



On general families of cyclic quadratic inequalities

Christophe Chesneau^{*}¹Department of Mathematics, University of Caen Normandie
Caen, France.

Received: 21 Aug 2026

•

Accepted: 29 Aug 2026

•

Published Online: 10 Sep 2026

Abstract: This paper introduces general families of parameterized cyclic quadratic inequalities in three positive real variables. We first establish eight- and nine-parameter algebraic inequalities, both of which are subsequently generalized to functional forms using convex and non-decreasing functions. We then formulate a broader ten-parameter weighted cyclic inequality that relaxes the monotonicity requirement, holding for any arbitrary convex function. By specializing these parameters and selecting standard convex functions, we derive numerous explicit inequalities, demonstrating the capacity to unify classical results and generate new bounds.

Key words: Cyclic inequalities, Nesbitt's inequality, Jensen's inequality, Titu's lemma.

1. Introduction

Cyclic inequalities play an important role in algebra, real analysis, and mathematical competitions. A cyclic inequality in three positive real variables, say x , y , and z , is an inequality that remains invariant under the cyclic permutation $(x, y, z) \mapsto (y, z, x)$. A classic example is the Nesbitt inequality, which states that, for any $x, y, z > 0$,

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}.$$

The equality holds if $x = y = z$. Beyond their theoretical appeal, such inequalities provide useful bounds in areas such as optimization and geometric analysis. Among cyclic inequalities, cyclic quadratic forms are of particular interest. In these expressions, the numerators and denominators consist of quadratic forms, squared linear terms, or cross-products such as x^2 , xy , and $(\kappa x + \mu y + \nu z)^2$, where κ , μ , and ν are real parameters. A classic example follows from the well-known inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$, which holds for any $x, y, z > 0$. Dividing by the right-hand side yields

$$\frac{x^2 + y^2 + z^2}{xy + yz + zx} \geq 1,$$

or, equivalently,

$$\frac{x^2}{xy + yz + zx} + \frac{y^2}{xy + yz + zx} + \frac{z^2}{xy + yz + zx} \geq 1. \quad (1)$$

The equality holds if $x = y = z$. Such inequalities are valuable because they capture non-linear interactions between variables that purely linear forms cannot model. For further details, see [1, 2, 5–8]. For recent developments, particularly regarding parameterized families of cyclic inequalities, we refer to [3, 4].

While many specific cyclic quadratic inequalities exist, comprehensive theoretical frameworks remain rare. Typically, these inequalities are approached on a case-by-case basis using ad hoc algebraic methods. Developing parameterized theorems that unify broad families of such inequalities is therefore an important objective. To address this gap, we establish a unified framework of parameterized cyclic inequalities in three positive real variables. Specifically, we begin by introducing eight- and nine-parameter algebraic inequalities, which we subsequently generalize to functional forms via convex and non-decreasing functions. Next, we formulate a broader ten-parameter weighted inequality that relaxes the monotonicity requirement, holding for any arbitrary convex function. Finally, we demonstrate the utility of these theorems by recovering classical inequalities and generating new bounds as special cases.

The remainder of this paper is organized as follows: Section 2 establishes the eight-parameter algebraic theorem, provides its proof, discusses several concrete examples, and extends the result to functional forms using convex and non-decreasing functions. Section 3 follows a similar structure to introduce and generalize the nine-parameter algebraic theorem. Section 4 presents the broader ten-parameter weighted framework for general convex functions, detailing its proof and relevant applications. Finally, Section 5 offers concluding remarks.

2. Eight-parameter framework

2.1. Eight-parameter algebraic theorem

The theorem below introduces an eight-parameter cyclic quadratic inequality for three positive real variables.

Theorem 2.1. *For any variables $x, y, z > 0$, numerator parameters $\kappa, \mu, \nu \in \mathbb{R}$ satisfying $\kappa + \mu + \nu > 0$, and denominator parameters $\alpha, \beta, \gamma, \delta, \epsilon \geq 0$ satisfying $\alpha + \beta + \gamma > 0$ and $\delta + \epsilon \leq 2(\alpha + \beta + \gamma)$, the following cyclic inequality holds:*

$$\begin{aligned} & \frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx} \\ & + \frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy} \\ & + \frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz} \\ & \geq \frac{(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma}. \end{aligned}$$

The eight independent parameters are $\kappa, \mu, \nu, \alpha, \beta, \gamma, \delta$, and ϵ . The equality holds if $x = y = z$.

Proof. Let us denote

$$a_1 = \kappa x + \mu y + \nu z,$$

$$a_2 = \kappa y + \mu z + \nu x,$$

$$a_3 = \kappa z + \mu x + \nu y,$$

$$b_1 = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx,$$

$$b_2 = \alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy$$

and

$$b_3 = \alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz.$$

Therefore, we can write

$$\begin{aligned}
& \frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx} \\
& + \frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy} \\
& + \frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz} \\
& = \sum_{i=1}^3 \frac{a_i^2}{b_i}.
\end{aligned}$$

The following sums hold:

$$\begin{aligned}
\sum_{i=1}^3 a_i &= \kappa x + \mu y + \nu z + \kappa y + \mu z + \nu x + \kappa z + \mu x + \nu y \\
&= (\kappa + \mu + \nu)(x + y + z)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^3 b_i &= \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx \\
&+ \alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy \\
&+ \alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz \\
&= (\alpha + \beta + \gamma)(x^2 + y^2 + z^2) + 2(\alpha + \beta + \gamma)(xy + yz + zx) \\
&= (\alpha + \beta + \gamma)(x + y + z)^2.
\end{aligned}$$

The Titu lemma and the sums above give

$$\sum_{i=1}^3 \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^3 a_i\right)^2}{\sum_{i=1}^3 b_i} = \frac{(\kappa + \mu + \nu)^2(x + y + z)^2}{(\alpha + \beta + \gamma)(x + y + z)^2} = \frac{(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma}.$$

Combining the equations above, we get

$$\begin{aligned}
& \frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx} \\
& + \frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy} \\
& + \frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz} \\
& \geq \frac{(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma}.
\end{aligned}$$

The equality holds if $x = y = z$, with both sides reducing to $(\kappa + \mu + \nu)^2/(\alpha + \beta + \gamma)$. This completes the proof. $\square$

A notable feature of this theorem is that its proof relies on a single inequality, namely, the Titu lemma (recalled in the Appendix). We also highlight the contrast between the sophistication of the upper bound and the simplicity of the lower bound. Additionally, the presence of eight adjustable parameters is relatively rare in the literature on cyclic inequalities.

2.2. Examples

To illustrate Theorem 2.1, we examine several examples for specific choices of the eight parameters involved.

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, and $\epsilon = 0$, the theorem yields

$$\frac{(x+y+z)^2}{x^2+xy+zx} + \frac{(x+y+z)^2}{y^2+yz+xy} + \frac{(x+y+z)^2}{z^2+zx+yz} \geq 9,$$

so that

$$\frac{1}{x^2+xy+zx} + \frac{1}{y^2+yz+xy} + \frac{1}{z^2+zx+yz} \geq \frac{9}{(x+y+z)^2}.$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\delta = 0$, and $\epsilon = 0$, the theorem gives

$$\frac{x^2}{y^2+2zx} + \frac{y^2}{z^2+2xy} + \frac{z^2}{x^2+2yz} \geq 1.$$

For $\kappa = 0$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, and $\epsilon = 0$, the theorem yields

$$\frac{(y+z)^2}{x^2+2zx} + \frac{(z+x)^2}{y^2+2xy} + \frac{(x+y)^2}{z^2+2yz} \geq 4.$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, and $\epsilon = 1$, the theorem gives

$$\frac{x^2}{x^2+xy+yz} + \frac{y^2}{y^2+yz+zx} + \frac{z^2}{z^2+zx+xy} \geq 1.$$

For $\kappa = 2$, $\mu = 1$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $\delta = 1$, and $\epsilon = 1$, the theorem yields

$$\frac{(2x+y)^2}{x^2+y^2+xy+yz+2zx} + \frac{(2y+z)^2}{y^2+z^2+yz+zx+2xy} + \frac{(2z+x)^2}{z^2+x^2+zx+xy+2yz} \geq \frac{9}{2}.$$

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $\epsilon = 1$, the theorem gives

$$\begin{aligned} & \frac{(x+y+z)^2}{x^2+y^2+z^2+xy+yz+4zx} + \frac{(x+y+z)^2}{x^2+y^2+z^2+yz+zx+4xy} \\ & + \frac{(x+y+z)^2}{x^2+y^2+z^2+zx+xy+4yz} \geq 3, \end{aligned}$$

so that

$$\begin{aligned} & \frac{1}{x^2+y^2+z^2+xy+yz+4zx} + \frac{1}{x^2+y^2+z^2+yz+zx+4xy} \\ & + \frac{1}{x^2+y^2+z^2+zx+xy+4yz} \geq \frac{3}{(x+y+z)^2}. \end{aligned}$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $\epsilon = 1$, the theorem yields

$$\begin{aligned} & \frac{x^2}{y^2 + z^2 + xy + yz + 2zx} + \frac{y^2}{z^2 + x^2 + yz + zx + 2xy} \\ & + \frac{z^2}{x^2 + y^2 + zx + xy + 2yz} \geq \frac{1}{2}. \end{aligned}$$

For $\kappa = 2$, $\mu = -1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = 0$, and $\epsilon = 2$, the theorem gives

$$\frac{(2x - y + z)^2}{x^2 + z^2 + 2yz + 2zx} + \frac{(2y - z + x)^2}{y^2 + x^2 + 2zx + 2xy} + \frac{(2z - x + y)^2}{z^2 + y^2 + 2xy + 2yz} \geq 2.$$

For $\kappa = 0$, $\mu = 2$, $\nu = 1$, $\alpha = 0$, $\beta = 2$, $\gamma = 1$, $\delta = 0$, and $\epsilon = 0$, the theorem yields

$$\frac{(2y + z)^2}{2y^2 + z^2 + 6zx} + \frac{(2z + x)^2}{2z^2 + x^2 + 6xy} + \frac{(2x + y)^2}{2x^2 + y^2 + 6yz} \geq 3.$$

Many additional examples can be derived directly from Theorem 2.1 by selecting specific parameter values.

2.3. Extension for convex and non-decreasing functions

Building on Theorem 2.1, the theorem below establishes a functional cyclic quadratic inequality in three positive variables by applying a convex and non-decreasing function to the eight-parameter rational expression.

Theorem 2.2. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex and non-decreasing function. For any variables $x, y, z > 0$, numerator parameters $\kappa, \mu, \nu \in \mathbb{R}$ satisfying $\kappa + \mu + \nu > 0$, and denominator parameters $\alpha, \beta, \gamma, \delta, \epsilon \geq 0$ satisfying $\alpha + \beta + \gamma > 0$ and $\delta + \epsilon \leq 2(\alpha + \beta + \gamma)$, the following cyclic inequality holds:*

$$\begin{aligned} & f\left(\frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx}\right) \\ & + f\left(\frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy}\right) \\ & + f\left(\frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz}\right) \\ & \geq 3f\left(\frac{(\kappa + \mu + \nu)^2}{3(\alpha + \beta + \gamma)}\right). \end{aligned}$$

The eight independent parameters are κ , μ , ν , α , β , γ , δ , and ϵ . The equality holds if $x = y = z$.

Proof. Let us denote

$$T_1 = \frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx},$$

$$T_2 = \frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy}$$

and

$$T_3 = \frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz}.$$

Since f is convex, the Jensen inequality gives

$$f(T_1) + f(T_2) + f(T_3) \geq 3f\left(\frac{T_1 + T_2 + T_3}{3}\right).$$

By Theorem 2.1, we have

$$T_1 + T_2 + T_3 \geq \frac{(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma}.$$

Using this and the fact that f is non-decreasing, we get

$$3f\left(\frac{T_1 + T_2 + T_3}{3}\right) \geq 3f\left(\frac{(\kappa + \mu + \nu)^2}{3(\alpha + \beta + \gamma)}\right).$$

Combining the inequalities above yield

$$f(T_1) + f(T_2) + f(T_3) \geq 3f\left(\frac{(\kappa + \mu + \nu)^2}{3(\alpha + \beta + \gamma)}\right),$$

so that

$$\begin{aligned} & f\left(\frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx}\right) \\ & + f\left(\frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy}\right) \\ & + f\left(\frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz}\right) \\ & \geq 3f\left(\frac{(\kappa + \mu + \nu)^2}{3(\alpha + \beta + \gamma)}\right). \end{aligned}$$

The equality holds if $x = y = z$, with both sides reducing to $3f((\kappa + \mu + \nu)^2 / (3(\alpha + \beta + \gamma)))$. This concludes the proof. $\square$

The Jensen inequality is thus the key to this proof (it is recalled in Appendix). For this result as well, we emphasize the contrast between the sophisticated upper bound and the simple lower bound. In particular, the lower bound remains tractable and easy to manipulate for a broad class of convex and decreasing functions f .

2.4. Examples

To illustrate Theorem 2.2, we examine several examples for specific choices of the eight parameters involved.

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, and $\epsilon = 0$, the theorem gives

$$f\left(\frac{(x + y + z)^2}{x^2 + xy + zx}\right) + f\left(\frac{(x + y + z)^2}{y^2 + yz + xy}\right) + f\left(\frac{(x + y + z)^2}{z^2 + zx + yz}\right) \geq 3f(3).$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\delta = 0$, and $\epsilon = 0$, the theorem yields

$$f\left(\frac{x^2}{y^2 + 2zx}\right) + f\left(\frac{y^2}{z^2 + 2xy}\right) + f\left(\frac{z^2}{x^2 + 2yz}\right) \geq 3f\left(\frac{1}{3}\right).$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 2$, and $\epsilon = 2$, the theorem gives

$$f\left(\frac{x^2}{(x+y+z)^2}\right) + f\left(\frac{y^2}{(x+y+z)^2}\right) + f\left(\frac{z^2}{(x+y+z)^2}\right) \geq 3f\left(\frac{1}{9}\right).$$

For $\kappa = 0$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, and $\epsilon = 0$, the theorem yields

$$f\left(\frac{(y+z)^2}{x^2 + 2zx}\right) + f\left(\frac{(z+x)^2}{y^2 + 2xy}\right) + f\left(\frac{(x+y)^2}{z^2 + 2yz}\right) \geq 3f\left(\frac{4}{3}\right).$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, and $\epsilon = 1$, the theorem gives

$$f\left(\frac{x^2}{x^2 + xy + yz}\right) + f\left(\frac{y^2}{y^2 + yz + zx}\right) + f\left(\frac{z^2}{z^2 + zx + xy}\right) \geq 3f\left(\frac{1}{3}\right).$$

For $\kappa = 2$, $\mu = 1$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $\delta = 1$, and $\epsilon = 1$, the theorem yields

$$\begin{aligned} & f\left(\frac{(2x+y)^2}{x^2 + y^2 + xy + yz + 2zx}\right) + f\left(\frac{(2y+z)^2}{y^2 + z^2 + yz + zx + 2xy}\right) \\ & + f\left(\frac{(2z+x)^2}{z^2 + x^2 + zx + xy + 2yz}\right) \geq 3f\left(\frac{3}{2}\right). \end{aligned}$$

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $\epsilon = 1$, the theorem gives

$$\begin{aligned} & f\left(\frac{(x+y+z)^2}{x^2 + y^2 + z^2 + xy + yz + 4zx}\right) + f\left(\frac{(x+y+z)^2}{x^2 + y^2 + z^2 + yz + zx + 4xy}\right) \\ & + f\left(\frac{(x+y+z)^2}{x^2 + y^2 + z^2 + zx + xy + 4yz}\right) \geq 3f(1). \end{aligned}$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $\epsilon = 1$, the theorem yields

$$\begin{aligned} & f\left(\frac{x^2}{y^2 + z^2 + xy + yz + 2zx}\right) + f\left(\frac{y^2}{z^2 + x^2 + yz + zx + 2xy}\right) \\ & + f\left(\frac{z^2}{x^2 + y^2 + zx + xy + 2yz}\right) \geq 3f\left(\frac{1}{6}\right). \end{aligned}$$

For $\kappa = 2$, $\mu = -1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = 0$, and $\epsilon = 2$, the theorem gives

$$\begin{aligned} & f\left(\frac{(2x-y+z)^2}{x^2 + z^2 + 2yz + 2zx}\right) + f\left(\frac{(2y-z+x)^2}{y^2 + x^2 + 2zx + 2xy}\right) \\ & + f\left(\frac{(2z-x+y)^2}{z^2 + y^2 + 2xy + 2yz}\right) \geq 3f\left(\frac{2}{3}\right). \end{aligned}$$

For $\kappa = 0$, $\mu = 2$, $\nu = 1$, $\alpha = 0$, $\beta = 2$, $\gamma = 1$, $\delta = 0$, and $\epsilon = 0$, the theorem yields

$$f\left(\frac{(2y+z)^2}{2y^2 + z^2 + 6zx}\right) + f\left(\frac{(2z+x)^2}{2z^2 + x^2 + 6xy}\right) + f\left(\frac{(2x+y)^2}{2x^2 + y^2 + 6yz}\right) \geq 3f(1).$$

For each of these examples, common choices for the convex and non-decreasing function $f : (0, \infty) \rightarrow \mathbb{R}$ include $f(x) = x^p$ for $p \geq 1$, $f(x) = e^{ax}$ for $a > 0$, $f(x) = a^x$ for $a > 1$, $f(x) = xe^x$, $f(x) = \ln(1 + e^{ax})$ for $a > 0$, $f(x) = \cosh(x)$, $f(x) = x \arctan(x)$, and $f(x) = \sqrt{1 + x^2}$.

For instance, for $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\delta = 0$, $\epsilon = 0$, and $f(x) = x^p$ for $p \geq 1$, the theorem yields

$$\left(\frac{x^2}{y^2 + 2zx}\right)^p + \left(\frac{y^2}{z^2 + 2xy}\right)^p + \left(\frac{z^2}{x^2 + 2yz}\right)^p \geq \frac{1}{3^{p-1}}.$$

Numerous additional examples can be obtained directly from Theorem 2.2 by specializing the parameters and the function f .

It is worth noting that, despite their generality, neither Theorem 2.1 nor Theorem 2.2 recovers the inequality in Equation (1). This limitation motivated the development of a nine-parameter framework, which is presented in the section below.

3. Nine-parameter framework

3.1. Nine-parameter algebraic theorem

The theorem below introduces a nine-parameter cyclic quadratic inequality for three positive real variables.

Theorem 3.1. *For any variables $x, y, z > 0$, numerator parameters $\kappa, \mu, \nu \in \mathbb{R}$ satisfying $\kappa + \mu + \nu > 0$, and denominator parameters $\alpha, \beta, \gamma, \delta, \epsilon, \omega \geq 0$ satisfying $\alpha + \beta + \gamma + \delta + \epsilon + \omega > 0$ and $\delta + \epsilon + \omega \geq 2(\alpha + \beta + \gamma)$, the following cyclic inequality holds:*

$$\begin{aligned} & \frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + \omega zx} + \frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + \omega xy} \\ & + \frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + \omega yz} \geq \frac{3(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma + \delta + \epsilon + \omega}. \end{aligned}$$

The nine independent parameters are κ , μ , ν , α , β , γ , δ , ϵ , and ω . The equality holds if $x = y = z$.

Proof. Let us denote

$$a_1 = \kappa x + \mu y + \nu z,$$

$$a_2 = \kappa y + \mu z + \nu x,$$

$$a_3 = \kappa z + \mu x + \nu y,$$

$$b_1 = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + \omega zx,$$

$$b_2 = \alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + \omega xy$$

and

$$b_3 = \alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + \omega yz.$$

Therefore, we can write

$$\begin{aligned} & \frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + \omega zx} + \frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + \omega xy} \\ & + \frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + \omega yz} = \sum_{i=1}^3 \frac{a_i^2}{b_i}. \end{aligned}$$

The following sums hold:

$$\begin{aligned}\sum_{i=1}^3 a_i &= \kappa x + \mu y + \nu z + \kappa y + \mu z + \nu x + \kappa z + \mu x + \nu y \\ &= (\kappa + \mu + \nu)(x + y + z)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^3 b_i &= \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + \omega zx \\ &\quad + \alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + \omega xy \\ &\quad + \alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + \omega yz \\ &= (\alpha + \beta + \gamma)(x^2 + y^2 + z^2) + (\delta + \epsilon + \omega)(xy + yz + zx).\end{aligned}$$

Since $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$, we obtain

$$x^2 + y^2 + z^2 = \frac{1}{3}(x + y + z)^2 + \frac{2}{3}(x^2 + y^2 + z^2 - (xy + yz + zx))$$

and

$$xy + yz + zx = \frac{1}{3}(x + y + z)^2 - \frac{1}{3}(x^2 + y^2 + z^2 - (xy + yz + zx)).$$

Substituting these identities into our sum, we have

$$\begin{aligned}\sum_{i=1}^3 b_i &= (\alpha + \beta + \gamma) \left(\frac{(x + y + z)^2 + 2(x^2 + y^2 + z^2 - (xy + yz + zx))}{3} \right) \\ &\quad + (\delta + \epsilon + \omega) \left(\frac{(x + y + z)^2 - (x^2 + y^2 + z^2 - (xy + yz + zx))}{3} \right) \\ &= \frac{\alpha + \beta + \gamma + \delta + \epsilon + \omega}{3} (x + y + z)^2 \\ &\quad - \frac{\delta + \epsilon + \omega - 2(\alpha + \beta + \gamma)}{3} (x^2 + y^2 + z^2 - (xy + yz + zx)).\end{aligned}$$

Using the assumption $\delta + \epsilon + \omega \geq 2(\alpha + \beta + \gamma)$ and the known inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$, we have

$$- \frac{\delta + \epsilon + \omega - 2(\alpha + \beta + \gamma)}{3} (x^2 + y^2 + z^2 - (xy + yz + zx)) \leq 0,$$

so that

$$\sum_{i=1}^3 b_i \leq \frac{\alpha + \beta + \gamma + \delta + \epsilon + \omega}{3} (x + y + z)^2.$$

The Titu lemma and the equations above give

$$\sum_{i=1}^3 \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^3 a_i \right)^2}{\sum_{i=1}^3 b_i} \geq \frac{(\kappa + \mu + \nu)^2 (x + y + z)^2}{\frac{\alpha + \beta + \gamma + \delta + \epsilon + \omega}{3} (x + y + z)^2} = \frac{3(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma + \delta + \epsilon + \omega}.$$

Combining the equations above, we get

$$\begin{aligned} & \frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + \omega zx} + \frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + \omega xy} \\ & + \frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + \omega yz} \geq \frac{3(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma + \delta + \epsilon + \omega}. \end{aligned}$$

The equality holds if $x = y = z$, with both sides reducing to $3(\kappa + \mu + \nu)^2/(\alpha + \beta + \gamma + \delta + \epsilon + \omega)$. This completes the proof. $\square$

For this result as well, we emphasize the contrast between the sophisticated upper bound and the simple lower bound. The presence of the nine adjustable parameters is also a notable contribution.

3.2. Examples

To illustrate Theorem 3.1, we examine several examples for specific choices of the nine parameters involved.

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, $\epsilon = 1$, and $\omega = 1$, the theorem yields

$$\frac{x^2}{xy + yz + zx} + \frac{y^2}{xy + yz + zx} + \frac{z^2}{xy + yz + zx} \geq 1.$$

This corresponds to the introductory example in Equation (1).

For $\kappa = 0$, $\mu = 1$, $\nu = 1$, $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, $\epsilon = 0$, and $\omega = 1$, the theorem gives

$$\frac{(y + z)^2}{xy + zx} + \frac{(z + x)^2}{yz + xy} + \frac{(x + y)^2}{zx + yz} \geq 6.$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, $\epsilon = 2$, and $\omega = 0$, the theorem yields

$$\frac{x^2}{x^2 + 2yz} + \frac{y^2}{y^2 + 2zx} + \frac{z^2}{z^2 + 2xy} \geq 1.$$

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, $\epsilon = 1$, and $\omega = 1$, the theorem gives

$$\frac{(x + y + z)^2}{x^2 + xy + yz + zx} + \frac{(x + y + z)^2}{y^2 + yz + zx + xy} + \frac{(x + y + z)^2}{z^2 + zx + xy + yz} \geq \frac{27}{4},$$

so that

$$\frac{1}{x^2 + xy + yz + zx} + \frac{1}{y^2 + yz + zx + xy} + \frac{1}{z^2 + zx + xy + yz} \geq \frac{27}{4(x + y + z)^2}.$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 2$, $\epsilon = 0$, and $\omega = 0$, the theorem yields

$$\frac{x^2}{x^2 + 2xy} + \frac{y^2}{y^2 + 2yz} + \frac{z^2}{z^2 + 2zx} \geq 1.$$

Numerous additional examples can be obtained directly from Theorem 3.1 by specializing the parameters.

3.3. Extension for convex and non-decreasing functions

Building on Theorem 3.1, the theorem below establishes a functional cyclic quadratic inequality in three positive variables by applying a convex and non-decreasing function to the nine-parameter rational expression.

Theorem 3.2. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex and non-decreasing function. For any variables $x, y, z > 0$, numerator parameters $\kappa, \mu, \nu \in \mathbb{R}$ satisfying $\kappa + \mu + \nu > 0$, and denominator parameters $\alpha, \beta, \gamma, \delta, \epsilon, \omega \geq 0$ satisfying $\alpha + \beta + \gamma + \delta + \epsilon + \omega > 0$ and $\delta + \epsilon + \omega \geq 2(\alpha + \beta + \gamma)$, the following cyclic inequality holds:*

$$\begin{aligned} & f\left(\frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + \omega zx}\right) + f\left(\frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + \omega xy}\right) \\ & + f\left(\frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + \omega yz}\right) \geq 3f\left(\frac{(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma + \delta + \epsilon + \omega}\right). \end{aligned}$$

The nine independent parameters are $\kappa, \mu, \nu, \alpha, \beta, \gamma, \delta, \epsilon$, and ω . The equality holds if $x = y = z$.

Proof. Let us denote

$$T_1 = \frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + \omega zx},$$

$$T_2 = \frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + \omega xy}$$

and

$$T_3 = \frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + \omega yz}.$$

Since f is convex, the Jensen inequality gives

$$f(T_1) + f(T_2) + f(T_3) \geq 3f\left(\frac{T_1 + T_2 + T_3}{3}\right).$$

By Theorem 3.1, we have

$$T_1 + T_2 + T_3 \geq \frac{3(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma + \delta + \epsilon + \omega}.$$

Using this and the fact that f is non-decreasing, we get

$$3f\left(\frac{T_1 + T_2 + T_3}{3}\right) \geq 3f\left(\frac{3(\kappa + \mu + \nu)^2}{3(\alpha + \beta + \gamma + \delta + \epsilon + \omega)}\right) = 3f\left(\frac{(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma + \delta + \epsilon + \omega}\right).$$

Combining the inequalities above yields

$$f(T_1) + f(T_2) + f(T_3) \geq 3f\left(\frac{(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma + \delta + \epsilon + \omega}\right),$$

so that

$$\begin{aligned} & f\left(\frac{(\kappa x + \mu y + \nu z)^2}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + \omega zx}\right) + f\left(\frac{(\kappa y + \mu z + \nu x)^2}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + \omega xy}\right) \\ & + f\left(\frac{(\kappa z + \mu x + \nu y)^2}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + \omega yz}\right) \geq 3f\left(\frac{(\kappa + \mu + \nu)^2}{\alpha + \beta + \gamma + \delta + \epsilon + \omega}\right). \end{aligned}$$

The equality holds if $x = y = z$, with both sides reducing to $3f((\kappa + \mu + \nu)^2/(\alpha + \beta + \gamma + \delta + \epsilon + \omega))$. This concludes the proof. $\square$

The Jensen inequality is once again fundamental to the proof. Furthermore, we emphasize the simplicity of the lower bound. Notably, it remains analytically tractable for a broad class of convex and non-decreasing functions f .

3.4. Examples

To illustrate Theorem 3.2, we examine several examples for specific choices of the nine parameters involved.

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, $\epsilon = 0$, and $\omega = 1$, the theorem gives

$$f\left(\frac{(x+y+z)^2}{x^2+xy+zx}\right) + f\left(\frac{(x+y+z)^2}{y^2+yz+xy}\right) + f\left(\frac{(x+y+z)^2}{z^2+zx+yz}\right) \geq 3f(3).$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\delta = 0$, $\epsilon = 0$, and $\omega = 2$, the theorem yields

$$f\left(\frac{x^2}{y^2+2zx}\right) + f\left(\frac{y^2}{z^2+2xy}\right) + f\left(\frac{z^2}{x^2+2yz}\right) \geq 3f\left(\frac{1}{3}\right).$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 2$, $\epsilon = 2$, and $\omega = 2$, the theorem gives

$$f\left(\frac{x^2}{(x+y+z)^2}\right) + f\left(\frac{y^2}{(x+y+z)^2}\right) + f\left(\frac{z^2}{(x+y+z)^2}\right) \geq 3f\left(\frac{1}{9}\right).$$

For $\kappa = 0$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, $\epsilon = 0$, and $\omega = 2$, the theorem yields

$$f\left(\frac{(y+z)^2}{x^2+2zx}\right) + f\left(\frac{(z+x)^2}{y^2+2xy}\right) + f\left(\frac{(x+y)^2}{z^2+2yz}\right) \geq 3f\left(\frac{4}{3}\right).$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$, $\epsilon = 1$, and $\omega = 0$, the theorem gives

$$f\left(\frac{x^2}{x^2+xy+yz}\right) + f\left(\frac{y^2}{y^2+yz+zx}\right) + f\left(\frac{z^2}{z^2+zx+xy}\right) \geq 3f\left(\frac{1}{3}\right).$$

For $\kappa = 2$, $\mu = 1$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $\delta = 1$, $\epsilon = 1$, and $\omega = 2$, the theorem yields

$$\begin{aligned} & f\left(\frac{(2x+y)^2}{x^2+y^2+xy+yz+2zx}\right) + f\left(\frac{(2y+z)^2}{y^2+z^2+yz+zx+2xy}\right) \\ & + f\left(\frac{(2z+x)^2}{z^2+x^2+zx+xy+2yz}\right) \geq 3f\left(\frac{3}{2}\right). \end{aligned}$$

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, $\epsilon = 1$, and $\omega = 4$, the theorem gives

$$\begin{aligned} & f\left(\frac{(x+y+z)^2}{x^2+y^2+z^2+xy+yz+4zx}\right) + f\left(\frac{(x+y+z)^2}{x^2+y^2+z^2+yz+zx+4xy}\right) \\ & + f\left(\frac{(x+y+z)^2}{x^2+y^2+z^2+zx+xy+4yz}\right) \geq 3f(1). \end{aligned}$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, $\epsilon = 1$, and $\omega = 2$, the theorem yields

$$\begin{aligned} & f\left(\frac{x^2}{y^2 + z^2 + xy + yz + 2zx}\right) + f\left(\frac{y^2}{z^2 + x^2 + yz + zx + 2xy}\right) \\ & + f\left(\frac{z^2}{x^2 + y^2 + zx + xy + 2yz}\right) \geq 3f\left(\frac{1}{6}\right). \end{aligned}$$

For $\kappa = 2$, $\mu = -1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = 0$, $\epsilon = 2$, and $\omega = 2$, the theorem gives

$$\begin{aligned} & f\left(\frac{(2x - y + z)^2}{x^2 + z^2 + 2yz + 2zx}\right) + f\left(\frac{(2y - z + x)^2}{y^2 + x^2 + 2zx + 2xy}\right) \\ & + f\left(\frac{(2z - x + y)^2}{z^2 + y^2 + 2xy + 2yz}\right) \geq 3f\left(\frac{2}{3}\right). \end{aligned}$$

For $\kappa = 0$, $\mu = 2$, $\nu = 1$, $\alpha = 0$, $\beta = 2$, $\gamma = 1$, $\delta = 0$, $\epsilon = 0$, and $\omega = 6$, the theorem yields

$$f\left(\frac{(2y + z)^2}{2y^2 + z^2 + 6zx}\right) + f\left(\frac{(2z + x)^2}{2z^2 + x^2 + 6xy}\right) + f\left(\frac{(2x + y)^2}{2x^2 + y^2 + 6yz}\right) \geq 3f(1).$$

For each of these examples, common choices for the convex and non-decreasing function $f : (0, \infty) \rightarrow \mathbb{R}$ include $f(x) = x^p$ for $p \geq 1$, $f(x) = e^{ax}$ for $a > 0$, $f(x) = a^x$ for $a > 1$, $f(x) = xe^x$, $f(x) = \ln(1 + e^{ax})$ for $a > 0$, $f(x) = \cosh(x)$, $f(x) = x \arctan(x)$, and $f(x) = \sqrt{1 + x^2}$.

For instance, setting $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\delta = 0$, $\epsilon = 0$, $\omega = 2$, and $f(x) = x^p$ for $p \geq 1$, the theorem yields

$$\left(\frac{x^2}{y^2 + 2zx}\right)^p + \left(\frac{y^2}{z^2 + 2xy}\right)^p + \left(\frac{z^2}{x^2 + 2yz}\right)^p \geq \frac{1}{3^{p-1}}.$$

Numerous additional examples can be obtained directly from Theorem 3.2 by specializing the parameters and the function f .

4. Ten-parameter weighted convex framework

4.1. Statement and proof

Inspired by the eight-parameter framework of Theorem 2.1, the theorem below introduces a ten-parameter weighted cyclic quadratic inequality. This generalized result applies to arbitrary convex functions evaluated at ratios of general quadratic forms in three positive real variables.

Theorem 4.1. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex function. For any variables $x, y, z > 0$, and parameters $\kappa, \mu, \nu, \alpha, \beta, \gamma, \rho, \sigma, \delta, \epsilon \geq 0$ satisfying $\alpha + \beta + \gamma > 0$, $\rho + \sigma \leq 2(\kappa + \mu + \nu)$, and $\delta + \epsilon \leq 2(\alpha + \beta + \gamma)$, the*

following cyclic inequality holds:

$$\begin{aligned}
& (\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx) \\
& \times f\left(\frac{\kappa x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma yz + (2(\kappa + \mu + \nu) - \rho - \sigma)zx}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx}\right) \\
& + (\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy) \\
& \times f\left(\frac{\kappa y^2 + \mu z^2 + \nu x^2 + \rho yz + \sigma zx + (2(\kappa + \mu + \nu) - \rho - \sigma)xy}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy}\right) \\
& + (\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz) \\
& \times f\left(\frac{\kappa z^2 + \mu x^2 + \nu y^2 + \rho zx + \sigma xy + (2(\kappa + \mu + \nu) - \rho - \sigma)yz}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz}\right) \\
& \geq (\alpha + \beta + \gamma)(x + y + z)^2 f\left(\frac{\kappa + \mu + \nu}{\alpha + \beta + \gamma}\right).
\end{aligned}$$

The ten independent parameters are $\kappa, \mu, \nu, \alpha, \beta, \gamma, \rho, \sigma, \delta$, and ϵ . The equality holds if $x = y = z$.

Proof. Let us denote

$$\begin{aligned}
a_1 &= \kappa x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma yz + (2(\kappa + \mu + \nu) - \rho - \sigma)zx, \\
a_2 &= \kappa y^2 + \mu z^2 + \nu x^2 + \rho yz + \sigma zx + (2(\kappa + \mu + \nu) - \rho - \sigma)xy, \\
a_3 &= \kappa z^2 + \mu x^2 + \nu y^2 + \rho zx + \sigma xy + (2(\kappa + \mu + \nu) - \rho - \sigma)yz, \\
b_1 &= \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx, \\
b_2 &= \alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy
\end{aligned}$$

and

$$b_3 = \alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz.$$

Therefore, we can write

$$\begin{aligned}
& (\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx) \\
& \times f\left(\frac{\kappa x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma yz + (2(\kappa + \mu + \nu) - \rho - \sigma)zx}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx}\right) \\
& + (\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy) \\
& \times f\left(\frac{\kappa y^2 + \mu z^2 + \nu x^2 + \rho yz + \sigma zx + (2(\kappa + \mu + \nu) - \rho - \sigma)xy}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy}\right) \\
& + (\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz) \\
& \times f\left(\frac{\kappa z^2 + \mu x^2 + \nu y^2 + \rho zx + \sigma xy + (2(\kappa + \mu + \nu) - \rho - \sigma)yz}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz}\right) \\
& = \sum_{i=1}^3 b_i f\left(\frac{a_i}{b_i}\right).
\end{aligned}$$

The following sums hold:

$$\begin{aligned}
 \sum_{i=1}^3 a_i &= \kappa x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma yz + (2(\kappa + \mu + \nu) - \rho - \sigma) zx \\
 &+ \kappa y^2 + \mu z^2 + \nu x^2 + \rho yz + \sigma zx + (2(\kappa + \mu + \nu) - \rho - \sigma) xy \\
 &+ \kappa z^2 + \mu x^2 + \nu y^2 + \rho zx + \sigma xy + (2(\kappa + \mu + \nu) - \rho - \sigma) yz \\
 &= (\kappa + \mu + \nu)(x^2 + y^2 + z^2) + 2(\kappa + \mu + \nu)(xy + yz + zx) \\
 &= (\kappa + \mu + \nu)(x + y + z)^2
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{i=1}^3 b_i &= \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon) zx \\
 &+ \alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon) xy \\
 &+ \alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon) yz \\
 &= (\alpha + \beta + \gamma)(x^2 + y^2 + z^2) + 2(\alpha + \beta + \gamma)(xy + yz + zx) \\
 &= (\alpha + \beta + \gamma)(x + y + z)^2.
 \end{aligned}$$

Since f is convex, the Jensen inequality gives

$$\begin{aligned}
 \sum_{i=1}^3 b_i f\left(\frac{a_i}{b_i}\right) &= \left(\sum_{i=1}^3 b_i\right) \sum_{i=1}^3 \left(\frac{b_i}{\sum_{i=1}^3 b_i}\right) f\left(\frac{a_i}{b_i}\right) \\
 &\geq \left(\sum_{i=1}^3 b_i\right) f\left(\frac{1}{\sum_{i=1}^3 b_i} \sum_{i=1}^3 b_i \times \frac{a_i}{b_i}\right) = \left(\sum_{i=1}^3 b_i\right) f\left(\frac{\sum_{i=1}^3 a_i}{\sum_{i=1}^3 b_i}\right).
 \end{aligned}$$

Replacing the sums by their expressions determined above, we get

$$\begin{aligned}
 \left(\sum_{i=1}^3 b_i\right) f\left(\frac{\sum_{i=1}^3 a_i}{\sum_{i=1}^3 b_i}\right) &= (\alpha + \beta + \gamma)(x + y + z)^2 f\left(\frac{(\kappa + \mu + \nu)(x + y + z)^2}{(\alpha + \beta + \gamma)(x + y + z)^2}\right) \\
 &= (\alpha + \beta + \gamma)(x + y + z)^2 f\left(\frac{\kappa + \mu + \nu}{\alpha + \beta + \gamma}\right).
 \end{aligned}$$

Combining the inequalities above, we obtain

$$\begin{aligned}
& (\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx) \\
& \times f\left(\frac{\kappa x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma yz + (2(\kappa + \mu + \nu) - \rho - \sigma)zx}{\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon yz + (2(\alpha + \beta + \gamma) - \delta - \epsilon)zx}\right) \\
& + (\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy) \\
& \times f\left(\frac{\kappa y^2 + \mu z^2 + \nu x^2 + \rho yz + \sigma zx + (2(\kappa + \mu + \nu) - \rho - \sigma)xy}{\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \epsilon zx + (2(\alpha + \beta + \gamma) - \delta - \epsilon)xy}\right) \\
& + (\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz) \\
& \times f\left(\frac{\kappa z^2 + \mu x^2 + \nu y^2 + \rho zx + \sigma xy + (2(\kappa + \mu + \nu) - \rho - \sigma)yz}{\alpha z^2 + \beta x^2 + \gamma y^2 + \delta zx + \epsilon xy + (2(\alpha + \beta + \gamma) - \delta - \epsilon)yz}\right) \\
& \geq (\alpha + \beta + \gamma)(x + y + z)^2 f\left(\frac{\kappa + \mu + \nu}{\alpha + \beta + \gamma}\right).
\end{aligned}$$

The equality holds if $x = y = z$, with both sides reducing to $9(\alpha + \beta + \gamma)x^2 f((\kappa + \mu + \nu)/(\alpha + \beta + \gamma))$. This completes the proof. $\square$

A notable feature of this theorem is that its proof relies on a single inequality, namely, the Jensen inequality. Consequently, the convexity is the only assumption required for the function f . For this result as well, we highlight the contrast between the sophisticated upper bound and the simple lower bound. In particular, the lower bound remains tractable and easy to manipulate for a broad class of convex functions f . The presence of ten adjustable parameters makes this result particularly flexible.

4.2. Examples

To illustrate Theorem 4.1, we examine several examples for specific choices of the ten parameters involved.

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\rho = 0$, $\sigma = 0$, $\delta = 0$, and $\epsilon = 0$, the theorem gives

$$\begin{aligned}
& (y^2 + 2zx)f\left(\frac{x^2 + 2zx}{y^2 + 2zx}\right) + (z^2 + 2xy)f\left(\frac{y^2 + 2xy}{z^2 + 2xy}\right) + (x^2 + 2yz)f\left(\frac{z^2 + 2yz}{x^2 + 2yz}\right) \\
& \geq (x + y + z)^2 f(1).
\end{aligned}$$

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\rho = 2$, $\sigma = 2$, $\delta = 1$, and $\epsilon = 0$, the theorem yields

$$\begin{aligned}
& (x^2 + xy + zx)f\left(\frac{(x + y + z)^2}{x^2 + xy + zx}\right) + (y^2 + yz + xy)f\left(\frac{(x + y + z)^2}{y^2 + yz + xy}\right) \\
& + (z^2 + zx + yz)f\left(\frac{(x + y + z)^2}{z^2 + zx + yz}\right) \geq (x + y + z)^2 f(3).
\end{aligned}$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\rho = 1$, $\sigma = 1$, $\delta = 2$, and $\epsilon = 2$, the theorem gives

$$\begin{aligned}
& (x + y + z)^2 f\left(\frac{x^2 + xy + yz}{(x + y + z)^2}\right) + (x + y + z)^2 f\left(\frac{y^2 + yz + zx}{(x + y + z)^2}\right) \\
& + (x + y + z)^2 f\left(\frac{z^2 + zx + xy}{(x + y + z)^2}\right) \geq 3(x + y + z)^2 f\left(\frac{1}{3}\right).
\end{aligned}$$

For $\kappa = 0$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\rho = 0$, $\sigma = 0$, $\delta = 0$, and $\epsilon = 0$, the theorem yields

$$\begin{aligned} & (x^2 + 2zx)f\left(\frac{y^2 + z^2 + 4zx}{x^2 + 2zx}\right) + (y^2 + 2xy)f\left(\frac{z^2 + x^2 + 4xy}{y^2 + 2xy}\right) \\ & + (z^2 + 2yz)f\left(\frac{x^2 + y^2 + 4yz}{z^2 + 2yz}\right) \geq (x + y + z)^2 f(2). \end{aligned}$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\rho = 2$, $\sigma = 0$, $\delta = 0$, and $\epsilon = 2$, the theorem gives

$$\begin{aligned} & (x^2 + 2yz)f\left(\frac{x^2 + 2xy}{x^2 + 2yz}\right) + (y^2 + 2zx)f\left(\frac{y^2 + 2yz}{y^2 + 2zx}\right) \\ & + (z^2 + 2xy)f\left(\frac{z^2 + 2zx}{z^2 + 2xy}\right) \geq (x + y + z)^2 f(1). \end{aligned}$$

For $\kappa = 2$, $\mu = 1$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $\rho = 1$, $\sigma = 1$, $\delta = 1$, and $\epsilon = 1$, the theorem yields

$$\begin{aligned} & (x^2 + y^2 + xy + yz + 2zx)f\left(\frac{2x^2 + y^2 + xy + yz + 4zx}{x^2 + y^2 + xy + yz + 2zx}\right) \\ & + (y^2 + z^2 + yz + zx + 2xy)f\left(\frac{2y^2 + z^2 + yz + zx + 4xy}{y^2 + z^2 + yz + zx + 2xy}\right) \\ & + (z^2 + x^2 + zx + xy + 2yz)f\left(\frac{2z^2 + x^2 + zx + xy + 4yz}{z^2 + x^2 + zx + xy + 2yz}\right) \\ & \geq 2(x + y + z)^2 f\left(\frac{3}{2}\right). \end{aligned}$$

For $\kappa = 1$, $\mu = 1$, $\nu = 1$, $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\rho = 1$, $\sigma = 1$, $\delta = 1$, and $\epsilon = 1$, the theorem gives

$$\begin{aligned} & (y^2 + z^2 + xy + yz + 2zx)f\left(\frac{x^2 + y^2 + z^2 + xy + yz + 4zx}{y^2 + z^2 + xy + yz + 2zx}\right) \\ & + (z^2 + x^2 + yz + zx + 2xy)f\left(\frac{x^2 + y^2 + z^2 + yz + zx + 4xy}{z^2 + x^2 + yz + zx + 2xy}\right) \\ & + (x^2 + y^2 + zx + xy + 2yz)f\left(\frac{x^2 + y^2 + z^2 + zx + xy + 4yz}{x^2 + y^2 + zx + xy + 2yz}\right) \\ & \geq 2(x + y + z)^2 f\left(\frac{3}{2}\right). \end{aligned}$$

For $\kappa = 0$, $\mu = 2$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\rho = 0$, $\sigma = 0$, $\delta = 0$, and $\epsilon = 0$, the theorem yields

$$\begin{aligned} & (x^2 + 2zx)f\left(\frac{2y^2 + z^2 + 6zx}{x^2 + 2zx}\right) + (y^2 + 2xy)f\left(\frac{2z^2 + x^2 + 6xy}{y^2 + 2xy}\right) \\ & + (z^2 + 2yz)f\left(\frac{2x^2 + y^2 + 6yz}{z^2 + 2yz}\right) \geq (x + y + z)^2 f(3). \end{aligned}$$

For $\kappa = 2$, $\mu = 0$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\rho = 0$, $\sigma = 2$, $\delta = 0$, and $\epsilon = 2$, the theorem gives

$$\begin{aligned} & (x^2 + z^2 + 2yz + 2zx)f\left(\frac{2x^2 + z^2 + 2yz + 4zx}{x^2 + z^2 + 2yz + 2zx}\right) \\ & + (y^2 + x^2 + 2zx + 2xy)f\left(\frac{2y^2 + x^2 + 2zx + 4xy}{y^2 + x^2 + 2zx + 2xy}\right) \\ & + (z^2 + y^2 + 2xy + 2yz)f\left(\frac{2z^2 + y^2 + 2xy + 4yz}{z^2 + y^2 + 2xy + 2yz}\right) \\ & \geq 2(x + y + z)^2 f\left(\frac{3}{2}\right). \end{aligned}$$

For $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\rho = 0$, $\sigma = 0$, $\delta = 1$, and $\epsilon = 1$, the theorem yields

$$\begin{aligned} & (x^2 + y^2 + z^2 + xy + yz + 4zx)f\left(\frac{x^2 + 2zx}{x^2 + y^2 + z^2 + xy + yz + 4zx}\right) \\ & + (x^2 + y^2 + z^2 + yz + zx + 4xy)f\left(\frac{y^2 + 2xy}{x^2 + y^2 + z^2 + yz + zx + 4xy}\right) \\ & + (x^2 + y^2 + z^2 + zx + xy + 4yz)f\left(\frac{z^2 + 2yz}{x^2 + y^2 + z^2 + zx + xy + 4yz}\right) \\ & \geq 3(x + y + z)^2 f\left(\frac{1}{3}\right). \end{aligned}$$

For each of these examples, common choices for the convex function $f : (0, \infty) \rightarrow \mathbb{R}$ include decreasing functions such as $f(x) = x^{-p}$ for $p > 0$, $f(x) = e^{-ax}$ for $a > 0$, $f(x) = -\ln(x)$, $f(x) = 1/(1+x)$, and $f(x) = 1/(e^x - 1)$, as well as the non-decreasing functions presented earlier, i.e., $f(x) = x^p$ for $p \geq 1$, $f(x) = e^{ax}$ for $a > 0$, $f(x) = a^x$ for $a > 1$, $f(x) = xe^x$, $f(x) = \ln(1 + e^{ax})$ for $a > 0$, $f(x) = \cosh(x)$, $f(x) = x \arctan(x)$, and $f(x) = \sqrt{1 + x^2}$.

For instance, for $\kappa = 1$, $\mu = 0$, $\nu = 0$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\rho = 2$, $\sigma = 0$, $\delta = 0$, $\epsilon = 2$, and $f(x) = -\ln(x)$, the theorem gives

$$\begin{aligned} & (x^2 + 2yz) \left(-\ln\left(\frac{x^2 + 2xy}{x^2 + 2yz}\right) \right) + (y^2 + 2zx) \left(-\ln\left(\frac{y^2 + 2yz}{y^2 + 2zx}\right) \right) \\ & + (z^2 + 2xy) \left(-\ln\left(\frac{z^2 + 2zx}{z^2 + 2xy}\right) \right) \geq (x + y + z)^2 (-\ln(1)) = 0, \end{aligned}$$

which is equivalent to the cyclic product inequality

$$\left(\frac{x^2 + 2xy}{x^2 + 2yz}\right)^{x^2 + 2yz} \left(\frac{y^2 + 2yz}{y^2 + 2zx}\right)^{y^2 + 2zx} \left(\frac{z^2 + 2zx}{z^2 + 2xy}\right)^{z^2 + 2xy} \leq 1.$$

As another example, for $\kappa = 0$, $\mu = 1$, $\nu = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\rho = 0$, $\sigma = 0$, $\delta = 0$, $\epsilon = 0$, and $f(x) = 1/x$, the theorem yields

$$\frac{(x^2 + 2zx)^2}{y^2 + z^2 + 4zx} + \frac{(y^2 + 2xy)^2}{z^2 + x^2 + 4xy} + \frac{(z^2 + 2yz)^2}{x^2 + y^2 + 4yz} \geq \frac{(x + y + z)^2}{2}.$$

Numerous additional inequalities can be derived directly from Theorem 4.1 by specializing the parameters and the function f .

5. Concluding remarks

This paper established unified eight-, nine-, and ten-parameter frameworks for cyclic quadratic inequalities in three variables, generalized via convex functions. By using the Titu lemma and the Jensen inequality, this approach generates classical and new inequalities contrasting sophisticated upper bounds with tractable lower bounds. Future research must extend these frameworks to n variables with $n \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and higher-degree polynomials. Furthermore, exploring non-convex functional extensions and applying these parameterized bounds to algebraic geometry and optimization problems remain next steps.

Acknowledgment

The author would like to thank the reviewers for their constructive comments.

References

- [1] Bencze M, Pop OT. Generalizations and refinements for Nesbitt's inequality. J Math Inequal 2011; 5: 13-20.
- [2] Bencze M, Zhao C. A refinement of Nesbitt's inequality. Octagon Math Mag 2008; 16: 275-276.
- [3] Chesneau C. On three one-parameter families of cyclic inequalities. J Comput Sci App Math 2026; 8: 207-212.
- [4] Chesneau C. A collection of six two-parameter families of cyclic inequalities. J Comput Sci App Math 2026; 8: 239-248.
- [5] Cîrtoaje V. Mathematical Inequalities – Volume 3: Cyclic and Noncyclic Inequalities. Lambert Academic Publishing, 2018.
- [6] Drâmbe MO. Inequalities – Ideas and Methods. Zalău, Romania: Ed. Gil, 2003.
- [7] Nesbitt AM. Problem 15114. Educational Times 1903; 3: 37-38.
- [8] Wei FH, Wu SH. Generalizations and analogues of Nesbitt's inequality. Octagon Math Mag 2009; 17: 215-220.

Appendix

In this section, we present the formal statements of two classical results used throughout this paper: the Titu lemma and the Jensen inequality.

Lemma 5.1 (Titu's Lemma). *Let $n \in \mathbb{N} \setminus \{0\}$, $a_1, \dots, a_n \in \mathbb{R}$, and $b_1, \dots, b_n > 0$. Then we have*

$$\sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{(\sum_{i=1}^n a_i)^2}{\sum_{i=1}^n b_i}.$$

Lemma 5.2 (Jensen's inequality). *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \rightarrow \mathbb{R}$ be a convex function, $n \in \mathbb{N} \setminus \{0\}$, and $x_1, \dots, x_n \in I$. If $w_1, \dots, w_n \geq 0$ satisfy $\sum_{i=1}^n w_i = 1$, then we have*

$$f\left(\sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n w_i f(x_i).$$