



Quasi Zariski-like topology on quasi-primary spectrum of lattice modules

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Abstract: Let L denote a multiplicative lattice and M be a lattice module over L . A proper element K of a lattice module M is called quasi-primary, whenever the relation $aA \leq K$ implies either $a \leq \sqrt{(K : 1_M)}$ or $A \leq \sqrt{K}$. In this work, we define the set $\text{Spec}^{qp}(M)$ as the family of all quasi-primary elements K of M satisfying the condition $(\sqrt{K} : 1_M) = \sqrt{(K : 1_M)}$ and this collection is referred to as the quasi-primary spectrum. Furthermore, we establish Zariski-type topology on quasi-primary spectrum. We then examine several topological characteristics of this space, with particular emphasis on compactness and irreducibility. In addition, we prove that the space is T_0 and spectral whenever the set $\text{Spec}_p^{qp}(M) = \{K \in \text{Spec}^{qp}(M) | \sqrt{(K : 1_M)} = p, p \in \text{Spec}(L)\}$ consists of a single element.

Key words: Lattice modules, Quasi-primary elements, Quasi-primary spectrum, Zariski-like topology, Spectral spaces.

1. Introduction

The study of Zariski-type topologies has been extended beyond the prime spectrum of commutative rings to the prime submodules of modules, leading to several important generalizations in recent years [6, 17]. One of the contributions in this direction was made by Behboodi and Noori [7], who introduced a Zariski-type topology on the classical prime spectrum of modules. Subsequently, Girase et al.[13] adapted this framework to the setting of lattice modules by developing a corresponding topology on their classical prime spectrum. Likewise, Borkar et al.[8] established a Zariski-type topological structure for the prime spectrum associated with lattice modules. Furthermore, M. Samiei et al.[21] introduced the notion of quasi-primary submodules and investigated the topological properties of their spectrum. Inspired by these developments, the present work focuses on the spectrum of quasi-primary elements in a lattice module M over a C -lattice L . We introduce a Zariski-like topology on this spectrum and examine its fundamental properties. In addition, several known results concerning prime and primary spectra are extended to the quasi-primary setting.

Throughout this work, let L denote a complete multiplicative lattice with least element 0_L and the greatest element 1_L . The lattice L is assumed to possess a multiplication operation “.” satisfying the conditions:

- (1) $x.y = y.x$, for all $x, y \in L$;
- (2) $x.(y.z) = (x.y).z$, for all $x, y, z \in L$;
- (3) $x.(\bigvee_{\alpha} y_{\alpha}) = \bigvee_{\alpha} (x.y_{\alpha})$, for all $x, y_{\alpha} \in L$;
- (4) $x.1_L = x$, for all $x \in L$.

An element x of the lattice L is called compact if whenever $x \leq \bigvee_{\alpha \in I} y_\alpha$, $x \leq y_{\alpha_1} \vee y_{\alpha_2} \vee \cdots \vee y_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of I . A C -lattice is a multiplicative lattice generated by its compact elements under arbitrary joins, where the multiplicatively closed set of compact elements contains the identity element. For more details on C -lattice the reader may refer ([12],[16],[22]-[24]).

An element x is referred to as *proper* whenever it is strictly less than the greatest element 1_L of the lattice. A proper element q of a multiplicative lattice L is said to be *prime* if $xy \leq q$ implies $x \leq q$ or $y \leq q$ for $x, y \in L$. The collection of all prime elements of L is denoted by $\text{Spec}(L)$.

The Zariski topology on the set $\text{Spec}(L)$ of all prime elements in multiplicative lattices is being studied in [23], by Thakare, Manjarekar and Maeda and in [22], by Thakare and Manjarekar as a generalization of the Zariski topology of a commutative ring with unity.

A proper element m of a multiplicative lattice L is said to be *maximal* if for every $x \in L$ with $m < x \leq 1_L$ implies $x = 1_L$. The radical of an element $a \in L$ is defined as $\sqrt{a} = \bigvee \{x \in L \mid x \text{ is compact and } x^n \leq a, \text{ for some positive integer } n\}$ [3].

A complete lattice M with smallest element 0_M and greatest element 1_M is said to be a *lattice module* over the multiplicative lattice L or L -module if there is a multiplication between elements of M and L , denoted by $xA \in M$, for $x \in L$ and $A \in M$, which satisfies the following properties:

1. $(xy)A = x(yA)$;
2. $(\bigvee_\alpha x_\alpha)(\bigvee_\beta A_\beta) = (\bigvee_{\alpha\beta} x_\alpha A_\beta)$;
3. $1_L A = A$;
4. $0_L A = 0_M$; for all $x, y, x_\alpha \in L$, and for all $A, A_\beta \in M$.

Assume that M is a lattice module over a C -lattice L . For $A \in M, y \in L$, denote $(A : y) = \bigvee \{K \in M \mid yK \leq A\}$. If $x, y \in L$, we write $(x : y) = \bigvee \{a \in L \mid ya \leq x\}$. If $A, B \in M$, then $(A : B) = \bigvee \{x \in L \mid xB \leq A\}$. An element $A \in M$ is said to be compact if $A \leq \bigvee_{\alpha \in I} A_\alpha$ (I is an indexed set) implies $A \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of I .

An element $N \in M$ is said to be *meet principal* (respectively *join principal*) if it satisfies the identity $A \wedge aN = (a \wedge (A : N))N$ (respectively $((aN \vee A) : N) = (a \vee (A : N))$) for all $a \in L$ and for all $A \in M$. Also N is said to be *principal* if it is both join as well as meet principal. If each element of M is the join of principal (compact) elements of M , then M is called the *principally generated* (*compactly generated*) lattice module.

An element $A \in M$ is said to be proper, if $A < 1_M$. A proper element A of a lattice module M is said to be *prime* if $aX \leq A$ implies $X \leq A$ or $a1_M \leq A$, i.e., $a \leq (A : 1_M)$ for every $a \in L$ and $X \in M$. The prime spectrum of a lattice module M is the set of all prime elements of M and it is denoted by $\text{Spec}(M)$. In [5], S. Ballal and V. Kharat studied the Zariski topology over $\text{Spec}(M)$ as a generalization of the results carried out in [[23], [22]]. Also in [11], Fethi Callialp et al. studied the Zariski topology over $\text{Spec}(M)$ over multiplicative lattice L .

For an element $N \in M$, the radical of N is defined as $\sqrt{N} = \bigwedge \{P \in \text{Spec}(M) \mid N \leq P\}$ [18]. If there is no prime which contains N , then $\sqrt{N} = 1_M$. Note that, $N \leq \sqrt{N}$.

An element $N < 1_M$ of M is said to be *maximal* if $N \leq B$ implies either $N = B$ or $B = 1_M, B \in M$. If 1_M is compact, then M has a maximal element [20].

For further details on these concepts and more information on multiplicative lattices, lattice modules, modules and topology, the reader may refer ([1], [2], [15], [19]).

2. Quasi Zariski-like Topology on $\text{Spec}^{qp}(M)$

We begin this section with the following definition.

Definition 2.1. Let M be a lattice module over a C -lattice L . An element $K < 1_M$ is said to be quasi-primary, if $aA \leq K$ implies that $a \leq \sqrt{(K : 1_M)}$ or $A \leq \sqrt{K}$.

Note that, if K is quasi-primary element of M , then $(K : 1_M)$ is primary element of L and $\sqrt{(K : 1_M)} = p$ is a prime element of L . In this case, K is called p -quasi-primary element of M . We use $\text{Spec}^{qp}(M)$ to denote the collection of all quasi-primary elements K of M , for which $\sqrt{(K : 1_M)} = (\sqrt{K} : 1_M)$. We consider $\mathcal{Q}^q(H) = \{K \in \text{Spec}^{qp}(M) \mid \sqrt{(H : 1_M)} \leq \sqrt{(K : 1_M)}\}$, for $H \in M$.

Lemma 2.1. [15] Let M be a lattice module over a C -lattice L . Then for $N \in M$, $(N : 1_M) = ((N : 1_M)1_M : 1_M)$.

The following properties will be used throughout this paper.

Lemma 2.2. [15] Let M be a lattice module over a multiplicative lattice L . Then for $x \in L$ and $A, B, C, A_\alpha \in M$, following hold.

1. $x \leq (0_M : (0_M : x))$.
2. $A \leq (0_M : (0_M : A))$.
3. If $A \leq B$ then $(C : B) \leq (C : A)$.
4. If $A \leq B$ then $(A : C) \leq (B : C)$.
5. $\vee_\alpha (A_\alpha : B) \leq ((\vee_\alpha A_\alpha) : B)$.
6. If $a \leq b$ then $aA \leq bA$.
7. $(0_M : A) = (0_M : (0_M : (0_M : A)))$.
8. $(A : B \vee C) = (A : B) \wedge (A : C)$.
9. $x(A : x) \leq A$.
10. $(A : B)B \leq A$.
11. $a \leq (aA : A)$.
12. $(B \wedge C : A) = (B : A) \wedge (C : A)$.

Some of the results in this paper require the following properties.

Properties 2.1. ([22] and [23]) Let L be a multiplicative lattice and $a, b \in L$. Then

1. $a \leq \sqrt{a}$.

2. $a \leq b$ implies $\sqrt{a} \leq \sqrt{b}$.
3. For $x \in L_*$, $x \leq \sqrt{a}$ if and only if $x^n \leq a$ for some integer $n \geq 1$.
4. $\sqrt{\sqrt{a}} = \sqrt{a}$.
5. $\sqrt{a \wedge b} = \sqrt{a} \wedge \sqrt{b} = \sqrt{ab}$.
6. $\sqrt{a \vee b} = \sqrt{\sqrt{a} \vee \sqrt{b}}$.
7. If $1 \in L$ is compact, then $a < 1$ implies $\sqrt{a} < 1$.
8. $\sqrt{a} = \wedge \{p \in L \mid p \text{ is prime and } a \leq p\} = \wedge \{p \in L \mid p \text{ is minimal prime over } a\}$.
9. If $a \in L$ is prime, then $\sqrt{a} = a$.

We begin by stating the following lemma.

Lemma 2.3. *Let M be a lattice module over a C -lattice L . Then the following conditions hold:*

1. $\mathcal{Q}^q(0_M) = \text{Spec}^{qp}(M)$ and $\mathcal{Q}^q(1_M) = \emptyset$
2. $\cap_{i \in I} \mathcal{Q}^q(H_i) = \mathcal{Q}^q(\vee_{i \in I} (H_i : 1_M) 1_M)$, H_i , for $i \in I$ (Indexed set) are elements of M .
3. $\mathcal{Q}^q(H) \cup \mathcal{Q}^q(F) = \mathcal{Q}^q(H \wedge F)$, H, F are any elements of M .

Proof. 1. We have, $0_M \leq K$, therefore by Lemma 2.2(4), $(0_M : 1_M) \leq (K : 1_M)$, and hence, $\sqrt{(0_M : 1_M)} \leq \sqrt{(K : 1_M)}$, by Properties 2.1(2). This yields $\mathcal{Q}^q(0_M) = \text{Spec}^{qp}(M)$. Obviously, $\mathcal{Q}^q(1_M) = \emptyset$.

2. Take $K \in \cap_{i \in I} \mathcal{Q}^q(H_i)$. Then $K \in \mathcal{Q}^q(H_i)$, for each $i \in I$. It implies that, $\sqrt{(H_i : 1_M)} \leq \sqrt{(K : 1_M)}$. Using Properties 2.1(1), $(H_i : 1_M) \leq \sqrt{(H_i : 1_M)} \leq \sqrt{(K : 1_M)}$, for each $i \in I$. Therefore, we have $\vee_{i \in I} (H_i : 1_M) \leq \sqrt{(K : 1_M)}$. By Lemma 2.2(6), it further implies that $\vee_{i \in I} (H_i : 1_M) 1_M \leq \sqrt{(K : 1_M)} 1_M = (\sqrt{K} : 1_M) 1_M$. According to Lemma 2.2(10), $(\sqrt{K} : 1_M) 1_M \leq \sqrt{K}$. One obtains, $\vee_{i \in I} (H_i : 1_M) 1_M \leq \sqrt{K}$. Hence, using Lemma 2.2(4), we conclude that $(\vee_{i \in I} (H_i : 1_M) 1_M : 1_M) \leq (\sqrt{K} : 1_M)$. Consequently, $\sqrt{(\vee_{i \in I} (H_i : 1_M) 1_M : 1_M)} \leq \sqrt{(K : 1_M)}$, by Properties 2.1(2) and (4). Hence, we get $K \in \mathcal{Q}^q(\vee_{i \in I} (H_i : 1_M) 1_M)$. To prove the reverse inclusion, let us consider $K \in \mathcal{Q}^q(\vee_{i \in I} (H_i : 1_M) 1_M)$. Thus $\sqrt{(\vee_{i \in I} (H_i : 1_M) 1_M : 1_M)} \leq \sqrt{(K : 1_M)}$. Hence, we get $(\vee_{i \in I} (H_i : 1_M) 1_M : 1_M) \leq \sqrt{(K : 1_M)}$, by Properties 2.1(1). Therefore $\vee_{i \in I} ((H_i : 1_M) 1_M : 1_M) \leq \sqrt{(K : 1_M)}$, by Lemma 2.2(5). It follows that $((H_i : 1_M) 1_M : 1_M) \leq \sqrt{(K : 1_M)}$ for all $i \in I$. Hence $(H_i : 1_M) \leq \sqrt{(K : 1_M)}$, by Lemma 2.1. From this, we obtain $\sqrt{(H_i : 1_M)} \leq \sqrt{(K : 1_M)}$, for each $i \in I$. Therefore $K \in \mathcal{Q}^q(H_i)$, for each $i \in I$. Accordingly, $K \in \cap_{i \in I} \mathcal{Q}^q(H_i)$.

3. Let $K \in \mathcal{Q}^q(H) \cup \mathcal{Q}^q(F)$. Then $K \in \mathcal{Q}^q(H)$ or $K \in \mathcal{Q}^q(F)$. Therefore $\sqrt{(H : 1_M)} \leq \sqrt{(K : 1_M)}$ or $\sqrt{(F : 1_M)} \leq \sqrt{(K : 1_M)}$. Since, $H \wedge F \leq H, F$, it follows that $((H \wedge F) : 1_M) \leq (H : 1_M)$ and $((H \wedge F) : 1_M) \leq (F : 1_M)$, by Lemma 2.2(4). Consequently, $\sqrt{((H \wedge F) : 1_M)} \leq \sqrt{(K : 1_M)}$, by Properties 2.1(2). Hence $K \in \mathcal{Q}^q(H \wedge F)$. For the opposite inclusion, let $K \in \mathcal{Q}^q(H \wedge F)$. This gives $((H \wedge F) : 1_M) \leq \sqrt{(K : 1_M)}$. Applying Lemma 2.2(12), it follows that $(H : 1_M) \wedge (F : 1_M) \leq \sqrt{(K : 1_M)}$. The primality of $\sqrt{(K : 1_M)}$ in L implies that $(H : 1_M) \leq \sqrt{(K : 1_M)}$ or $(F : 1_M) \leq \sqrt{(K : 1_M)}$. According

to Properties 2.1(2) and (4), we conclude that $\sqrt{(H : 1_M)} \leq \sqrt{(K : 1_M)}$ or $\sqrt{(F : 1_M)} \leq \sqrt{(K : 1_M)}$. Hence, $K \in \mathcal{Q}^q(H)$ or $K \in \mathcal{Q}^q(F)$. $\square$

Lemma 2.3 shows that the sets $\mathcal{Q}^q(H)$ satisfy the axioms of closed sets for a topology, called as quasi Zariski-like topology on $\text{Spec}^{qp}(M)$.

For each $p \in \text{Spec}(L)$, take $\text{Spec}_p^{qp}(M) = \{K \in \text{Spec}^{qp}(M) \mid \sqrt{(K : 1_M)} = p\}$.

In what follows, L/a denotes the interval $[a, 1]$ which is multiplicative lattice with $x \circ y = xy \vee a$, where $x, y \in L$ [10]. We denote $\text{Spec}(L/(0_M : 1_M))$ by $\text{Spec}(\bar{L})$.

Let $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ be defined by $\psi^{qp}(K) = \overline{\sqrt{(K : 1_M)}}$. Then ψ^{qp} is called the natural map on $\text{Spec}^{qp}(M)$.

Proposition 2.1. *Suppose that M is a lattice module over a C -lattice L and the map $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ is the natural map. Let $K, K' \in \text{Spec}^{qp}(M)$. Then the following statements are equivalent:*

1. $\mathcal{Q}^q(K) = \mathcal{Q}^q(K')$ implies $K = K'$.
2. $|\text{Spec}_p^{qp}(M)| \leq 1$, for each $p \in \text{Spec}(L)$.
3. ψ^{qp} is injective.

Proof. (1) $\implies$ (2) : Take $K, K' \in \text{Spec}_p^{qp}(M)$. Then $\sqrt{(K : 1_M)} = \sqrt{(K' : 1_M)} = p$. Consider $P \in \mathcal{Q}^q(K)$. Thus, $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$. It follows that $\sqrt{(K' : 1_M)} \leq \sqrt{(P : 1_M)}$. This yields $P \in \mathcal{Q}^q(K')$. Hence $\mathcal{Q}^q(K) \subseteq \mathcal{Q}^q(K')$. Now we show the converse inclusion, so take $P \in \mathcal{Q}^q(K')$. Therefore $\sqrt{(K' : 1_M)} \leq \sqrt{(P : 1_M)}$ and hence $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$. One obtains $P \in \mathcal{Q}^q(K)$ and $\mathcal{Q}^q(K') \subseteq \mathcal{Q}^q(K)$. We conclude that, $\mathcal{Q}^q(K) = \mathcal{Q}^q(K')$ and hence $K = K'$. Consequently, $|\text{Spec}_p^{qp}(M)| \leq 1$.

(2) $\implies$ (3) : Let $K, K' \in \text{Spec}^{qp}(M)$ and $\psi^{qp}(K) = \psi^{qp}(K')$. Then $\overline{\sqrt{(K : 1_M)}} = \overline{\sqrt{(K' : 1_M)}}$, i.e., $\sqrt{(K : 1_M)} = \sqrt{(K' : 1_M)} = p$. Therefore $K, K' \in \text{Spec}_p^{qp}(M)$ and $K = K'$. Thus, the statement holds.

(3) $\implies$ (1) : Assuming that ψ^{qp} is an injective map and $\mathcal{Q}^q(K) = \mathcal{Q}^q(K')$, we obtain $\overline{\sqrt{(K : 1_M)}} = \overline{\sqrt{(K' : 1_M)}}$. From this, it follows $\psi^{qp}(K) = \psi^{qp}(K')$ and $K = K'$. $\square$

Corollary 2.1. *Suppose that M is a lattice module over a C -lattice L and the map $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ is the natural map. If for each $p \in \text{Spec}(L)$, $|\text{Spec}_p^{qp}(M)| \leq 1$, then ψ^{qp} is a bijective map.*

Proof. It follows from Proposition 2.1. $\square$

Let $V(a) = \{p \in \text{Spec}(L) \mid a \leq p\}$. Then the collection of all subsets $V(a)$ of $\text{Spec}(L)$ forms a Zariski topology on $\text{Spec}(L)$ [11]. Note that, $V(\bar{a}) = \{\bar{p} \in \text{Spec}(\bar{L}) \mid \bar{a} \leq \bar{p}\}$.

Proposition 2.2. *Suppose that M is a lattice module over a C -lattice L and the map $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ is the natural map. Then $(\psi^{qp})^{-1}(V(\bar{a})) = \mathcal{Q}^q(a1_M)$, for each element $a \in L$, such that $(0_M : 1_M) \leq a$. Therefore the map ψ^{qp} is continuous.*

Proof. Let $K \in (\psi^{qp})^{-1}(V(\bar{a}))$. Then $\psi^{qp}(K) \in V(\bar{a})$. It implies that $\sqrt{(K : 1_M)} \in V(\bar{a})$. Therefore $\bar{a} \leq \sqrt{(K : 1_M)}$ and $a \leq \sqrt{(K : 1_M)} = (\sqrt{K} : 1_M)$. From this, it follows that $a1_M \leq \sqrt{K}$. According to Lemma 2.2(4), $(a1_M : 1_M) \leq (\sqrt{K} : 1_M)$ which implies that $\sqrt{(a1_M : 1_M)} \leq \sqrt{(K : 1_M)}$ by using Properties 2.1(2) and (4). Hence, $K \in \mathcal{Q}^q(a1_M)$. Consequently, $(\psi^{qp})^{-1}(V(\bar{a})) \subseteq \mathcal{Q}^q(a1_M)$. It remains to prove the other inclusion, so take $K \in \mathcal{Q}^q(a1_M)$. Then $\sqrt{(a1_M : 1_M)} \leq \sqrt{(K : 1_M)}$. From this, we get $(a1_M : 1_M) \leq (\sqrt{K} : 1_M)$. Applying Lemma 2.2(11), we obtain $a \leq (\sqrt{K} : 1_M)$. Hence $\sqrt{(K : 1_M)} \in V(\bar{a})$ and this yields $K \in (\psi^{qp})^{-1}(V(\bar{a}))$. Thus, we have $\mathcal{Q}^q(a1_M) \subseteq (\psi^{qp})^{-1}(V(\bar{a}))$. This completes the proof. $\square$

Lemma 2.4. *Suppose that M is a lattice module over a C -lattice L and H, F are elements of M . Then the following statements hold:*

1. $\sqrt{(H : 1_M)} = \sqrt{(F : 1_M)}$ implies $\mathcal{Q}^q(H) = \mathcal{Q}^q(F)$. The converse holds when H and F are quasi-primary.
2. $\mathcal{Q}^q((H : 1_M)1_M) = \mathcal{Q}^q(H)$.

Proof. 1. Consider $\sqrt{(H : 1_M)} = \sqrt{(F : 1_M)}$. Then $K \in \mathcal{Q}^q(H)$ if and only if $\sqrt{(H : 1_M)} \leq \sqrt{(K : 1_M)}$ if and only if $\sqrt{(F : 1_M)} \leq \sqrt{(K : 1_M)}$ if and only if $K \in \mathcal{Q}^q(F)$. Consequently, $\mathcal{Q}^q(H) = \mathcal{Q}^q(F)$. To prove the converse, let us assume that H and F are quasi-primary. Because H is quasi-primary, $H \in \mathcal{Q}^q(H) = \mathcal{Q}^q(F)$ and hence $\sqrt{(F : 1_M)} \leq \sqrt{(H : 1_M)}$. Again, as F is quasi-primary, $F \in \mathcal{Q}^q(F) = \mathcal{Q}^q(H)$. As a consequence, $\sqrt{(H : 1_M)} \leq \sqrt{(F : 1_M)}$. This establishes the result.

2. Assume that $K \in \mathcal{Q}^q(H)$. Then $\sqrt{(H : 1_M)} \leq \sqrt{(K : 1_M)}$. Hence $(H : 1_M) \leq (\sqrt{K} : 1_M)$ by Properties 2.1(1). By Lemma 2.1, we have $(H : 1_M) = ((H : 1_M)1_M : 1_M)$. Therefore $((H : 1_M)1_M : 1_M) \leq (\sqrt{K} : 1_M)$. It further implies that $\sqrt{((H : 1_M)1_M : 1_M)} \leq \sqrt{(K : 1_M)}$ by Properties 2.1(2) and (4). Hence, we conclude that $K \in \mathcal{Q}^q((H : 1_M)1_M)$. To prove the reverse inclusion, take $K \in \mathcal{Q}^q((H : 1_M)1_M)$. Then, $\sqrt{((H : 1_M)1_M : 1_M)} \leq \sqrt{(K : 1_M)}$. Therefore $((H : 1_M)1_M : 1_M) \leq (\sqrt{K} : 1_M)$ by Properties 2.1(1). According to Lemma 2.1, we have $(H : 1_M) = ((H : 1_M)1_M : 1_M)$, hence $(H : 1_M) \leq (\sqrt{K} : 1_M) = \sqrt{(K : 1_M)}$. Thus $\sqrt{(H : 1_M)} \leq \sqrt{(K : 1_M)}$, by Properties 2.1(2) and (4). Consequently, $K \in \mathcal{Q}^q(H)$. Therefore, the theorem is proved. $\square$

Theorem 2.1. *Suppose that M is a lattice module over a C -lattice L and the map $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ is the surjective natural map. Then the map ψ^{qp} is both closed and open, i.e., $\psi^{qp}(\mathcal{Q}^q(H)) = V(\overline{(H : 1_M)})$ and $\psi^{qp}(\text{Spec}^{qp}(M) - \mathcal{Q}^q(H)) = \text{Spec}(\bar{L}) - V(\overline{(H : 1_M)})$, for $H \in M$.*

Proof. By applying Proposition 2.2, we obtain, $(\psi^{qp})^{-1}(V(\overline{(H : 1_M)})) = \mathcal{Q}^q((H : 1_M)1_M) = \mathcal{Q}^q(H)$. Hence, $\psi^{qp}(\mathcal{Q}^q(H)) = \psi^{qp}((\psi^{qp})^{-1}(V(\overline{(H : 1_M)}))) = V(\overline{(H : 1_M)})$. Consider $\psi^{qp}(\text{Spec}^{qp}(M) - \mathcal{Q}^q(H)) = \psi^{qp}((\psi^{qp})^{-1}(\text{Spec}(\bar{L}) - V(\overline{(H : 1_M)}))) = \text{Spec}(\bar{L}) - V(\overline{(H : 1_M)})$. This completes the proof. $\square$

We fix, $cl(X)$ = the closure of a set X .

Proposition 2.3. *If M is a lattice module over a C -lattice L and $X = \{K_1, K_2, \dots, K_n\}$ is a subset of $\text{Spec}^{qp}(M)$, then $cl(X) = \bigcup_{i=1}^n \mathcal{Q}^q(K_i)$.*

Proof. It is easy to verify that $X \subseteq \cup_{i=1}^n \mathcal{Q}^q(K_i)$. Suppose that Y is any closed subset of $\text{Spec}^{qp}(M)$ containing X . Then $Y = \mathcal{Q}^q(H)$ for the element H in M . For any $K \in \cup_{i=1}^n \mathcal{Q}^q(K_i)$, it implies that $K \in \mathcal{Q}^q(K_i)$ for some i . Thus $\sqrt{(K_i : 1_M)} \leq \sqrt{(K : 1_M)}$. Since, K_i is a member of Y , $\sqrt{(H : 1_M)} \leq \sqrt{(K_i : 1_M)} \leq \sqrt{(K : 1_M)}$ and hence $K \in Y$. It follows immediately that $\cup_{i=1}^n \mathcal{Q}^q(K_i) \subseteq Y$. Consequently, $\cup_{i=1}^n \mathcal{Q}^q(K_i)$ is the smallest closed set of $\text{Spec}^{qp}(M)$ containing X . Hence, $cl(X) = \cup_{i=1}^n \mathcal{Q}^q(K_i)$. $\square$

Corollary 2.2. *Assume that M is a lattice module over a C -lattice L . Then the following statements hold:*

1. $\mathcal{Q}^q(K) = cl(\{K\})$, for some $K \in \text{Spec}^{qp}(M)$.
2. $\mathcal{Q}^q(Q) \subseteq \mathcal{Q}^q(K)$ if and only if $\sqrt{(K : 1_M)} \leq \sqrt{(Q : 1_M)}$ if and only if $Q \in cl(\{K\})$.

Proof. 1. According to Proposition 2.3, we have, $cl(X) = \cup_{i=1}^n \mathcal{Q}^q(K_i)$, for subset $X = \{K_1, K_2, \dots, K_n\}$ of $\text{Spec}^{qp}(M)$. Let $X = \{K\}$. It implies that $cl(X) = cl(\{K\}) = \mathcal{Q}^q(K)$.

2. $Q \in cl(\{K\}) = \mathcal{Q}^q(K)$ if and only if $\sqrt{(K : 1_M)} \leq \sqrt{(Q : 1_M)}$ if and only if $\mathcal{Q}^q(Q) \subseteq \mathcal{Q}^q(K)$. $\square$

Let $\mathcal{C}_a = \text{Spec}^{qp}(M) - \mathcal{Q}^q(a1_M)$ and $\mathcal{D}_{\bar{a}} = \text{Spec}(\bar{L}) - V(\bar{a})$, $\mathcal{D}_a = \text{Spec}(L) - V(a)$, for each $a \in L$. Then it is observed that $\mathcal{C}_a$ is open set in $\text{Spec}^{qp}(M)$ and $\mathcal{D}_a$ is open set in $\text{Spec}(L)$. Moreover $\{\mathcal{D}_a | a \in L\}$ forms a basis for the Zariski topology on $\text{Spec}(L)$ [11].

Lemma 2.5. *Assume that M is a lattice module over a C -lattice L and $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ be the natural map. Then the following statements hold:*

1. $(\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}}) = \mathcal{C}_a$.
2. $\psi^{qp}(\mathcal{C}_a) \subseteq \mathcal{D}_{\bar{a}}$.
3. $\psi^{qp}(\mathcal{C}_a) = \mathcal{D}_{\bar{a}}$, if the map ψ^{qp} is surjective.
4. $\mathcal{C}_{ab} = \mathcal{C}_a \cap \mathcal{C}_b$, for any elements $a, b \in L$.

Proof. 1. Consider $K \in \text{Spec}^{qp}(M) - (\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}})$. Then $a \leq \sqrt{(K : 1_M)} = (\sqrt{K} : 1_M)$. Thus $a1_M \leq \sqrt{K}$. Hence $(a1_M : 1_M) \leq (\sqrt{K} : 1_M) = \sqrt{(K : 1_M)}$, by Lemma 2.2(4). Further by Properties 2.1(2) and (4), $\sqrt{(a1_M : 1_M)} \leq \sqrt{(K : 1_M)}$. Therefore, $K \in \mathcal{Q}^q(a1_M)$ and so $K \in \text{Spec}^{qp}(M) - \mathcal{C}_a$. Consequently, $\mathcal{C}_a \subseteq (\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}})$. Conversely, suppose that $K \in (\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}})$. Then $\psi^{qp}(K) \in \mathcal{D}_{\bar{a}}$ and therefore $\bar{p} = \sqrt{(\psi^{qp}(K) : 1_{\bar{L}})} \in \mathcal{D}_{\bar{a}}$. It follows that $a \not\leq p$, i.e., $a \not\leq \sqrt{(K : 1_M)}$. Hence, we get $a \leq (a1_M : 1_M) \not\leq \sqrt{(K : 1_M)}$, by Lemma 2.2(11). Using Properties 2.1(2) and (4), $\sqrt{(a1_M : 1_M)} \not\leq \sqrt{(K : 1_M)}$. Hence, $K \notin \mathcal{Q}^q(a1_M)$. Consequently, $(\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}}) \subseteq \mathcal{C}_a$. Now, we conclude that $(\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}}) = \mathcal{C}_a$.

2. Let $a \in L$ and $K \in \text{Spec}^{qp}(M)$. Take $\sqrt{(K : 1_M)} = p$. If $\bar{p} \in V(\bar{a})$, then $a \leq p = \sqrt{(K : 1_M)}$ and hence $a1_M \leq \sqrt{K}$. Using Lemma 2.2(4), we obtain $(a1_M : 1_M) \leq (\sqrt{K} : 1_M)$ and further $\sqrt{(a1_M : 1_M)} \leq \sqrt{(K : 1_M)}$, by Properties 2.1(2) and (4). Consequently, $K \in \mathcal{Q}^q(a1_M)$. Therefore, if $K \in \mathcal{C}_a$, then $\psi^{qp}(K) = \sqrt{(\psi^{qp}(K) : 1_{\bar{L}})} \in \mathcal{D}_{\bar{a}}$. Thus $\psi^{qp}(\mathcal{C}_a) \subseteq \mathcal{D}_{\bar{a}}$.

3. According to part (2), $\psi^{qp}(\mathcal{C}_a) \subseteq \mathcal{D}_{\bar{a}}$. For converse inclusion, let $p \in \mathcal{D}_{\bar{a}}$, then $a \not\leq p$. By surjectivity of ψ^{qp} , there exists $K \in \text{Spec}^{qp}(M)$ such that $\sqrt{(K : 1_M)} = p$. Hence, $a \not\leq \sqrt{(K : 1_M)}$ and $(a1_M : 1_M) \not\leq \sqrt{(K : 1_M)}$,

by Lemma 2.2(11). Therefore $\sqrt{(a1_M : 1_M)} \not\leq \sqrt{(K : 1_M)}$, by Properties 2.1(2) and (4). Thus, it follows that $K \in \mathcal{C}_a$. As a consequence, $\mathcal{D}_{\bar{a}} \subseteq \psi^{qp}(\mathcal{C}_a)$.

4. From part (1), $\mathcal{C}_a = (\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}})$. Therefore, $\mathcal{C}_{ab} = (\psi^{qp})^{-1}(\mathcal{D}_{\overline{ab}}) = (\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}} \cap \mathcal{D}_{\bar{b}}) = (\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}}) \cap (\psi^{qp})^{-1}(\mathcal{D}_{\bar{b}}) = \mathcal{C}_a \cap \mathcal{C}_b$, for any elements $a, b \in L$. $\square$

Lemma 2.6. *Suppose that M is a lattice module over a C -lattice L . Then the collection $\mathcal{C} = \{\mathcal{C}_a | a \in L\}$ forms a basis for the quasi Zariski-like topology on $\text{Spec}^{qp}(M)$.*

Proof. Suppose that Y is an open set in $\text{Spec}^{qp}(M)$. Then Y is the complement of $\mathcal{Q}^q(H)$ in $\text{Spec}^{qp}(M)$ for some $H \in M$. Thus $Y = \text{Spec}^{qp}(M) - \mathcal{Q}^q(H)$. By Lemma 2.4(2), we have $\mathcal{Q}^q(H) = \mathcal{Q}^q((H : 1_M)1_M)$. Therefore, $Y = \text{Spec}^{qp}(M) - \mathcal{Q}^q((H : 1_M)1_M)$. It follows that $Y = \text{Spec}^{qp}(M) - \mathcal{Q}^q(\bigvee_{a_i \leq (H : 1_M)} a_i 1_M) = \text{Spec}^{qp}(M) - \bigcap_{a_i \leq (H : 1_M)} \mathcal{Q}^q(a_i 1_M) = \bigcup_{a_i \leq (H : 1_M)} (\text{Spec}^{qp}(M) - \mathcal{Q}^q(a_i 1_M)) = \bigcup_{a_i \leq (H : 1_M)} \mathcal{C}_{a_i}$. Consequently, the collection $\mathcal{C} = \{\mathcal{C}_a | a \in L\}$ forms a basis for the quasi Zariski-like topology on $\text{Spec}^{qp}(M)$. $\square$

Theorem 2.2. *Suppose that M is a lattice module over a C -lattice L and the map $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ is the natural map. If ψ^{qp} is surjective, then $\mathcal{C}_a$ and $\mathcal{C}_{a_1} \cap \dots \cap \mathcal{C}_{a_n}$ are quasi-compact subsets of $\text{Spec}^{qp}(M)$.*

Proof. Let us assume that $\mathcal{C}_a$ is covered by an open covering in $\text{Spec}^{qp}(M)$. Since, the quasi Zariski-like topology on $\text{Spec}^{qp}(M)$ has a basis $\mathcal{C} = \{\mathcal{C}_a | a \in L\}$, $\mathcal{C}_a \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{C}_{a_\lambda}$, for some open cover $\{\mathcal{C}_{a_\lambda} | \lambda \in \Lambda\} \subseteq \mathcal{C}$. From the Lemma 2.5(3), $\mathcal{D}_{\bar{a}} = \psi^{qp}(\mathcal{C}_a) \subseteq \bigcup_{\lambda \in \Lambda} \psi^{qp}(\mathcal{C}_{a_\lambda}) = \bigcup_{\lambda \in \Lambda} \mathcal{D}_{\bar{a}_{\lambda}}$. Now, by the fact that $\text{Spec}(\bar{L})$ is compact space with respect to Zariski topology [11], we have, $\mathcal{D}_{\bar{a}} \subseteq \bigcup_{i=1}^n \mathcal{D}_{\bar{a}_{\lambda_i}}$ for some $\lambda_i (1 \leq i \leq n)$. Therefore by Lemma 2.5(1), $\mathcal{C}_a = (\psi^{qp})^{-1}(\psi^{qp}(\mathcal{C}_a)) \subseteq \bigcup_{i=1}^n (\psi^{qp})^{-1}(\mathcal{D}_{\bar{a}_{\lambda_i}}) = \bigcup_{i=1}^n \mathcal{C}_{a_{\lambda_i}}$. Hence, $\mathcal{C}_a$ is quasi-compact in $\text{Spec}^{qp}(M)$, for each $a \in L$.

Let $\mathcal{W}$ be any open covering of $\mathcal{C}_{a_1} \cap \mathcal{C}_{a_2}$. Then $\mathcal{W}$ covers each of $\mathcal{C}_{a_1}$ and $\mathcal{C}_{a_2}$ which are quasi-compact. Consequently, each of $\mathcal{C}_{a_1}$ and $\mathcal{C}_{a_2}$ has a finite subcover and therefore $\mathcal{C}_{a_1} \cap \mathcal{C}_{a_2}$ has a finite subcover. Hence, $\mathcal{C}_{a_1} \cap \mathcal{C}_{a_2} \cap \dots \cap \mathcal{C}_{a_n}$ is quasi-compact subset of $\text{Spec}^{qp}(M)$. $\square$

Corollary 2.3. *Suppose that M is a lattice module over a C -lattice L and the map $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ is the natural map. If ψ^{qp} is surjective, then $\text{Spec}^{qp}(M)$ is a compact space with respect to quasi Zariski-like topology and has a basis of quasi-compact open subsets.*

Proof. By applying Lemma 2.6 and Theorem 2.2, the result follows. $\square$

Proposition 2.4. *Let M be a lattice module over a C -lattice L , then $\mathcal{Q}^q(K) = \mathcal{Q}^q(\sqrt{K})$, for $K \in \text{Spec}^{qp}(M)$.*

Proof. Consider $P \in \mathcal{Q}^q(\sqrt{K})$. Then $\sqrt{(\sqrt{K} : 1_M)} \leq \sqrt{(P : 1_M)}$. It follows that $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$ by Properties 2.1(4). Therefore $P \in \mathcal{Q}^q(K)$. Hence $\mathcal{Q}^q(\sqrt{K}) \subseteq \mathcal{Q}^q(K)$. Conversely, suppose that $P \in \mathcal{Q}^q(K)$. Then $\sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$. Now $(\sqrt{K} : 1_M) = \sqrt{(K : 1_M)} \leq \sqrt{(P : 1_M)}$. Thus $\sqrt{(\sqrt{K} : 1_M)} \leq \sqrt{(P : 1_M)}$, by Properties 2.1(2). Hence $P \in \mathcal{Q}^q(\sqrt{K})$. Thus, the statement holds. $\square$

Let $\mathcal{T}(Y)$ be the meet of all $\sqrt{K}$, for $K \in Y$, i.e., $\mathcal{T}(Y) = \bigwedge \{\sqrt{K} | K \in Y\}$.

Lemma 2.7. *Suppose that M is a lattice module over a C -lattice L . If Y is a subset of $\text{Spec}^{qp}(M)$, then $cl(Y) = \mathcal{Q}^q(\mathcal{T}(Y))$. In particular, if Y contains 0_M , then Y is dense in $\text{Spec}^{qp}(M)$, that is, $cl(Y) = \text{Spec}^{qp}(M)$.*

Proof. Let Y be a subset of $\text{Spec}^{qp}(M)$. Take $K \in Y$. Then $\mathcal{T}(Y) = \bigwedge_{K \in Y} \sqrt{K} < \sqrt{K}$ and hence $(\mathcal{T}(Y) : 1_M) \leq (\sqrt{K} : 1_M)$ by Lemma 2.2(4). Applying Properties 2.1(2) and (4), we obtain $\sqrt{(\mathcal{T}(Y) : 1_M)} \leq \sqrt{(K : 1_M)}$. Thus $K \in \mathcal{Q}^q(\mathcal{T}(Y))$. Hence $Y \subseteq \mathcal{Q}^q(\mathcal{T}(Y))$ and $cl(Y) \subseteq \mathcal{Q}^q(\mathcal{T}(Y))$. For the reverse inclusion, let H be any element such that $Y \subseteq \mathcal{Q}^q(H)$. Thus for each $K \in \mathcal{Q}^q(H)$, $\sqrt{(H : 1_M)} \leq \sqrt{(K : 1_M)}$. Let $K' \in \mathcal{Q}^q(\mathcal{T}(Y))$, then $\sqrt{(\mathcal{T}(Y) : 1_M)} \leq \sqrt{(K' : 1_M)}$. It follows that $\sqrt{(\bigwedge_{K \in \mathcal{Q}^q(H)} \sqrt{K} : 1_M)} \leq \sqrt{(K' : 1_M)} = (\sqrt{K'} : 1_M)$. Therefore $(\bigwedge_{K \in \mathcal{Q}^q(H)} \sqrt{K} : 1_M) \leq (\sqrt{K'} : 1_M)$ by Properties 2.1(1). Applying the Lemma 2.2(12), we get $\bigwedge_{K \in \mathcal{Q}^q(H)} (\sqrt{K} : 1_M) \leq (\sqrt{K'} : 1_M)$. Hence $(H : 1_M) \leq \sqrt{(H : 1_M)} \leq \bigwedge_{K \in \mathcal{Q}^q(H)} (\sqrt{K} : 1_M) \leq (\sqrt{K'} : 1_M)$, by the use of Properties 2.1(1). Consequently, $\sqrt{(H : 1_M)} \leq \sqrt{(K' : 1_M)}$, by Properties 2.1(2) and (4). As a consequence, $K' \in \mathcal{Q}^q(H)$. Thus we conclude that $\mathcal{Q}^q(\mathcal{T}(Y))$ is the smallest closed subset of $\text{Spec}^{qp}(M)$ containing Y . This yields, $cl(Y) = \mathcal{Q}^q(\mathcal{T}(Y))$. Now if $0_M \in Y$, then $\mathcal{T}(Y) = \sqrt{0_M}$. From, $cl(Y) = \mathcal{Q}^q(\mathcal{T}(Y))$, it implies that $cl(Y) = \mathcal{Q}^q(\mathcal{T}(Y)) = \mathcal{Q}^q(\sqrt{0_M}) = \mathcal{Q}^q(0_M) = \text{Spec}^{qp}(M)$. Thus Y is dense in $\text{Spec}^{qp}(M)$. $\square$

For the topological space $\mathcal{Y}$ and its closed subsets $\mathcal{Y}_1$ and $\mathcal{Y}_2$, if $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$ implies $\mathcal{Y} = \mathcal{Y}_1$ or $\mathcal{Y} = \mathcal{Y}_2$, then $\mathcal{Y}$ is called irreducible. A subset $\mathcal{X}$ of $\mathcal{Y}$ is irreducible, if it is irreducible as a subspace of $\mathcal{Y}$ [9].

An element y of a topological space $\mathcal{Y}$ is called a generic point of closed subset $\mathcal{X}$ of $\mathcal{Y}$, if $\mathcal{X} = cl(\{y\})$ (see [4]).

Theorem 2.3. *Suppose that M is a lattice module over a C -lattice L . Then for each $K \in \text{Spec}^{qp}(M)$, the closed set $\mathcal{Q}^q(K)$ is irreducible in $\text{Spec}^{qp}(M)$.*

Proof. From the Lemma 2.7, we have, $cl(Y) = \mathcal{Q}^q(\mathcal{T}(Y))$. Let $Y = \{K\}$, then $cl(Y) = cl(\{K\}) = \mathcal{Q}^q(\mathcal{T}(\{K\})) = \mathcal{Q}^q(\sqrt{K}) = \mathcal{Q}^q(K)$. By applying the fact that the closure of irreducible set is irreducible and singleton sets are irreducible, $\mathcal{Q}^q(K)$ is irreducible in $\text{Spec}^{qp}(M)$. $\square$

Corollary 2.4. *Suppose that M is a lattice module over a C -lattice L . For $K, K' \in \text{Spec}^{qp}(M)$, if $\sqrt{(K : 1_M)} = \sqrt{(K' : 1_M)}$, then K is a generic point for the irreducible closed subset $\mathcal{Q}^q(K')$.*

Proof. According to the Lemma 2.4(1), if $\sqrt{(K : 1_M)} = \sqrt{(K' : 1_M)}$, then $\mathcal{Q}^q(K) = \mathcal{Q}^q(K')$. By Lemma 2.7, we get $cl(\{K\}) = \mathcal{Q}^q(K) = \mathcal{Q}^q(K')$. In view of the Theorem 2.3, $\mathcal{Q}^q(K')$ is an irreducible closed subset of $\text{Spec}^{qp}(M)$. Consequently, K is a generic point of $\mathcal{Q}^q(K')$. $\square$

Theorem 2.4. *Suppose that M is a lattice module over a C -lattice L and Y is a subset of $\text{Spec}^{qp}(M)$. If $\mathcal{T}(Y)$ is a quasi-primary element of M , then Y is irreducible. Conversely, if Y is irreducible, then $Y' = \{\sqrt{(K : 1_M)} | K \in Y\}$ is an irreducible subset of $\text{Spec}(L)$, that is, $\mathcal{T}(Y') = \sqrt{(\mathcal{T}(Y) : 1_M)}$ is a prime element of L .*

Proof. Suppose that Y is a subset of $\text{Spec}^{qp}(M)$ and $\mathcal{T}(Y)$ is a quasi-primary element of M . By an application of the Theorem 2.3, for each quasi-primary element K of M , $\mathcal{Q}^q(K)$ is irreducible, therefore it follows that $\mathcal{Q}^q(\mathcal{T}(Y))$ is an irreducible subset of $\text{Spec}^{qp}(M)$. We have, $cl(Y) = \mathcal{Q}^q(\mathcal{T}(Y))$, by Lemma 2.7. Therefore $cl(Y)$ is irreducible. Consider $Y \subseteq \mathcal{Y}_1 \cup \mathcal{Y}_2$, where $\mathcal{Y}_1, \mathcal{Y}_2$ are closed subsets of $\text{Spec}^{qp}(M)$. It follows that $cl(Y) \subseteq \mathcal{Y}_1 \cup \mathcal{Y}_2$. Since, $cl(Y)$ is irreducible, $Y \subseteq cl(Y) \subseteq \mathcal{Y}_1$ or $Y \subseteq cl(Y) \subseteq \mathcal{Y}_2$. Consequently, Y is irreducible.

To prove the reverse inclusion, suppose that Y is an irreducible subset of $\text{Spec}^{qp}(M)$. Then by Proposition 2.2, it implies that the image $\psi^{qp}(Y) = \{\sqrt{(K : 1_M)} | K \in Y\}$ of Y is an irreducible subset of $\text{Spec}(\bar{L})$, since the natural map ψ^{qp} is continuous. By using Proposition 7(3) in [11], $\overline{\wedge_{K \in Y} \sqrt{(K : 1_M)}} = \overline{\wedge_{K \in Y} (\sqrt{K} : 1_M)} = \overline{(\wedge_{K \in Y} \sqrt{K} : 1_M)} = \overline{(\mathcal{T}(Y) : 1_M)}$ is a prime element of $\bar{L}$. Thus, $\mathcal{T}(Y') = \sqrt{(\mathcal{T}(Y) : 1_M)}$ is a prime element of L . Hence, Y' is an irreducible subset of $\text{Spec}(L)$. $\square$

Theorem 2.5. *Suppose that M is a lattice module over a C -lattice L and $Y \subseteq \text{Spec}^{qp}(M)$. Let $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ be the surjective natural map. Then the following conditions are equivalent:*

1. Y is an irreducible closed subset of $\text{Spec}^{qp}(M)$.
2. $Y = \mathcal{Q}^q(K)$, for some $K \in \text{Spec}^{qp}(M)$.

Proof. (1) $\implies$ (2) : Let Y be an irreducible closed subset of $\text{Spec}^{qp}(M)$. Then $Y = \mathcal{Q}^q(H)$, for some $H \in M$. According to the Theorem 2.4, $\sqrt{(\mathcal{T}(\mathcal{Q}^q(H)) : 1_M)} = \sqrt{(\mathcal{T}(Y) : 1_M)} = p$ is a prime element of L . By surjectivity of the natural map ψ^{qp} , there exists $K \in \text{Spec}^{qp}(M)$ such that $\psi^{qp}(K) = \overline{\sqrt{(K : 1_M)}} = \bar{p}$. Thus $\sqrt{(K : 1_M)} = p = \sqrt{(\mathcal{T}(\mathcal{Q}^q(H)) : 1_M)}$. By Lemma 2.7 and Lemma 2.4, we obtain $Y = \mathcal{Q}^q(H) = \mathcal{Q}^q(\mathcal{T}(\mathcal{Q}^q(H))) = \mathcal{Q}^q(K)$.

(2) $\implies$ (1) : According to the Theorem 2.3, for each $K \in \text{Spec}^{qp}(M)$, $\mathcal{Q}^q(K)$ is an irreducible closed subset of $\text{Spec}^{qp}(M)$. Therefore $Y = \mathcal{Q}^q(K)$ is an irreducible closed subset of $\text{Spec}^{qp}(M)$. $\square$

A topological space X is called a T_0 -space when, for any two distinct points in X , at least one possesses an open neighborhood that excludes the other. Equivalently, X satisfies the T_0 axiom precisely when distinct points have distinct closures. For multiplicative lattice L , the prime spectrum $\text{Spec}(L)$ with the Zariski topology is known to satisfy the T_0 condition.

A topological space X is termed spectral if it is homeomorphic to the spectrum of a commutative ring equipped with the Zariski topology. According to Hochster's characterization ([14]) a space X is spectral if it fulfills the following conditions:

1. X is compact.
2. Compact open subsets of X constitute a basis for the topology and remain closed under finite intersections.
3. Every irreducible closed subset of X is the closure of a singleton set.
4. X is a T_0 -space.

We investigate the space $\text{Spec}^{qp}(M)$ endowed with quasi Zariski-like topology and study whether it satisfies the properties of a spectral space.

Theorem 2.6. *Suppose that M is a lattice module over a C -lattice L and $Y \subseteq \text{Spec}^{qp}(M)$. Let $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ be the surjective natural map. Then $\text{Spec}^{qp}(M)$ is a T_0 -space if and only if $|\text{Spec}_p^{qp}(M)| \leq 1$, for every $p \in \text{Spec}(L)$.*

Proof. Assume that $\text{Spec}^{qp}(M)$ is a T_0 -space. Take $K_1 \neq K_2 \in \text{Spec}_p^{qp}(M)$. Then $\sqrt{(K_1 : 1_M)} = \sqrt{(K_2 : 1_M)} = p$. Hence by Lemma 2.4, $\mathcal{Q}^q(K_1) = \mathcal{Q}^q(K_2)$. Thus, the closures of two distinct points are not distinct. Since, $\text{Spec}^{qp}(M)$ is a T_0 -space, we conclude that $K_1 = K_2$. Consequently, $|\text{Spec}_p^{qp}(M)| \leq 1$. For the converse part, let us assume that $|\text{Spec}_p^{qp}(M)| \leq 1$. Let K_1, K_2 be two distinct quasi-primary elements of M . Then $\sqrt{(K_1 : 1_M)} \neq \sqrt{(K_2 : 1_M)}$ be two distinct prime elements of L . Thus $K_1 \notin \mathcal{Q}^q(K_2)$ and $K_2 \notin \mathcal{Q}^q(K_1)$. Hence $cl(\{K_1\}) \neq cl(\{K_2\})$. Therefore, the closures of distinct points are distinct. Consequently, $\text{Spec}^{qp}(M)$ is a T_0 -space. $\square$

Theorem 2.7. *Suppose that M is a lattice module over a C -lattice L and $\psi^{qp} : \text{Spec}^{qp}(M) \rightarrow \text{Spec}(\bar{L})$ be the surjective natural map. If $|\text{Spec}_p^{qp}(M)| \leq 1$, for every $p \in \text{Spec}(L)$, then $\text{Spec}^{qp}(M)$ with quasi Zariski-like topology is a spectral space.*

Proof. According to the Theorem 2.2, Corollary 2.3, Theorem 2.5, and Theorem 2.6, $\text{Spec}^{qp}(M)$ with quasi Zariski-like topology satisfies the conditions (1),(2),(3) and (4) of Hochsters characterization. Hence, $\text{Spec}^{qp}(M)$ is a spectral space. $\square$

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References

- [1] F. Alarcon, D. D. Anderson and C. Jayaram. Some results on abstract commutative ideal theory. Period Math Hung 1995; 30(1): 1-26.
- [2] E. A. AL-Khouja. Maximal elements and prime elements in lattice modules. Damascus Univ Basic Sci 2003; 19: 9-20.
- [3] D. D. Anderson, C. Jayaram. Regular lattices. Studia Scientiarum Mathematicarum Hungarica 1995; 30: 379-388.
- [4] M. Atiyah and I. MacDonald. Introduction to commutative algebra. Reading: Addison-Wesley. 1969.
- [5] S. Ballal and V. Kharat. Zariski topology on lattice modules. Asian Eur J Math 2015; 8(1550066): 10 pages. doi:1142/S1793557115500667.
- [6] M. Behboodi and M. R. Haddadi. Classical Zariski topology of modules and spectral spaces I. Int Electron J Algebra 2008; 4: 104-130.
- [7] M. Behboodi and M. J. Noori. Zariski-like topology on the classical prime spectrum of a module. Bulletin of the Iranian Math Society 2009; 35(1): 253-269.
- [8] V. Borkar, P. Girase and N. Phadatare. Classical Zariski topology on prime spectrum of lattice modules. J Algebra Relat Top 2018; 6(2): 1-14.
- [9] N. Bourbaki. Commutative algebra. Hermann, Paris 1961.
- [10] F. Callialp and U. Tekir. Multiplication lattice modules. Iran J Sci Technol 2011; A4: 309-313.
- [11] F. Callialp, G. Ulucak and U. Tekir. On the Zariski topology over an L -Module M . Turk J Math 2017; 41(2): 326-336.

- [12] R. P. Dilworth. Abstract commutative ideal theory. *Pacific J Math* 1962; 12: 481-498.
- [13] P. Girase, V. Borkar and N. Phadatare. On the classical prime spectrum of lattice modules. *Int Electron J Algebra* 2019; 25: 186-198.
- [14] M. Hochster. Prime ideal structure in commutative rings. *Trans Amer Math Soc* 1969; 142: 43-60.
- [15] J. A. Johnson. a -adic completions of Noetherian lattice modules. *Fund Math* 1970; 66(3): 341-371.
- [16] E. W. Johnson. A -transforms of noether lattices, Ph.D. Dissertation, Univ of California, Riverside, (1966).
- [17] C. P. Lu. The Zariski topology on the prime spectrum of a module. *Houston J Math* 1999; 25: 417-425.
- [18] C. S. Manjarekar and A. V. Bingi. Absorbing elements in lattice modules. *Int Electron J Algebra* 2016; 19: 58-76.
- [19] J. R. Munkres. *Topology, a first course*, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1975.
- [20] H. M. Nakkar, I. A. Al-Khouja. Nakayama's lemma and the principal elements in lattice modules over multiplicative lattices. *Research Journal of Aleppo University* 1985; 7: 1-16.
- [21] M. Samiei and H. F. Moghimi. Quasi-Zariski topology on the quasi-primary spectrum of a module. *Jordan J Math Stat* 2017; 10(4): 319-345.
- [22] N. K. Thakare and C. S. Manjarekar. Abstract spectral theory: Multiplicative lattices in which every character is contained in a unique maximal character. *Algebra and Its applications* (Marcel Dekker, New York) 1984; 265-276.
- [23] N. K. Thakare, C. S. Manjarekar and S. Maeda. Abstract spectral theory II: minimal characters and minimal spectrums of multiplicative lattices. *Acta Sci Math* 1988; 52: 53-67.
- [24] M. Ward and R. P. Dilworth. Residuated lattices. *Trans Amer Math Soc* 1939; 45: 335-354.