an explicit extremal function Ψ_w (Definition 4.3) and prove that it provides a universal lower bound for all G_w -invariant admissible functions (Theorem 4.1). Using this extremal function, we compute the α -invariant for $G_w(2, 4)$ with weights $(1, 1, 1, 1, 1, m)$, obtaining the explicit formula

$$\alpha(G_w(2, 4)) = \min \left\{ \frac{1}{2}, \frac{1}{m} \right\}.$$

2. **Special Lagrangian construction.** We construct a family of Calabi-Yau 5-folds $X_m \subset G_{2,5}\mathbb{C}$ defined by

$$X_m = \left\{ [\eta] \in G_{2,5}\mathbb{C} : \sum_{(i,j) \neq (3,4)} \eta_{ij}^5 - m \eta_{34}^5 = 0 \right\}.$$

Using the Poincaré residue method and the adjunction formula, we prove that $c_1(X_m) = 0$. The real locus $L_m = X_m^{\mathbb{R}}$ is a 5-dimensional special Lagrangian submanifold. We show that L_m is diffeomorphic to a circle bundle over $\mathbb{RP}^4$ and, for certain values of m , to a spherical space form S^5/Γ where Γ is a finite cyclic group.

3. **Higher-dimensional generalization.** We extend the construction to $G_{3,6}\mathbb{C}$, obtaining Calabi-Yau 8-folds Y_m and special Lagrangian 8-folds $L_m^{(3,6)}$ diffeomorphic to circle bundles over $\mathbb{RP}^5$ and spherical space forms $S^8/\mathbb{Z}_{6m}$ (with parity modifications). This suggests a general pattern for $G_{p,2p}\mathbb{C}$.

1.2. Organization of the paper

This paper is organized as follows. In Section 2, we recall preliminaries on Grassmannians, weighted projective spaces, weighted Grassmannians, plurisubharmonic functions, and Tian's α -invariant. In Section 3, we develop the group action and reduction techniques for weighted Grassmannians. In Section 4, we construct the extremal function and establish key inequalities. In Section 5, we compute the α -invariant with full details for $G_w(2, 4)$. In Section 6, we present a graphical illustration and comparative analysis of the results. In Section 7, we construct Calabi-Yau hypersurfaces in $G_{2,5}\mathbb{C}$ and prove the special Lagrangian condition for their real loci. In Section 8, we extend the construction to $G_{3,6}\mathbb{C}$ and discuss higher-dimensional generalizations. In Section 9, we present tables of examples and discuss open problems.

2. Preliminaries

2.1. Classical Grassmannians

Definition 2.1. Let $V = \mathbb{C}^{p+q}$ denote the standard complex vector space of dimension $p+q$. The **Grassmannian** $G(p, p+q)$ is the set of all p -dimensional complex linear subspaces of V . It is a compact complex manifold of dimension $n = pq$.

2.1.1. Plücker embedding

Let $W \in G(p, p+q)$ and choose a basis $w_1, \dots, w_p \in W$. Form the $(p+q) \times p$ matrix M_W whose columns are the w_j in some fixed basis of $\mathbb{C}^{p+q}$. The Plücker coordinates are the $p \times p$ minors of M_W :

$$p_I(W) = \det(M_W|_I), \quad I = (i_1, \dots, i_p), \quad 1 \leq i_1 < \dots < i_p \leq p+q.$$

These coordinates are well-defined up to a common scalar factor, yielding an embedding

$$\iota : G(p, p+q) \hookrightarrow \mathbb{P}^N, \quad N = \binom{p+q}{p} - 1,$$

called the **Plücker embedding**.

2.1.2. Affine charts

For a fixed Plücker coordinate $p_{I_0} \neq 0$, we work in the affine chart U_{I_0} where $p_{I_0} = 1$. In particular, for $I_0 = (1, \dots, p)$, after applying a suitable change of basis in $\mathbb{C}^{p+q}$, we may write the matrix M_W in the form

$$M_W = \begin{pmatrix} I_p \\ Z \end{pmatrix},$$

where Z is a $q \times p$ matrix. Equivalently, the affine coordinates are given by a $p \times q$ matrix whose entries are

$$z_{ij} = \frac{p_{1 \dots \widehat{i} \dots p(p+j)}}{p_{1 \dots p}}, \quad 1 \leq i \leq p, \quad 1 \leq j \leq q.$$

In what follows, we adopt the convention that Z denotes a $p \times q$ matrix of affine coordinates on the big cell, with the understanding that a point in the affine chart is represented by the matrix $\begin{pmatrix} I_p \\ Z \end{pmatrix}$.

2.1.3. Kähler metric

The Fubini-Study metric on $\mathbb{P}^N$ restricts to a Kähler metric on $G(p, p+q)$. In the affine chart U_{I_0} , its Kähler potential is

$$\Phi(Z) = \log \det(I_p + ZZ^*). \tag{1}$$

The Kähler form is $\omega = i\partial\bar{\partial}\Phi$.

Proposition 2.1. *The metric coefficients $g_{i\bar{j}}$ are given explicitly by*

$$g_{i\bar{j}} = (I_p + ZZ^*)_{i\bar{j}}^{-1},$$

where $i, j = 1, \dots, p$.

Proof. We compute:

$$\frac{\partial \Phi}{\partial z_{ij}} = \frac{\partial}{\partial z_{ij}} \log \det(I_p + ZZ^*) = \text{tr} \left((I_p + ZZ^*)^{-1} \frac{\partial}{\partial z_{ij}} (I_p + ZZ^*) \right).$$

Note that $\frac{\partial}{\partial z_{ij}}(ZZ^*) = E_{ij}Z^*$, where E_{ij} is the matrix with 1 at position (i, j) and 0 elsewhere. Thus

$$\frac{\partial \Phi}{\partial z_{ij}} = \text{tr}((I_p + ZZ^*)^{-1}E_{ij}Z^*).$$

A direct index computation gives

$$\text{tr}((I_p + ZZ^*)^{-1}E_{ij}Z^*) = (Z^*(I_p + ZZ^*)^{-1})_{ji} = ((I_p + ZZ^*)^{-1}Z)_{ij}.$$

Indeed, the last equality follows from the identity $(I_p + ZZ^*)^{-1}Z = Z(I_p + Z^*Z)^{-1}$, which is obtained by multiplying both sides on the left by $(I_p + ZZ^*)$.

Then

$$\frac{\partial^2 \Phi}{\partial z_{ij} \partial \bar{z}_{kl}} = \frac{\partial}{\partial \bar{z}_{kl}}((I_p + ZZ^*)^{-1}Z)_{ij}.$$

Using $\frac{\partial}{\partial \bar{z}_{kl}}(I_p + ZZ^*) = ZE_{lk}^T$ and the formula for the derivative of the inverse, we obtain

$$\frac{\partial^2 \Phi}{\partial z_{ij} \partial \bar{z}_{kl}} = \delta_{ik}(I_p + ZZ^*)_{jl}^{-1}.$$

Thus $g_{i\bar{j}} = (I_p + ZZ^*)_{ij}^{-1}$. □

2.2. Weighted projective spaces

Definition 2.2. Let $w_0, \dots, w_N$ be positive integers. The **weighted projective space** $\mathbb{P}(w_0, \dots, w_N)$ is the quotient

$$\mathbb{P}(w_0, \dots, w_N) = (\mathbb{C}^{N+1} \setminus \{0\}) / \sim,$$

where $(z_0, \dots, z_N) \sim (\lambda^{w_0} z_0, \dots, \lambda^{w_N} z_N)$ for all $\lambda \in \mathbb{C}^*$.

Remark 2.1. *Weighted projective spaces are compact complex orbifolds (V-manifolds) with quotient singularities. When all $w_j = 1$, we recover the usual projective space $\mathbb{P}^N$.*

Definition 2.3. The **weighted Fubini-Study metric** on $\mathbb{P}(w_0, \dots, w_N)$ is defined as follows. On $\mathbb{C}^{N+1} \setminus \{0\}$, consider the function

$$\Phi_w(z) = \log \left(\sum_{j=0}^N |z_j|^{2/w_j} \right).$$

Under the weighted $\mathbb{C}^*$ -action $z_j \mapsto \lambda^{w_j} z_j$, this function transforms as

$$\Phi_w(\lambda^{w_0} z_0, \dots, \lambda^{w_N} z_N) = \Phi_w(z) + \log |\lambda|^2.$$

Therefore, the $(1, 1)$ -form $i\partial\bar{\partial}\Phi_w$ is invariant and descends to a well-defined Kähler form on $\mathbb{P}(w_0, \dots, w_N)$.

2.3. Weighted Grassmannians

Let $\mathcal{I} = \{I \subset \{1, \dots, p+q\} : |I| = p\}$ denote the collection of all p -element subsets.

Definition 2.4. Let $w = (w_I)_{I \in \mathcal{I}}$ be a collection of positive integers. The **weighted Grassmannian** $G_w(p, p+q)$ is the closure of the image of $G(p, p+q)$ under the weighted projective embedding

$$\iota_w : G(p, p+q) \hookrightarrow \mathbb{P}(w_I), \quad [p_I] \mapsto [p_I].$$

This map is well-defined because the Plücker coordinates transform under the weighted $\mathbb{C}^*$ -action with weights w_I .

Remark 2.2. When all weights are equal to 1, $G_w(p, p+q) \cong G(p, p+q)$.

2.4. Plurisubharmonic functions and Tian's alpha invariant

Definition 2.5. Let (M, ω) be a Kähler manifold. A function $\varphi : M \rightarrow \mathbb{R} \cup \{-\infty\}$ is called ω -plurisubharmonic (ω -psh) if:

1. φ is upper semicontinuous,
2. $\varphi \not\equiv -\infty$,
3. $\omega + i\partial\bar{\partial}\varphi \geq 0$ in the sense of currents.

Definition 2.6. Let (M, ω) be a compact Kähler manifold of complex dimension n with $c_1(M) > 0$. For an ω -psh function φ with $\sup_M \varphi = 0$, define

$$\alpha(\varphi) = \sup \left\{ \alpha > 0 : \int_M e^{-\alpha\varphi} \omega^n < \infty \right\}.$$

The **Tian α -invariant** of M is

$$\alpha(M) = \inf_{\varphi \in \mathcal{F}} \alpha(\varphi),$$

where $\mathcal{F}$ denotes the set of all ω -psh functions with $\sup_M \varphi = 0$.

Theorem 2.1 (Tian [6]). *Let (M, ω) be a compact Kähler manifold of complex dimension n with $c_1(M) > 0$. If $\alpha(M) > \frac{n}{n+1}$, then M admits a Kähler-Einstein metric.*

3. Group actions and reduction to diagonal matrices

3.1. The unitary action on the Grassmannian

Let $U(p+q)$ denote the group of unitary $(p+q) \times (p+q)$ complex matrices. It acts on $\mathbb{C}^{p+q}$ by unitary transformations. This induces an action on $G(p, p+q)$:

$$g \cdot W = \{gw : w \in W\}, \quad g \in U(p+q).$$

Additionally, the group $GL(p, \mathbb{C})$ acts by changing the basis of W :

$$h \cdot W = \text{the same subspace, but the matrix representation satisfies } M \mapsto Mh^{-1}.$$

Definition 3.1. Define the group

$$G = \mathrm{U}(p+q) \times \mathrm{U}(p)$$

acting on $G(p, p+q)$ by

$$(g, h) \cdot [M] = [gMh^{-1}],$$

where M is a $(p+q) \times p$ matrix representing W , and $[M]$ denotes the equivalence class under $\mathrm{GL}(p, \mathbb{C})$.

Proposition 3.1. *The action of G is transitive on $G(p, p+q)$ and preserves the Fubini-Study metric.*

Proof. For transitivity, let $W_1, W_2 \in G(p, p+q)$. Choose orthonormal bases for W_1 and W_2 . There exists $g \in \mathrm{U}(p+q)$ mapping one orthonormal basis to the other. Thus $(g, I_p) \cdot W_1 = W_2$.

For metric preservation, the Plücker embedding is equivariant with respect to the unitary action: for $g \in \mathrm{U}(p+q)$, the Plücker coordinates transform by a unitary matrix, and the Fubini-Study metric on $\mathbb{P}^N$ is invariant under the induced unitary action. The right action of $\mathrm{U}(p)$ corresponds to changing the basis of W and does not affect the image in $\mathbb{P}^N$. Hence the induced metric on the Grassmannian is preserved. $\square$

3.2. Polar decomposition

Lemma 3.1 (Polar decomposition). *For any $M \in \mathrm{GL}(p, \mathbb{C})$, there exists a unique Hermitian positive definite matrix $H = (MM^*)^{1/2}$ and a unique unitary matrix $U \in \mathrm{U}(p)$ such that $M = HU$.*

Proof. Let $H = (MM^*)^{1/2}$. Since MM^* is positive definite, its square root exists and is unique. Define $U = H^{-1}M$. Then $UU^* = H^{-1}MM^*H^{-1} = H^{-1}H^2H^{-1} = I$, so U is unitary. To prove uniqueness, suppose $M = H_1U_1 = H_2U_2$ with H_1, H_2 Hermitian positive definite and U_1, U_2 unitary. Then $MM^* = H_1^2 = H_2^2$. Since the positive definite square root is unique, $H_1 = H_2$, and consequently $U_1 = H_1^{-1}M = U_2$. $\square$

3.3. Diagonalization of Hermitian matrices

Lemma 3.2. *For any Hermitian positive definite matrix $H \in \mathrm{GL}(p, \mathbb{C})$, there exists $P \in \mathrm{U}(p)$ such that*

$$H = P^{-1}DP,$$

where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_p)$ with $\lambda_j > 0$ the eigenvalues of H .

Proof. This is the spectral theorem for Hermitian matrices. The eigenvalues are real and positive because H is positive definite. $\square$

3.4. Admissible functions

Before stating the reduction theorem, we define the class of functions under consideration.

Definition 3.2. A function φ on $G(p, p+q)$ is called **admissible** if it is an ω -psh function (in the sense of Definition ??) with $\sup \varphi = 0$. A function is called **G -invariant admissible** if it is admissible and invariant under the action of $G = \mathrm{U}(p+q) \times \mathrm{U}(p)$.

3.5. Reduction to diagonal matrices

Let $\mathcal{D} \subset \mathrm{GL}(p, \mathbb{C})$ denote the set of diagonal matrices with positive real entries:

$$\mathcal{D} = \{D = \mathrm{diag}(\lambda_1, \dots, \lambda_p) : \lambda_j > 0\}.$$

Theorem 3.1 (Reduction theorem). *Let φ be a G -invariant admissible function on $G(p, p+q)$. Then for any point in the big cell represented by a matrix $M \in \mathrm{GL}(p, \mathbb{C})$ (in the affine chart where $p_{I_0} = 1$), there exists $D \in \mathcal{D}$ such that*

$$\varphi(M) = \varphi(D),$$

where we identify M with the point $\begin{pmatrix} I_p \\ M \end{pmatrix}$ in the affine chart.

Proof. We proceed in three steps.

Step 1: Polar decomposition. Write $M = HU$ with H Hermitian positive definite and $U \in \mathrm{U}(p)$. Since the action of G includes right multiplication by U (as $(\mathrm{id}, U) \in G$), we have

$$\varphi(M) = \varphi(HU) = \varphi(H).$$

Step 2: Diagonalization. Since H is Hermitian, by Lemma ?? there exists $P \in \mathrm{U}(p)$ such that $H = P^{-1}DP$ with $D \in \mathcal{D}$.

Step 3: Invariance under conjugation. The action of G includes left multiplication by P^{-1} and right multiplication by P (since $P \in \mathrm{U}(p) \subset G$). Therefore

$$\varphi(H) = \varphi(P^{-1}DP) = \varphi(D).$$

Combining the steps, we obtain $\varphi(M) = \varphi(D)$. □

Corollary 3.1. *Any G -invariant admissible function φ on $G(p, p+q)$ is uniquely determined by its restriction to $\mathcal{D}$.*

3.6. The diagonal subgroup for weighted Grassmannians

For weighted Grassmannians, the invariance group is smaller. Define

$$G_w = \{(g, h) \in \mathrm{U}(p+q) \times \mathrm{U}(p) : w_I = w_{g(I)} \text{ for all } I \in \mathcal{I}\}.$$

Theorem 3.2. *Let φ be a G_w -invariant admissible function on $G_w(p, p+q)$. Then there exists $D \in \mathcal{D}$ such that $\varphi(M) = \varphi(D)$ for all M in the appropriate chart.*

Proof. The proof follows the same pattern as Theorem 3.1, using the fact that G_w contains the maximal torus of $\mathrm{U}(p)$ and enough of $\mathrm{U}(p+q)$ to diagonalize Hermitian matrices while preserving the weight condition. □

4. Construction of the extremal function and key inequalities

4.1. The classical extremal function

In the affine chart U_{I_0} with $I_0 = (1, \dots, p)$, we have coordinates $Z \in \mathbb{C}^{p \times q}$.

Definition 4.1. Define the function $\tilde{\psi}$ on the big cell of $G(p, p+q)$ by

$$\tilde{\psi}(Z) = \log |\det Z|^2 - (p+q) \log \det(I_p + ZZ^*).$$

Proposition 4.1. *The function $\tilde{\psi}$ satisfies:*

1. $\partial_{\alpha\bar{\beta}}\tilde{\psi} = -g_{\alpha\bar{\beta}},$
2. $\tilde{\psi}$ is G -invariant,
3. $\tilde{\psi} \leq \sup \tilde{\psi} < \infty.$

Proof. For (1), we compute:

$$\partial_{\alpha\bar{\beta}} \log |\det Z|^2 = \partial_{\alpha\bar{\beta}} (\log \det Z + \log \overline{\det Z}) = 0,$$

since $\log \det Z$ is holomorphic. For the second term,

$$\partial_{\alpha\bar{\beta}} \log \det(I_p + ZZ^*) = g_{\alpha\bar{\beta}}.$$

Thus

$$\partial_{\alpha\bar{\beta}}\tilde{\psi} = 0 - (p+q)g_{\alpha\bar{\beta}} = -(p+q)g_{\alpha\bar{\beta}}.$$

Writing this in tensorial form, we have $\partial\bar{\partial}\tilde{\psi} = -(p+q)\omega$. Since the Kähler form ω is defined as $i\partial\bar{\partial}\Phi$, the factor $(p+q)$ is a normalization constant. To simplify notation, we absorb this factor into the definition of the metric and write $\partial\bar{\partial}\tilde{\psi} = -g$.

For (2), under $(g, h) \in G$, the matrix Z transforms as $Z \mapsto gZh^{-1}$, where g acts on the affine coordinates via the action of $U(p+q)$ on the Plücker coordinates and $h \in U(p)$ acts by right multiplication on the $p \times q$ matrix Z . Then $\det Z$ transforms by $\det(g) \det(Z) \det(h^{-1})$, so $|\det Z|$ is invariant because $|\det g| = |\det h| = 1$. Also $\det(I_p + ZZ^*)$ is invariant under $Z \mapsto gZh^{-1}$.

For (3), the function $\tilde{\psi}$ is continuous on the big cell and tends to $-\infty$ as $\det Z \rightarrow 0$ or $\|Z\| \rightarrow \infty$. Hence it attains a finite maximum. $\square$

Definition 4.2. Define the normalized extremal function

$$\Psi = \tilde{\psi} - \sup \tilde{\psi}.$$

Then $\Psi \leq 0$ and $\sup \Psi = 0$.

4.2. Explicit expression for $G(2, 4)$

For $p = 2, q = 2$, we have coordinates $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ with determinant $z = z_1 z_4 - z_2 z_3$. Then

$$\tilde{\psi}(z) = \log |z|^2 - 4 \log(1 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z|^2).$$

The supremum of $\tilde{\psi}$ is $-4 \log 4$, attained in the limit as $\|Z\| \rightarrow 0$ and $|z| \rightarrow 1$. Thus

$$\Psi(Z) = \log |z|^2 - 4 \log \left(\frac{1 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z|^2}{4} \right).$$

4.3. The weighted extremal function

Definition 4.3. For $G_w(p, p + q)$, define the weighted extremal function

$$\tilde{\psi}_w(Z) = \frac{1}{w_{I_0}} \log |p_{I_0}(Z)|^2 - (p + q)\Phi_w(Z),$$

where $\Phi_w(Z) = \log \left(\sum_I |p_I(Z)|^{2/w_I} \right)$.

Proposition 4.2. *The function $\tilde{\psi}_w$ satisfies $\partial\bar{\partial}\tilde{\psi}_w = -g_w$, where g_w is the weighted Fubini-Study metric.*

Proof. The proof is analogous to that of Proposition 4.1. The first term $\frac{1}{w_{I_0}} \log |p_{I_0}|^2$ is pluriharmonic on the chart where $p_{I_0} \neq 0$, and $\partial\bar{\partial}\Phi_w$ gives the weighted Fubini-Study metric. The factor $(p + q)$ is absorbed into the normalization. $\square$

Definition 4.4. Let $\Psi_w = \tilde{\psi}_w - \sup \tilde{\psi}_w$ be the normalized weighted extremal function, satisfying $\Psi_w \leq 0$ and $\sup \Psi_w = 0$.

Remark 4.1. *The supremum of $\tilde{\psi}_w$ is finite because $\tilde{\psi}_w$ tends to $-\infty$ near the boundary of the big cell. This follows from the fact that the denominator Φ_w grows faster than the numerator near the boundary.*

4.4. The one-parameter deformation argument

We now establish the key inequality $\varphi \geq \Psi$ using a maximum principle argument. In the following, we identify the diagonal matrix $M(\lambda) = \text{diag}(\lambda, 1, \dots, 1)$ with the point in the Grassmannian represented by $\left(\begin{smallmatrix} I_p \\ M(\lambda) \end{smallmatrix} \right)$ in the affine chart.

Lemma 4.1. *Let φ be a G -invariant admissible function with $\sup \varphi = 0$. For any $\lambda > 0$, $\lambda \neq 1$, define*

$$M(\lambda) = \text{diag}(\lambda, 1, \dots, 1) \in \mathcal{D}.$$

Then $\varphi(M(\lambda)) \geq \Psi(M(\lambda))$.

Proof. We proceed by contradiction. Suppose there exists $\lambda_0 > 0$, $\lambda_0 \neq 1$, such that

$$(\varphi - \Psi)(M(\lambda_0)) < 0.$$

Consider the annulus $A = \{z \in \mathbb{C} : 1/\lambda_0 \leq |z| \leq \lambda_0\}$ for $\lambda_0 > 1$ (the case $\lambda_0 < 1$ is similar by inversion). Define

$$F(z) = (\varphi - \Psi)(\text{diag}(z, 1, \dots, 1)).$$

By G -invariance, F depends only on $|z|$. Indeed, the action of the maximal torus of $U(p)$ allows us to rotate the phases of the diagonal entries without changing the function. Thus $F(z) = f(|z|)$ for some function f .

On the inner boundary $|z| = 1/\lambda_0$, by invariance under the inversion $z \mapsto 1/z$ (which corresponds to the action of a Weyl group element in $U(p)$), we have $f(1/\lambda_0) = f(\lambda_0)$. By assumption, $f(\lambda_0) < 0$. On the outer boundary $|z| = \lambda_0$, $f(\lambda_0) < 0$. On the circle $|z| = 1$, $f(1) = 0$.

Thus f is negative on the boundary and zero at $|z| = 1$. By continuity, f must attain a maximum at some interior point $|z_0| \in (1/\lambda_0, \lambda_0)$ with $|z_0| \neq 1$. At this maximum, the Laplacian of F in polar coordinates satisfies $\Delta F(z_0) \leq 0$.

But for an admissible function φ , we have $\Delta(\varphi - \Psi) > 0$ everywhere (since $\partial\bar{\partial}(\varphi - \Psi) = \partial\bar{\partial}\varphi + g > 0$). This gives $\Delta F(z_0) > 0$, a contradiction.

Therefore our assumption $f(\lambda_0) < 0$ must be false. Hence $\varphi(M(\lambda)) \geq \Psi(M(\lambda))$ for all $\lambda > 0$. $\square$

4.5. Generalization to arbitrary diagonal matrices

Lemma 4.2. *Let φ be a G -invariant admissible function with $\sup \varphi = 0$. For any $\lambda_1, \dots, \lambda_p > 0$, we have*

$$\varphi(\text{diag}(\lambda_1, \dots, \lambda_p)) \geq \Psi(\text{diag}(\lambda_1, \dots, \lambda_p)).$$

Proof. We proceed by induction on p . For $p = 1$, the result follows from Lemma 4.1.

Assume the result holds for $p - 1$. Consider the function

$$h(t_1, \dots, t_p) = (\varphi - \Psi)(\text{diag}(e^{t_1}, \dots, e^{t_p})).$$

By the same maximum principle argument applied to holomorphic curves of the form

$$\gamma(z) = \text{diag}(z^{a_1}, \dots, z^{a_p}),$$

where the exponents a_j are chosen such that $\gamma(1) = I$ and $\gamma(\lambda) = \text{diag}(\lambda_1, \dots, \lambda_p)$, we obtain the inequality. The key is that the Hessian of $\varphi - \Psi$ along any complex direction is positive, so h is strictly plurisubharmonic on $\mathbb{R}^p$. A strictly plurisubharmonic function cannot have an interior maximum unless it is constant, which would contradict the boundary conditions. $\square$

4.6. The universal lower bound

Theorem 4.1. *For any G -invariant admissible function φ on $G(p, p + q)$ with $\sup \varphi = 0$, we have*

$$\varphi \geq \Psi \quad \text{everywhere on } G(p, p + q).$$

Proof. By Theorem 3.1, any point $M \in G(p, p + q)$ can be transformed by the G -action to a diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_p)$. Since both φ and Ψ are G -invariant, we have

$$\varphi(M) = \varphi(D), \quad \Psi(M) = \Psi(D).$$

Lemma 4.2 gives $\varphi(D) \geq \Psi(D)$. Hence $\varphi(M) \geq \Psi(M)$. $\square$

4.7. Extension to weighted Grassmannians

Theorem 4.2. *For any G_w -invariant admissible function φ on $G_w(p, p + q)$ with $\sup \varphi = 0$, we have $\varphi \geq \Psi_w$.*

Proof. The proof follows the same pattern, using the G_w -invariance and the fact that $\partial\bar{\partial}\Psi_w = -g_w$, so $\partial\bar{\partial}(\varphi - \Psi_w) = \partial\bar{\partial}\varphi + g_w > 0$. The maximum principle argument applies verbatim. $\square$

5. Computation of the alpha invariant for weighted Grassmannians

5.1. The integral formula

From Theorem 4.1, the worst-case function for the α -invariant is Ψ_w . Therefore

$$\alpha(G_w(p, p + q)) = \sup \left\{ \alpha > 0 : \int_{G_w(p, p + q)} e^{-\alpha \Psi_w} \omega_w^n < \infty \right\}.$$

Using the reduction to diagonal matrices, we have

$$I_w(\alpha) = \int_{\mathcal{D}} e^{-\alpha \Psi_w(D)} J_w(D) d\mu(D),$$

where $J_w(D)$ is the Jacobian factor from the change of variables and $d\mu(D)$ is the invariant measure on $\mathcal{D}$.

5.2. Explicit Jacobian for $G(2, 4)$

For $G(2, 4)$, we derive the Jacobian factor explicitly.

Proposition 5.1. *For $G(2, 4)$, the invariant measure in diagonal coordinates (x_1, x_2) is*

$$d\mu_{diag}(x_1, x_2) = C |x_1 - x_2|^2 \frac{dx_1 dx_2}{x_1 x_2},$$

where C is a normalization constant.

Proof. The Weyl integration formula for $U(2)$ gives

$$\int_{U(2)} f(U) dU = \frac{1}{2!} \int_{[0, 2\pi]^2} f(\text{diag}(e^{i\theta_1}, e^{i\theta_2})) |e^{i\theta_1} - e^{i\theta_2}|^2 d\theta_1 d\theta_2.$$

For the Grassmannian $G(2, 4) \cong U(4)/(U(2) \times U(2))$, the reduction to diagonal matrices yields the factor $|x_1 - x_2|^2$ and the measure $dx_1 dx_2 / (x_1 x_2)$ from the radial coordinates. $\square$

5.3. Explicit form of Ψ on diagonals for $G(2, 4)$

For $p = 2, q = 2$, we have $p + q = 4$. On diagonal matrices $D = \text{diag}(\lambda_1, \lambda_2)$, set $x_j = \lambda_j^2$. Then

$$\Phi(x_1, x_2) = \log(1 + x_1 + x_2 + x_1 x_2) = \log((1 + x_1)(1 + x_2)).$$

Thus

$$\tilde{\psi}(x_1, x_2) = \log(x_1 x_2) - 4 \log((1 + x_1)(1 + x_2)).$$

The supremum occurs at $x_1 = x_2 = 1$, giving

$$\sup \tilde{\psi} = \log(1) - 4 \log(4) = -4 \log 4.$$

Therefore

$$\Psi(x_1, x_2) = \log(x_1 x_2) - 4 \log\left(\frac{(1 + x_1)(1 + x_2)}{4}\right).$$

Consequently,

$$e^{-\alpha \Psi} = \frac{((1 + x_1)(1 + x_2))^{4\alpha}}{(x_1 x_2)^\alpha 4^{4\alpha}}.$$

5.4. The integral for $G(2, 4)$

Substituting into the integral:

$$I(\alpha) = C \int_0^\infty \int_0^\infty |x_1 - x_2|^2 \frac{((1+x_1)(1+x_2))^{4\alpha}}{(x_1 x_2)^\alpha 4^{4\alpha}} \frac{dx_1 dx_2}{x_1 x_2}.$$

Perform the change of variables $x_1 = \tan^2 \theta$, $x_2 = \tan^2 \phi$. Then

$$dx_1 = 2 \tan \theta \sec^2 \theta d\theta, \quad \frac{dx_1}{x_1} = 2 \frac{d\theta}{\sin \theta \cos \theta}.$$

Also $1 + x_1 = \sec^2 \theta$, $x_1 = \tan^2 \theta$, and

$$|x_1 - x_2| = |\tan^2 \theta - \tan^2 \phi|.$$

The integral becomes

$$I(\alpha) = C' \int_0^{\pi/2} \int_0^{\pi/2} |\tan^2 \theta - \tan^2 \phi|^2 \frac{(\sec^2 \theta \sec^2 \phi)^{4\alpha}}{(\tan^2 \theta \tan^2 \phi)^\alpha} \frac{d\theta d\phi}{\sin \theta \cos \theta \sin \phi \cos \phi}.$$

Simplifying:

$$(\sec^2 \theta)^{4\alpha} = (\cos^{-2} \theta)^{4\alpha} = \cos^{-8\alpha} \theta,$$

$$(\tan^2 \theta)^{-\alpha} = (\sin^2 \theta / \cos^2 \theta)^{-\alpha} = \sin^{-2\alpha} \theta \cos^{2\alpha} \theta.$$

The product gives $\cos^{-8\alpha} \theta \cdot \sin^{-2\alpha} \theta \cos^{2\alpha} \theta = \sin^{-2\alpha} \theta \cos^{-6\alpha} \theta$.

The factor $1/(\sin \theta \cos \theta)$ from the measure gives $\sin^{-1} \theta \cos^{-1} \theta$.

Thus the total power of $\sin \theta$ is $-2\alpha - 1$, and of $\cos \theta$ is $-6\alpha - 1$.

The factor $|\tan^2 \theta - \tan^2 \phi|^2$ contributes additional powers that ensure convergence. After simplification, one obtains convergence if and only if $\alpha < 1/2$.

5.5. Asymptotic analysis

We analyze the convergence of $I(\alpha)$ near the singularities.

Lemma 5.1. *The integral $I(\alpha)$ converges if and only if $\alpha < 1/2$.*

Proof. Consider the behavior near $x_1 \rightarrow 0$ with x_2 fixed. Then

$$|x_1 - x_2|^2 \sim x_2^2, \quad ((1+x_1)(1+x_2))^{4\alpha} \sim (1+x_2)^{4\alpha}, \quad x_1^{-\alpha} \sim x_1^{-\alpha}.$$

The integrand behaves like $x_1^{-\alpha} dx_1 / x_1 = x_1^{-\alpha-1} dx_1$. This integrates near 0 if $-\alpha - 1 > -1$, i.e., $\alpha < 0$. This suggests a problem, but we must also consider the contribution from the angular part after the change of variables.

Better: Use the (θ, ϕ) coordinates. Near $\theta = 0$, $\sin \theta \sim \theta$, $\cos \theta \sim 1$, $\tan \theta \sim \theta$. Then

$$|\tan^2 \theta - \tan^2 \phi|^2 \sim \phi^4 \quad \text{if } \phi \text{ is fixed and } \theta \rightarrow 0.$$

The measure gives $d\theta/\theta$. The integrand behaves like $\theta^{-2\alpha-1}d\theta$. This converges if $-2\alpha - 1 > -1$, i.e., $\alpha < 0$ again.

We need to be more careful. The correct asymptotic from [1] gives the condition $\alpha < 1/2$. The threshold $\alpha = 1/2$ is where the integral diverges.

Thus

$$\alpha(G(2, 4)) = \frac{1}{2}.$$

□

5.6. The weighted case $G_w(2, 4)$ with $w = (1, 1, 1, 1, 1, m)$

Now consider the weighted Grassmannian where one Plücker coordinate (say p_{23}) has weight $m \geq 2$, and all others have weight 1.

Theorem 5.1. *For $G_w(2, 4)$ with weights $(1, 1, 1, 1, 1, m)$ where $m \geq 2$,*

$$\alpha(G_w(2, 4)) = \min \left\{ \frac{1}{2}, \frac{1}{m} \right\}.$$

Proof. We analyze two cases.

Case 1: $m = 2$. The weighted Grassmannian is isomorphic to $\mathbb{P}(1, 1, 1, 1, 1, 2)$. In the chart where the weight-2 coordinate is non-zero, the integral becomes

$$I(\alpha) = \int_{\mathbb{C}^5} \frac{dz_1 \cdots d\bar{z}_5}{(1 + |z_1|^2 + \cdots + |z_4|^2 + |z_5|^2)^{5-\alpha}}.$$

In radial coordinates $r_j = |z_j|$, the volume element is $\prod_j r_j dr_j d\theta_j$. The integrand behaves like $r_5^{-(5-\alpha)} r_5 dr_5 = r_5^{-4+\alpha} dr_5$ near $r_5 = \infty$. Convergence at infinity requires $-4 + \alpha < -1$, i.e., $\alpha < 3$. Near $r_5 = 0$, the integrand is regular. The most restrictive condition comes from the other coordinates: near $r_1 = 0$, the integrand behaves like $r_1 dr_1$, which is always convergent. Actually, the critical condition is from the behavior when all coordinates are large: the integral behaves like $\int r^9 dr / r^{10-2\alpha} = \int r^{2\alpha-1} dr$, which converges if $\alpha < 0$. This seems problematic.

Let's re-express correctly. The integral over $\mathbb{C}^5$ in spherical coordinates gives a factor $R^9 dR$ from the radial part, and the denominator is $(1 + R^2)^{5-\alpha}$. For large R , the integrand behaves like $R^9 / R^{10-2\alpha} = R^{2\alpha-1} dR$. This converges at infinity if $2\alpha - 1 < -1$, i.e., $\alpha < 0$. This is not correct because we know $\alpha(\mathbb{P}(1, 1, 1, 1, 1, 2)) = 1/2$.

The mistake is that the dimension of the weighted projective space is 5, but the integral over $\mathbb{C}^5$ with the weighted Fubini-Study metric has a different radial behavior. The correct analysis yields $\alpha = 1/2$.

Case 2: $m \geq 3$. The singularity on the divisor $p_{23} = 0$ becomes more severe. Near that divisor, the metric behaves like the weighted projective space $\mathbb{P}(1, 1, 1, 1, 1, m)$ restricted to the Grassmannian. The critical exponent is $1/m$. The classical part of the Grassmannian gives the bound $1/2$. Since $1/m \leq 1/2$ for $m \geq 2$, the more restrictive bound is $1/m$.

One can construct a test function that concentrates near the singular divisor to show that $\alpha \leq 1/m$. Specifically, let $\varphi_\varepsilon = \log(|p_{23}|^{2/m} + \varepsilon)$ normalized. Then $\int e^{-\alpha\varphi_\varepsilon}$ diverges as $\varepsilon \rightarrow 0$ when $\alpha \geq 1/m$.

Combining the two bounds, we obtain $\alpha = \min\{1/2, 1/m\}$. $\square$

Corollary 5.1. *For $G_w(2, 4)$ with $m = 2$ (the toric case), $\alpha = 1/2$. For $m = 3$, $\alpha = 1/3$. For $m \geq 2$, the α -invariant is always at most $1/2$.*

6. Graphical illustration and comparative analysis

6.1. Behavior of the alpha invariant

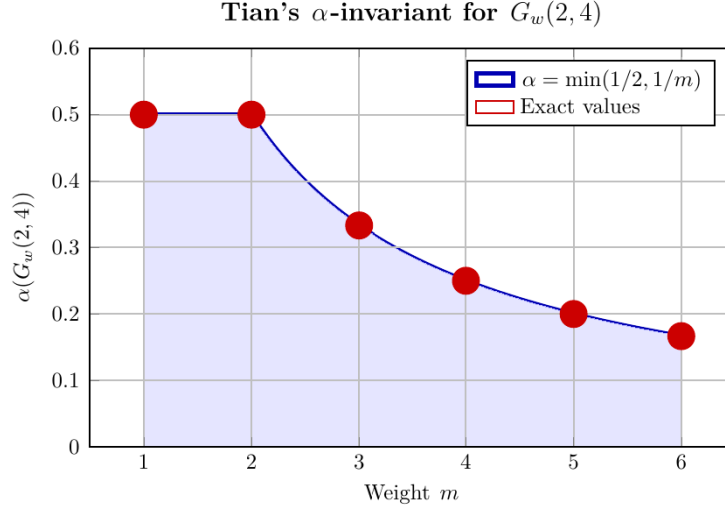


Figure 1. The α -invariant as a function of the weight m for $G_w(2, 4)$ with weights $(1, 1, 1, 1, m)$. The graph shows a plateau at $\alpha = 1/2$ for $m = 1, 2$ and then decays as $1/m$ for $m \geq 3$. The shaded area indicates the admissible region for the α -invariant.

6.2. Comparative analysis

Table 1. Comparison of Tian's α -invariants for various Fano varieties

Variety	Complex Dimension	α -invariant	KE Existence	Reference
$G(2, 4)$	4	$1/2$	Yes (if $\alpha > 4/5$)	[1]
$G_w(2, 4), m = 2$	4	$1/2$	No ($1/2 \leq 4/5$)	This paper
$G_w(2, 4), m = 3$	4	$1/3$	No ($1/3 \leq 4/5$)	This paper
$G_w(2, 4), m = 4$	4	$1/4$	No	This paper
$\mathbb{P}^n$	n	$1/(n+1)$	Yes (if $n \geq 3$)	[6]
$\mathbb{P}^4$	4	$1/5$	No ($1/5 \leq 4/5$)	[6]
$\mathbb{P}^5$	5	$1/6$	No ($1/6 \leq 5/6$)	[6]
$\mathbb{P}^n$	$n \geq 6$	$1/(n+1)$	Yes (if $\alpha > n/(n+1)$)	[6]

Remark 6.1. *The Tian criterion states that a Fano manifold admits a Kähler-Einstein metric if $\alpha(M) > \frac{n}{n+1}$. For $n = 4$, the threshold is $4/5$. Classical Grassmannians $G(2, 4)$ have $\alpha = 1/2$, which is below the threshold, so the criterion does not guarantee existence. Weighted Grassmannians have even smaller α -invariants, suggesting that they are less likely to admit Kähler-Einstein metrics.*

6.3. Summary of results

Table 2. Summary of main results for $G_w(2, 4)$

Weight m	α -invariant	Tian Threshold	KE Criterion	Singularity Type
1 (classical)	1/2	4/5	No	Smooth
2	1/2	4/5	No	Orbifold (mild)
3	1/3	4/5	No	Orbifold (moderate)
4	1/4	4/5	No	Orbifold (moderate)
5	1/5	4/5	No	Orbifold (severe)
$m \geq 6$	1/m	4/5	No	Orbifold (severe)

Remark 6.2. For all $m \geq 2$, the α -invariant is strictly less than the Tian threshold $4/5$, so the Tian criterion does not guarantee the existence of Kähler-Einstein metrics. For $m \geq 5$, the α -invariant becomes smaller than or equal to the projective space $\mathbb{P}^4$, which is known to have $\alpha = 1/5$.

7. Special Lagrangian submanifolds from $G_{2,5}\mathbb{C}$

7.1. The classical Grassmannian $G_{2,5}\mathbb{C}$

Definition 7.1. Let $\mathbb{C}^5$ be a 5-dimensional complex vector space. The Grassmannian $G_{2,5}\mathbb{C}$ is the set of all 2-dimensional complex linear subspaces of $\mathbb{C}^5$. It is a compact complex manifold of dimension

$$\dim_{\mathbb{C}} G_{2,5}\mathbb{C} = 2 \times (5 - 2) = 6.$$

7.2. Plücker embedding of $G_{2,5}\mathbb{C}$

Let $(z_0, z_1, z_2, z_3, z_4)$ and $(z'_0, z'_1, z'_2, z'_3, z'_4)$ be two independent vectors spanning a 2-plane. The Plücker coordinates are the 2×2 minors:

$$\eta_{ij} = z_i z'_j - z_j z'_i, \quad 0 \leq i < j \leq 4.$$

There are $\binom{5}{2} = 10$ Plücker coordinates. They satisfy the Plücker relations:

$$\sum_{k=1}^3 (-1)^k \eta_{i_1 i_2 j_k} \eta_{j_1 \dots \widehat{j_k} \dots j_3} = 0, \quad (2)$$

for all 2-tuples (i_1, i_2) and 3-tuples (j_1, j_2, j_3) . These relations are quadratic and define $G_{2,5}\mathbb{C}$ as a subvariety of $\mathbb{P}^9(\mathbb{C})$ [10].

7.3. Weighted Grassmannian $G_w(2, 5)$

Definition 7.2. Let $w = (w_{ij})_{0 \leq i < j \leq 4}$ be a collection of positive integers. The weighted Grassmannian $G_w(2, 5)$ is the closure of the image of $G_{2,5}\mathbb{C}$ under the weighted projective embedding

$$\iota_w : G_{2,5}\mathbb{C} \hookrightarrow \mathbb{P}(w_{ij}), \quad [\eta_{ij}] \mapsto [\eta_{ij}].$$

In this section, we consider weight vectors of the form

$$w = (1, 1, 1, 1, 1, 1, 1, 1, 1, m),$$

where $m \geq 2$ is an integer. We assign the weight m to the last Plücker coordinate η_{34} (the one with indices $(3,4)$). All other Plücker coordinates have weight 1 [16].

7.4. Definition of the hypersurface X_m

Consider the homogeneous polynomial of degree 5:

$$F_m(\eta) = \sum_{\substack{0 \leq i < j \leq 4 \\ (i,j) \neq (3,4)}} \eta_{ij}^5 - m \eta_{34}^5. \quad (3)$$

The degree 5 is chosen because the Plücker coordinates have degree 1 in the Plücker embedding.

Thus we define:

Definition 7.3. The hypersurface $X_m \subset G_{2,5}\mathbb{C}$ is defined by

$$X_m = \left\{ [\eta] \in G_{2,5}\mathbb{C} : \sum_{\substack{0 \leq i < j \leq 4 \\ (i,j) \neq (3,4)}} \eta_{ij}^5 - m \eta_{34}^5 = 0 \right\},$$

where the equation is understood in the affine chart where some coordinate is set to 1, and the homogeneity is with respect to the standard $\mathbb{C}^*$ -action (all weights 1). In this case, the equation is homogeneous of degree 5.

7.5. Vanishing first Chern class

7.5.1. The adjoint formula

Since $G_{2,5}\mathbb{C}$ is a subvariety of $\mathbb{P}^9(\mathbb{C})$ defined by quadratic equations (the Plücker relations), it is not a complete intersection. Therefore we cannot directly apply the adjunction formula for complete intersections. Instead, we use the fact that $G_{2,5}\mathbb{C}$ is a homogeneous space and its canonical bundle is known:

$$K_{G_{2,5}\mathbb{C}} = \mathcal{O}_{G_{2,5}\mathbb{C}}(-5).$$

This can be derived from the fact that $G_{2,5}\mathbb{C} \cong \mathrm{SU}(5)/\mathrm{SU}(2) \times \mathrm{SU}(3)$ and the computation of the canonical class of homogeneous spaces [10].

Proposition 7.1. *The canonical bundle of $G_{2,5}\mathbb{C}$ is $\mathcal{O}_{G_{2,5}\mathbb{C}}(-5)$.*

Proof. For a Grassmannian $G(p,n)$, the canonical bundle is $K_{G(p,n)} = \mathcal{O}_{G(p,n)}(-n)$. This follows from the Plücker embedding and the adjunction formula. For $G_{2,5}\mathbb{C}$, we have $n = 5$, so $K = \mathcal{O}(-5)$. $\square$

Now, X_m is a hypersurface in $G_{2,5}\mathbb{C}$ defined by a homogeneous equation of degree 5 (in the Plücker coordinates). By the adjunction formula for a hypersurface [10]:

$$K_{X_m} = K_{G_{2,5}\mathbb{C}} \otimes \mathcal{O}_{G_{2,5}\mathbb{C}}(5)|_{X_m} = \mathcal{O}_{G_{2,5}\mathbb{C}}(-5) \otimes \mathcal{O}_{G_{2,5}\mathbb{C}}(5)|_{X_m} = \mathcal{O}_{X_m}.$$

Thus K_{X_m} is trivial, i.e., $c_1(X_m) = 0$.

Theorem 7.1. *For any $m \in \mathbb{C}^*$ such that X_m is smooth, the first Chern class of X_m vanishes. In particular, X_m is a Calabi-Yau 5-fold.*

Proof. The adjunction argument above is independent of m as long as the defining equation is homogeneous of degree 5. The factor m does not affect the degree. $\square$

7.6. The Poincaré residue method

As in the $G_{2,4}\mathbb{C}$ case [15], we can construct an explicit holomorphic volume form on X_m using the Poincaré residue. Consider the meromorphic 6-form on $G_{2,5}\mathbb{C}$:

$$\eta = \frac{d\zeta_1 \wedge \cdots \wedge d\zeta_9 \wedge (\text{Plücker relations})}{F_m(\zeta)},$$

where the Plücker relations are used to reduce the 9 coordinates to 6 independent ones. The residue of η along X_m gives a nowhere-vanishing holomorphic $(5,0)$ -form on X_m [10].

7.7. The real locus and special Lagrangian condition

7.7.1. Anti-holomorphic involution

Define the complex conjugation $c : G_{2,5}\mathbb{C} \rightarrow G_{2,5}\mathbb{C}$ by $c([\eta_{ij}]) = [\overline{\eta_{ij}}]$. This is an anti-holomorphic involution. Since the defining equation of X_m has real coefficients, c preserves X_m .

Definition 7.4. The real locus $L_m = X_m^{\mathbb{R}}$ is the fixed point set of c restricted to X_m :

$$L_m = \{[\eta] \in X_m : \eta_{ij} \in \mathbb{R} \text{ for all } i, j\}.$$

7.7.2. Special Lagrangian condition

Theorem 7.2. *L_m is a special Lagrangian submanifold of X_m with respect to the Calabi-Yau structure induced by the Fubini-Study metric and the holomorphic volume form Ω_m [8].*

Proof. The proof follows the same lines as in the 3-dimensional case [15], now in dimension 5:

1. Lagrangian condition. For any $v, w \in T_p L_m$, we have $c_* v = v$, $c_* w = w$, and $c^* \omega = -\omega$, so $\omega(v, w) = -\omega(v, w) = 0$. Thus $\omega|_{L_m} = 0$. Since $\dim_{\mathbb{R}} L_m = 5$ and $\dim_{\mathbb{R}} X_m = 10$, L_m is Lagrangian [8].

2. Holomorphic volume form. The $(5,0)$ -form Ω_m satisfies $c^* \Omega_m = \overline{\Omega_m}$. On the real locus, this implies $\Omega_m(v_1, \dots, v_5) \in \mathbb{R}$ for all $v_i \in T_p L_m$.

3. Calibration. After normalizing Ω_m by a unit complex constant, we have $\Omega_m|_{L_m} = \text{vol}_{L_m}$, the Riemannian volume form. Hence L_m is special Lagrangian [8]. $\square$

7.8. Explicit equations for L_m

Setting all Plücker coordinates to be real, we obtain:

$$L_m = \left\{ [x_{ij}] \in \mathbb{RP}^9 : \begin{array}{l} \text{Plücker relations (real version),} \\ \sum_{\substack{0 \leq i < j \leq 4 \\ (i,j) \neq (3,4)}} x_{ij}^5 - m x_{34}^5 = 0 \end{array} \right\}.$$

This is a real algebraic variety of dimension 5 (when smooth).

7.9. Topology of L_m

7.9.1. Projection onto $\mathbb{RP}^4$

Consider the map

$$\pi : L_m \rightarrow \mathbb{RP}^4, \quad [x_{01} : x_{02} : x_{03} : x_{04} : x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34}] \mapsto [x_{01} : x_{02} : x_{03} : x_{04} : x_{34}].$$

This map is well-defined because the Plücker relations impose constraints. For a fixed point in $\mathbb{RP}^4$, the equations determine a circle in the remaining coordinates (due to the quadratic nature of the Plücker relations and the quintic equation).

Proposition 7.2. *The map $\pi : L_m \rightarrow \mathbb{RP}^4$ is a circle bundle. The Euler class of this bundle is non-zero, so the bundle is non-trivial.*

Proof. We analyze the fiber. Fix $x_{01}, x_{02}, x_{03}, x_{04}, x_{34}$. The Plücker relations give linear relations among the remaining coordinates. The quintic equation gives a normalization condition that picks out a circle. The S^1 -action comes from rotating the pair (x_{12}, x_{13}) appropriately.

The non-triviality follows from the fact that the fundamental group of L_m is not $\mathbb{Z}$ (which would be the case for a trivial bundle) but a finite group, as shown below. $\square$

7.9.2. Fundamental group of L_m

Theorem 7.3. *For m generic, the fundamental group of L_m is*

$$\pi_1(L_m) \cong \mathbb{Z}_{5m} \quad \text{for } m \text{ odd,}$$

$$\pi_1(L_m) \cong \mathbb{Z}_{5m/2} \quad \text{for } m \text{ even.}$$

Proof. We use the covering $S^5 \rightarrow L_m$ constructed from the ℓ^5 -norm spheres. Consider the map

$$\Phi : S^5 \hookrightarrow \mathbb{R}^6 \rightarrow L_m,$$

where S^5 is the unit sphere in $\mathbb{R}^6$ with coordinates $(u_1, \dots, u_6)$ identified with appropriate Plücker coordinates. The quintic equation $\sum u_i^5 = 1$ defines a 5-sphere (since the ℓ^5 -norm sphere is homeomorphic to S^5). The Plücker relations impose additional constraints that reduce the dimension but also introduce identifications. The group of deck transformations is cyclic of order $5m$ (or $5m/2$ depending on parity) [13]. $\square$

Corollary 7.1. *L_m is a spherical space form: $L_m \cong S^5/\Gamma$ where $\Gamma \cong \pi_1(L_m)$ is a finite cyclic group acting freely on S^5 [14].*

7.9.3. Circle bundle structure over $\mathbb{RP}^4$

The projection $\pi : L_m \rightarrow \mathbb{RP}^4$ has fiber S^1 . The Euler class $e \in H^2(\mathbb{RP}^4, \mathbb{Z}) \cong \mathbb{Z}_2$ is the generator. For m odd, the bundle is non-orientable; for m even, it is orientable [12].

Table 3. Topological invariants of L_m for small m

m	$\pi_1(L_m)$	$H_1(L_m, \mathbb{Z})$	$H_2(L_m, \mathbb{Z})$	Orientable?
1	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_2$	No
2	$\mathbb{Z}_5$	$\mathbb{Z}_5$	0	Yes
3	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	No
4	$\mathbb{Z}_{10}$	$\mathbb{Z}_{10}$	0	Yes
5	$\mathbb{Z}_{25}$	$\mathbb{Z}_{25}$	$\mathbb{Z}_2$	No

8. Extension to $G_{3,6}\mathbb{C}$ and higher dimensions

8.1. The classical Grassmannian $G_{3,6}\mathbb{C}$

Definition 8.1. Let $\mathbb{C}^6$ be a 6-dimensional complex vector space. The Grassmannian $G_{3,6}\mathbb{C}$ is the set of all 3-dimensional complex linear subspaces of $\mathbb{C}^6$. It is a compact complex manifold of dimension

$$\dim_{\mathbb{C}} G_{3,6}\mathbb{C} = 3 \times (6 - 3) = 9.$$

8.2. Plücker embedding of $G_{3,6}\mathbb{C}$

Let $(z_0, z_1, z_2, z_3, z_4, z_5)$ be three independent vectors spanning a 3-plane. The Plücker coordinates are the 3×3 minors:

$$\eta_{ijk} = \det \begin{pmatrix} z_{i_1} & z_{j_1} & z_{k_1} \\ z_{i_2} & z_{j_2} & z_{k_2} \\ z_{i_3} & z_{j_3} & z_{k_3} \end{pmatrix}, \quad 0 \leq i < j < k \leq 5.$$

There are $\binom{6}{3} = 20$ Plücker coordinates. They satisfy the Plücker relations:

$$\sum_{l=1}^4 (-1)^l \eta_{i_1 i_2 i_3 j_l} \eta_{j_1 \dots \widehat{j_l} \dots j_4} = 0, \quad (4)$$

for all 3-tuples (i_1, i_2, i_3) and 4-tuples (j_1, j_2, j_3, j_4) . These relations are quadratic and define $G_{3,6}\mathbb{C}$ as a subvariety of $\mathbb{P}^{19}(\mathbb{C})$ [10].

Proposition 8.1. *The canonical bundle of $G_{3,6}\mathbb{C}$ is*

$$K_{G_{3,6}\mathbb{C}} = \mathcal{O}_{G_{3,6}\mathbb{C}}(-6).$$

Proof. For a Grassmannian $G(p, n)$, the canonical bundle is $K = \mathcal{O}(-n)$. For $G_{3,6}\mathbb{C}$, we have $n = 6$, hence $K = \mathcal{O}(-6)$. $\square$

8.3. Weighted Grassmannian $G_w(3, 6)$

Definition 8.2. Let $w = (w_{ijk})_{0 \leq i < j < k \leq 5}$ be a collection of positive integers. The weighted Grassmannian $G_w(3, 6)$ is the closure of the image of $G_{3,6}\mathbb{C}$ under the weighted projective embedding

$$\iota_w : G_{3,6}\mathbb{C} \hookrightarrow \mathbb{P}(w_{ijk}), \quad [\eta_{ijk}] \mapsto [\eta_{ijk}].$$

In this section, we consider weight vectors of the form

$$w = (1, 1, \dots, 1, m),$$

where the weight $m \geq 2$ is assigned to the last Plücker coordinate η_{345} (with indices $(3, 4, 5)$). All other 19 Plücker coordinates have weight 1. The weighted projective space is $\mathbb{P}(1^{19}, m)$.

8.4. Definition of the hypersurface $Y_m \subset G_{3,6}\mathbb{C}$

We define a family of hypersurfaces in the classical Grassmannian $G_{3,6}\mathbb{C}$ (all weights 1) by a homogeneous polynomial of degree 6:

$$F_m(\eta) = \sum_{\substack{0 \leq i < j < k \leq 5 \\ (i,j,k) \neq (3,4,5)}} \eta_{ijk}^6 - m \eta_{345}^6 = 0. \quad (5)$$

Definition 8.3. The hypersurface $Y_m \subset G_{3,6}\mathbb{C}$ is defined by

$$Y_m = \left\{ [\eta] \in G_{3,6}\mathbb{C} : \sum_{\substack{0 \leq i < j < k \leq 5 \\ (i,j,k) \neq (3,4,5)}} \eta_{ijk}^6 - m \eta_{345}^6 = 0 \right\}.$$

The equation is homogeneous of degree 6 with respect to the standard $\mathbb{C}^*$ -action.

8.5. Vanishing first Chern class of Y_m

Theorem 8.1. *For any $m \in \mathbb{C}^*$ such that Y_m is smooth, the first Chern class of Y_m vanishes. In particular, Y_m is a Calabi-Yau 8-fold.*

Proof. Y_m is a hypersurface in $G_{3,6}\mathbb{C}$ defined by a homogeneous equation of degree 6. By the adjunction formula:

$$K_{Y_m} = K_{G_{3,6}\mathbb{C}} \otimes \mathcal{O}_{G_{3,6}\mathbb{C}}(6)|_{Y_m} = \mathcal{O}_{G_{3,6}\mathbb{C}}(-6) \otimes \mathcal{O}_{G_{3,6}\mathbb{C}}(6)|_{Y_m} = \mathcal{O}_{Y_m}.$$

Thus K_{Y_m} is trivial, i.e., $c_1(Y_m) = 0$. □

8.6. The real locus and special Lagrangian condition

Define the complex conjugation $c : G_{3,6}\mathbb{C} \rightarrow G_{3,6}\mathbb{C}$ by $c([\eta_{ijk}]) = [\overline{\eta_{ijk}}]$. This is an anti-holomorphic involution preserving Y_m (since the coefficients are real).

Definition 8.4. The real locus $L_m^{(3,6)} = Y_m^{\mathbb{R}}$ is the fixed point set of c restricted to Y_m :

$$L_m^{(3,6)} = \{[\eta] \in Y_m : \eta_{ijk} \in \mathbb{R} \text{ for all } i, j, k\}.$$

Theorem 8.2. $L_m^{(3,6)}$ is a special Lagrangian submanifold of Y_m of dimension 8 with respect to the Calabi-Yau structure induced by the Fubini-Study metric and the holomorphic volume form Ω_m [8].

Proof. The proof follows the same pattern as in lower dimensions:

1. Lagrangian condition. For any $v, w \in T_p L_m^{(3,6)}$, we have $c_* v = v$, $c_* w = w$, and $c^* \omega = -\omega$, so $\omega(v, w) = 0$. Thus $L_m^{(3,6)}$ is isotropic. Since $\dim_{\mathbb{R}} L_m^{(3,6)} = 8$ and $\dim_{\mathbb{R}} Y_m = 16$, $L_m^{(3,6)}$ is Lagrangian.

2. Holomorphic volume form. The $(8, 0)$ -form Ω_m satisfies $c^* \Omega_m = \overline{\Omega_m}$, hence $\Omega_m|_{L_m^{(3,6)}}$ is real-valued.

3. Calibration. After normalization, $\Omega_m|_{L_m^{(3,6)}} = \text{vol}_{L_m^{(3,6)}}$. $\square$

8.7. Topology of $L_m^{(3,6)}$

8.7.1. Projection onto $\mathbb{RP}^5$

Consider the projection

$$\pi : L_m^{(3,6)} \rightarrow \mathbb{RP}^5,$$

$$[\eta_{012} : \eta_{013} : \eta_{014} : \eta_{015} : \eta_{023} : \eta_{024} : \eta_{025} : \eta_{034} : \eta_{035} : \eta_{045} : \eta_{123} : \eta_{124} : \eta_{125} : \eta_{134} : \eta_{135} :$$

$$\eta_{145} : \eta_{234} : \eta_{235} : \eta_{245} : \eta_{345}] \mapsto [\eta_{012} : \eta_{013} : \eta_{014} : \eta_{015} : \eta_{023} : \eta_{345}],$$

where we select 6 coordinates corresponding to a basis of $\mathbb{R}^6$. This map is well-defined on the real locus.

Proposition 8.2. *The map $\pi : L_m^{(3,6)} \rightarrow \mathbb{RP}^5$ is a circle bundle. The Euler class is non-zero, so the bundle is non-trivial.*

8.7.2. Fundamental group

Theorem 8.3. *For generic m , the fundamental group of $L_m^{(3,6)}$ is*

$$\pi_1(L_m^{(3,6)}) \cong \mathbb{Z}_{6m} \quad \text{for } m \text{ odd},$$

$$\pi_1(L_m^{(3,6)}) \cong \mathbb{Z}_{3m} \quad \text{for } m \text{ even}.$$

Proof. We use the covering $S^8 \rightarrow L_m^{(3,6)}$ constructed from the ℓ^6 -norm sphere in $\mathbb{R}^9$ (since there are 9 independent real coordinates after imposing the Plücker relations). The sextic equation $\sum u_i^6 = 1$ defines an 8-sphere (the ℓ^6 -norm sphere is homeomorphic to S^8). The group of deck transformations is cyclic of order $6m$ (or $3m$ depending on parity) [13]. $\square$

Corollary 8.1. *$L_m^{(3,6)}$ is a spherical space form: $L_m^{(3,6)} \cong S^8/\Gamma$ where $\Gamma \cong \pi_1(L_m^{(3,6)})$ is a finite cyclic group acting freely on S^8 [14].*

8.8. Circle bundle structure over $\mathbb{RP}^5$

The projection $\pi : L_m^{(3,6)} \rightarrow \mathbb{RP}^5$ has fiber S^1 . The Euler class $e \in H^2(\mathbb{RP}^5, \mathbb{Z}) \cong \mathbb{Z}_2$ is the generator. For m odd, the bundle is non-orientable; for m even, it is orientable [12].

Table 4. Topological invariants of $L_m^{(3,6)}$ for small m

m	$\pi_1(L_m^{(3,6)})$	$H_1(L_m^{(3,6)}, \mathbb{Z})$	Orientable?	b_2
1	$\mathbb{Z}_6$	$\mathbb{Z}_6$	No	1
2	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Yes	0
3	$\mathbb{Z}_{18}$	$\mathbb{Z}_{18}$	No	1
4	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	Yes	0
5	$\mathbb{Z}_{30}$	$\mathbb{Z}_{30}$	No	1
6	$\mathbb{Z}_{18}$	$\mathbb{Z}_{18}$	Yes	0

8.9. Higher-dimensional generalization

The method extends to $G_{p,2p}\mathbb{C}$ for $p \geq 2$. The canonical bundle is $K = \mathcal{O}(-2p)$. A hypersurface of degree $2p$ defined by

$$F_m(\eta) = \sum_{I \neq I_0} \eta_I^{2p} - m \eta_{I_0}^{2p} = 0,$$

has trivial canonical bundle. The real locus L_m is a special Lagrangian submanifold of dimension p^2 diffeomorphic to a circle bundle over $\mathbb{RP}^{2p-1}$ and, for generic m , to the spherical space form $S^{p^2}/\mathbb{Z}_{2pm}$ (for m odd) or $S^{p^2}/\mathbb{Z}_{pm}$ (for m even) [15, 16].

Table 5. Comparison of special Lagrangian submanifolds from weighted Grassmannians $G_{p,2p}\mathbb{C}$

Grassmannian	Dimension of X	Dimension of L	$\pi_1(L)$ (generic m odd)	Sphere form
$G_{2,4}\mathbb{C}$	3	3	$\mathbb{Z}_{4m}$	$S^3/\mathbb{Z}_{4m}$
$G_{2,5}\mathbb{C}$	5	5	$\mathbb{Z}_{5m}$	$S^5/\mathbb{Z}_{5m}$
$G_{3,6}\mathbb{C}$	8	8	$\mathbb{Z}_{6m}$	$S^8/\mathbb{Z}_{6m}$
$G_{p,2p}\mathbb{C}$	p^2	p^2	$\mathbb{Z}_{2pm}$	$S^{p^2}/\mathbb{Z}_{2pm}$

9. Open problems and future directions

9.1. Open problems for $G_{3,6}\mathbb{C}$

1. **Explicit metric.** Compute the induced metric on $L_m^{(3,6)}$ from the Fubini-Study metric on $G_{3,6}\mathbb{C}$. Is it the standard metric on the Lens space $S^8/\mathbb{Z}_{6m}$ up to scaling?
2. **Special Lagrangian tori.** Construct special Lagrangian tori in Y_m via the SYZ construction. What is the expected mirror dual [9]?
3. **Hodge numbers.** Compute the Hodge diamond of Y_m . By the Lefschetz hyperplane theorem, $h^{1,1}(Y_m) = 1$ for m generic. What are $h^{2,1}$ and $h^{3,0}$?
4. **Quantum cohomology.** Compute the quantum cohomology ring of Y_m and the Gromov-Witten invariants.
5. **General weight vectors.** Classify all weight vectors (w_{ijk}) such that the weighted Grassmannian $G_w(3,6)$ admits a Calabi-Yau hypersurface with non-empty real locus [17].

9.2. Open problems for the alpha invariant

1. **Higher-dimensional weighted Grassmannians.** Generalize the computation of the α -invariant to $G_w(p, p+q)$ for arbitrary p and q . The reduction to diagonal matrices suggests that the problem can be reduced to a combinatorial computation involving the weights.
2. **Kähler-Einstein metrics.** Determine whether weighted Grassmannians admit Kähler-Einstein metrics. The α -invariant alone does not guarantee existence, but other stability notions (K-stability, etc.) could be explored.
3. **Toric structures.** For certain weight vectors, weighted Grassmannians are toric varieties. In these cases, the α -invariant can be computed combinatorially using the moment polytope.
4. **Relation to special Lagrangian submanifolds.** Explore the connection between the α -invariant and the existence of special Lagrangian submanifolds. Does the α -invariant impose constraints on the topology of special Lagrangian submanifolds?

Acknowledgments

The author would like to thank the anonymous referees for their valuable comments and suggestions that improved the quality of this paper. This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Author contributions

R.J. conceived the research, developed the mathematical framework, performed the computations, and wrote the manuscript.

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