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Nano $\mathcal{I}_{g\#}$-normal and nano $\mathcal{I}_{g\#}$-regular spaces

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Abstract: In this paper, we define $n\mathcal{I}_{g\#}$-normal and $n\mathcal{I}_{g\#}$-regular spaces are introduced and various characterizations and properties are given. Characterizations of nano normal, mildly nano normal, $ng^{\#}$-normal and regular spaces are also given.

Key words: $n\mathcal{I}_{g\#}$-normal, ${}_{g\#}n\mathcal{I}$-normal, $n\mathcal{I}_{g\#}$-regular and $ng^{\#}$-normal spaces

1. Introduction and Preliminaries

In 2013, Lellis Thivagar et al. [6] introduced a nano topological space. Then the notions of an ideal nano topological space was introduced by Parimala et al [10]. A nano topological space $(U, \mathcal{N})$ with an ideal I on U is called [10] an ideal nano topological space and is denoted by $(U, \mathcal{N}, I)$. Recently several authors were discussed and studied the concepts of an ideal nanotopological spaces for example[[1], [4], [14], [15] and [16]]. In this paper, we define $n\mathcal{I}_{g\#}$-normal, ${}_{g\#}n\mathcal{I}$-normal and $n\mathcal{I}_{g\#}$-regular spaces using $n\mathcal{I}_{g\#}$-open sets and give characterizations and properties of such spaces. Also, characterizations of nano normal, mildly nano normal, $ng^{\#}$-normal and regular spaces are given.

Definition 1.1. A subset H of a nano topological space $(U, \mathcal{N})$ is called;

1. nano g-closed (briefly, ng-closed) [2] if $ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is n-open.
2. nano regular closed (briefly, n-regular closed) [6] if $H = ncl(nint(H))$.
3. nano pre-open [6] if $H \subseteq nint(ncl(H))$.
4. nano αg-closed set (briefly, $n\alpha g$-closed) [19] if $n\alpha cl(H) \subseteq G$ whenever $H \subseteq G$ and G is n-open.
5. nano rg-closed set (briefly, nrg-closed) [18] if $ncl(H) \subseteq G$ whenever $H \subseteq G$ and G is n-regular open.
6. nano dense (briefly n-dense) [17] if $ncl(H) = U$,

Definition 1.2. [6] A subset H of a space $(U, \mathcal{N})$ is called nano α-open (briefly $n\alpha$-open) if $H \subseteq nint(ncl(nint(H)))$.

The family of all $n\alpha$-open sets in $(U, \mathcal{N})$ denoted by $n\tau^{\alpha}$.

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Definition 1.3. A subset H of a space $(U, \mathcal{N}, I)$ is called

1. nano $\star$-closed (briefly, $n\star$-closed) [11] if $H_n^* \subseteq H$.
2. nano $\tau^\star$-closed (briefly $n\tau^\star$-closed) [10] if $H_n^* \subseteq H$.
3. nano I_g-closed (briefly nI_g-closed) [11] if $H_n^* \subseteq G$ whenever $H \subseteq G$ and G is n-open.
4. nano I_{rg}-closed (briefly nI_{rg}-closed) [12] if $H_n^* \subseteq G$ whenever $H \subseteq G$ and G is n-regular open.
5. nano $\mathcal{I}_{g\#}$-closed (briefly $n\mathcal{I}_{g\#}$-closed) [3] if $H_n* \subseteq G$ whenever $H \subseteq G$ and G is nano αg-open.

Definition 1.4. [7] A subset E of a space $(U, \mathcal{N})$ is called n-nowhere dense if $nint(ncl(E)) = \phi$.

Definition 1.5. [11] A subset E of a space $(U, \mathcal{N}, I)$ is called $n\star$-closed if $E_n^\star \subseteq E$.

The following lemmas and proposition will be useful in the sequel.

Lemma 1.1. *[10] Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nano topological space. If $\mathcal{I}$ is completely nano codense, then $n\tau^\star \subseteq n\tau^\alpha$.*

Theorem 1.1. *[3] Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nano topological space where $\mathcal{I}$ is completely nano codense. Then the following are equivalent.*

1. *U is nano normal.*
2. *For any disjoint nano closed sets E and F, there exist disjoint $n\mathcal{I}_{g\#}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
3. *For any nano closed set E and open set Q containing E, there exists an $n\mathcal{I}_{g\#}$-open set P such that $E \subseteq P \subseteq ncl^\star(P) \subseteq Q$.*

Corollary 1.1. *[3] If $(U, \tau_R(X), \mathcal{I})$ is an ideal nano topological space and $E \subseteq U$, then if $\mathcal{I} = \{\phi\}$, then E is $n\mathcal{I}_{g\#}$-closed if and only if E is $ng^\#$-closed.*

Lemma 1.2. *[3] Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nano topological space and $E \subseteq U$. Then E is $n\mathcal{I}_{g\#}$-open if and only if $G \subseteq nint^\star(E)$ whenever G is $n\alpha g$-closed and $G \subseteq E$.*

Lemma 1.3. *[3] Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nano topological space. Then every subset of U is $n\mathcal{I}_{g\#}$-closed if and only if every $n\alpha g$-open set is $n\star$-closed.*

Lemma 1.4. *[12] Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nano topological space. A subset $A \subseteq U$ is $n\mathcal{I}_{rg}$-open if and only if $F \subseteq nint^\star(A)$ whenever F is nano regular closed and $F \subseteq A$.*

2. Nano $\mathcal{I}_{g\#}$-normal and nano $_{g\#}\mathcal{I}$-normal spaces

Definition 2.1. An ideal nano topological space $(U, \tau_R(X), \mathcal{I})$ is said to be an $n\mathcal{I}_{g\#}$-normal space if for every pair of disjoint closed sets E and F, there exist disjoint $n\mathcal{I}_{g\#}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.

Remark 2.1. *Every nano open set is an $n\mathcal{I}_{g\#}$-open set.*

2. *Every nano normal space is* $n\mathcal{I}_{g^\#}$*-normal.*

The following Example 2.1 shows that an $n\mathcal{I}_{g^\#}$-normal space is not necessarily a nano normal space. Theorem 2.1 below gives characterizations of $n\mathcal{I}_{g^\#}$-normal spaces. Theorem 2.2 below shows that the two concepts coincide for completely nano codense ideal topological spaces.

Example 2.1. *Let* $U = \{a_1, a_2, a_3, a_4\}$ *with* $U/R = \{\{a_2\}, \{a_4\}, \{a_1, a_3\}\}$ *and* $X = \{a_3, a_4\}$*. Then* $\mathcal{N} = \{\phi, \{a_4\}, \{a_1, a_3\}, \{a_1, a_3, a_4\}, U\}$ *and* $\mathcal{I} = \{\phi, \{a_4\}\}$*. Then* $\phi^* = \phi$*,* $(\{a_4\})^* = \{\phi\}$*,* $(\{a_1, a_3\})^* = \{a_1, a_2, a_3\}$*,* $(\{a_1, a_3, a_4\})^* = \{a_1, a_2, a_3\}$ *and* $U^* = \{a_1, a_2, a_3\}$*. Here every* $n\alpha g$*-open set is* $n\star$*-closed and so, by Lemma 1.3, every subset of* U *is* $n\mathcal{I}_{g^\#}$*-closed and hence every subset of* U *is* $n\mathcal{I}_{g^\#}$*-open. This implies that* $(U, \tau_R(X), \mathcal{I})$ *is* $n\mathcal{I}_{g^\#}$*-normal. Now* ϕ *and* $\{a_2, a_4\}$ *are disjoint nano closed subsets of* U *which are not separated by disjoint nano open sets and so* $(U, \tau_R(X))$ *is not nano normal.*

Theorem 2.1. *Let* $(U, \tau_R(X), \mathcal{I})$ *be an ideal nano topological space. Then the following are equivalent.*

1. U *is* $n\mathcal{I}_{g^\#}$*-normal.*
2. *For every nano closed set* E *and an nano open set* Q *containing* E*, there exists an* $n\mathcal{I}_{g^\#}$*-open set* P *such that* $E \subseteq P \subseteq ncl^\star(P) \subseteq Q$*.*

Proof. $(1) \Rightarrow (2)$. Let E be a nano closed set and Q be an nano open set containing E. Since E and $U - Q$ are disjoint nano closed sets, there exist disjoint $n\mathcal{I}_{g^\#}$-open sets P and S such that $E \subseteq P$ and $U - Q \subseteq S$. Again, $P \cap S = \phi$ implies that $P \cap nint^\star(S) = \phi$ and so $ncl^\star(P) \subseteq U - nint^\star(S)$. Since $U - Q$ is $n\alpha g$-closed and S is $n\mathcal{I}_{g^\#}$-open, $U - Q \subseteq S$ implies that $U - Q \subseteq nint^\star(S)$ and so $U - nint^\star(S) \subseteq Q$. Thus, we have $E \subseteq P \subseteq ncl^\star(P) \subseteq U - nint^\star(S) \subseteq Q$ which proves (2).

$(2) \Rightarrow (1)$. Let E and F be two disjoint nano closed subsets of U. By hypothesis, there exists an $n\mathcal{I}_{g^\#}$-open set P such that $E \subseteq P \subseteq ncl^\star(P) \subseteq U - F$. If $S = U - ncl^\star(P)$, then P and S are the required disjoint $n\mathcal{I}_{g^\#}$-open sets containing E and F respectively. So, $(U, \tau_R(X), \mathcal{I})$ is $n\mathcal{I}_{g^\#}$-normal. □

Theorem 2.2. *Let* $(U, \tau_R(X), \mathcal{I})$ *be an ideal nano topological space where* $n\mathcal{I}$ *is completely nano codense. If* $(U, \tau_R(X), \mathcal{I})$ *is* $n\mathcal{I}_{g^\#}$*-normal, then it is a nano normal space.*

Proof. It is obvious from Theorem 2.1 and Lemma 1.1. □

Theorem 2.3. *Let* $(U, \tau_R(X), \mathcal{I})$ *be an* $n\mathcal{I}_{g^\#}$*-normal space. If* F *is nano closed and* E *is a* $ng^\#$*-closed set such that* $E \cap F = \phi$*, then there exist disjoint* $n\mathcal{I}_{g^\#}$*-open sets* P *and* Q *such that* $E \subseteq P$ *and* $F \subseteq Q$*.*

Proof. Since $E \cap F = \phi$, $E \subseteq U - F$ where $U - F$ is $n\alpha g$-open. Therefore, by hypothesis, $ncl(A) \subseteq U - F$. Since $ncl(E) \cap F = \phi$ and U is $n\mathcal{I}_{g^\#}$-normal, there exist disjoint $n\mathcal{I}_{g^\#}$-open sets P and Q such that $ncl(E) \subseteq P$ and $F \subseteq Q$. Thus $E \subseteq P$ and $F \subseteq Q$.

The following Corollaries 2.1 and 2.2 give properties of nano normal spaces. If $\mathcal{I}=\{\phi\}$ in Theorem 2.3, then we have the following Corollary 2.1, the proof of which follows from Theorem 2.2 and Lemma 1.1, since $\{\phi\}$ is a completely nano codense ideal. If $\mathcal{I}=\mathcal{N}$ in Theorem 2.3, then we have the Corollary 2.2 below, since $n\tau^\star(\mathcal{N}) = n\tau^\alpha$ and $n\mathcal{I}_{g^\#}$-open sets coincide with $n\alpha g^\#$-open sets. □

Corollary 2.1. *Let $(U, \tau_R(X))$ be a nano normal space with $\mathcal{I} = \{\phi\}$. If F is a nano closed set and E is a $ng^{\#}$-closed set disjoint from F, then there exist disjoint $ng^{\#}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*

Corollary 2.2. *Let $(U, \tau_R(X), \mathcal{I})$ be a normal ideal nano topological space where $\mathcal{I} = \mathcal{N}$. If F is a nano closed set and E is a $ng^{\#}$-closed set disjoint from F, then there exist disjoint $n\alpha g^{\#}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*

Theorem 2.4. *Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nano topological space which is $n\mathcal{I}_{g\#}$-normal. Then the following hold.*

1. *For every nano closed set E and every $ng^{\#}$-open set F containing E, there exists an $n\mathcal{I}_{g\#}$-open set P such that $E \subseteq nint^{\star}(P) \subseteq P \subseteq F$.*
2. *For every $ng^{\#}$-closed set E and every nano open set F containing E, there exists an $n\mathcal{I}_{g\#}$-closed set P such that $E \subseteq P \subseteq ncl^{\star}(P) \subseteq F$.*

Proof. (1) Let E be a nano closed set and F be a $ng^{\#}$-open set containing E. Then $E \cap (U - F) = \phi$, where E is nano closed and $U - F$ is $ng^{\#}$-closed. By Theorem 2.3, there exist disjoint $n\mathcal{I}_{g\#}$-open sets P and Q such that $E \subseteq P$ and $U - F \subseteq Q$. Since $P \cap Q = \phi$, we have $P \subseteq U - Q$. By Lemma 1.2, $E \subseteq nint^{\star}(P)$. Therefore, $E \subseteq nint^{\star}(P) \subseteq P \subseteq U - Q \subseteq F$. This proves (1).

(2) Let E be a $ng^{\#}$-closed set and F be an nano open set containing E. Then $U - F$ is a nano closed set contained in the $ng^{\#}$-open set $U - E$. By (1), there exists an $n\mathcal{I}_{g\#}$-open set Q such that $U - F \subseteq nint^{\star}(Q) \subseteq Q \subseteq U - E$. Therefore, $E \subseteq U - Q \subseteq ncl^{\star}(U - Q) \subseteq F$. If $P = U - Q$, then $E \subseteq P \subseteq ncl^{\star}(P) \subseteq F$ and so P is the required $n\mathcal{I}_{g\#}$-closed set.

The following Corollaries 2.3 and 2.4 give some properties of nano normal spaces. If $\mathcal{I}=\{\phi\}$ in Theorem 2.4, then we have the following Corollary 2.3. If $\mathcal{I}=\mathcal{N}$ in Theorem 2.4, then we have the Corollary 2.4 below. □

Corollary 2.3. *Let $(U, \tau_R(X))$ be a nano normal space with $\mathcal{I}=\{\phi\}$. Then the following hold.*

1. *For every nano closed set E and every $ng^{\#}$-open set F containing E, there exists a $ng^{\#}$-open set P such that $E \subseteq nint(P) \subseteq P \subseteq F$.*
2. *For every $ng^{\#}$-closed set E and every nano open set F containing E, there exists a $ng^{\#}$-closed set P such that $E \subseteq P \subseteq ncl(P) \subseteq F$.*

Corollary 2.4. *Let $(U, \tau_R(X))$ be a nano normal space with $\mathcal{I} = \mathcal{N}$. Then the following hold.*

1. *For every nano closed set E and every $ng^{\#}$-open set F containing E, there exists an $n\alpha g^{\#}$-open set P such that $E \subseteq n\alpha\text{-}int(P) \subseteq P \subseteq F$.*
2. *For every $ng^{\#}$-closed set E and every nano open set F containing E, there exists an $n\alpha g^{\#}$-closed set P such that $E \subseteq P \subseteq n\alpha\text{-}cl(P) \subseteq F$.*

Definition 2.2. An ideal nano topological space $(U, \tau_R(X), \mathcal{I})$ is said to be $_{g\#}\, n\mathcal{I}$-normal if for each pair of disjoint $n\mathcal{I}_{g\#}$-closed sets E and F, there exist disjoint open sets P and Q in U such that $E \subseteq P$ and $F \subseteq Q$.

Remark 2.2. 1. *Every nano closed set is $n\mathcal{I}_{g^\#}$-closed.*

2. *Every ${}_{g^\#}n\mathcal{I}$-normal space is nano normal. But converse need not be true as the following Example 2.2 shows.*

Example 2.2. *Let $U=\{a_1,a_2,a_3\}$ with $U/R=\{\{a_1\},\{a_2,a_3\}\}$ and $X=\{a_1,a_3\}$. Then the nano topology $\mathcal{N}=\{\phi,\{a_1\},\{a_2,a_3\},U\}$ and $\mathcal{I}=\{\phi,\{a_1\},\{a_2\},\{a_3\},\ \{a_1,a_2\},\{a_1,a_3\},\{a_2,a_3\},U\}$. Every $n\alpha g$-open set is $n\star$-closed and so every subset of U is $n\mathcal{I}_{g^\#}$-closed. Now $E=\{a_1,a_2\}$ and $F=\{a_3\}$ are disjoint $n\mathcal{I}_{g^\#}$-closed sets, but they are not separated by disjoint nano open sets. So $(U,\tau_R(X),\mathcal{I})$ is not ${}_{g^\#}n\mathcal{I}$-normal. But $(U,\tau_R(X),\mathcal{I})$ is nano normal.*

Theorems 2.5 and 2.6 below give characterizations of ${}_{g^\#}n\mathcal{I}$-normal spaces.

Theorem 2.5. *In an ideal nano topological space $(U,\tau_R(X),\mathcal{I})$, the following are equivalent.*

1. *U is ${}_{g^\#}n\mathcal{I}$-normal.*
2. *For every $n\mathcal{I}_{g^\#}$-closed set E and every $n\mathcal{I}_{g^\#}$-open set F containing E, there exists an nano open set P of U such that $E\subseteq P\subseteq ncl(P)\subseteq F$.*

Proof. It is similar to the proof of Theorem 2.1.

If $\mathcal{I}=\{\phi\}$, then ${}_{g^\#}n\mathcal{I}$-normal spaces coincide with $ng^\#$-normal spaces and so if we take $\mathcal{I}=\{\phi\}$, in Theorem 2.5 then we have the following characterization for $ng^\#$-normal spaces. □

Corollary 2.5. *In a space $(U,\tau_R(X))$, the following are equivalent.*

1. *U is $ng^\#$-normal.*
2. *For every $ng^\#$-closed set E and every $ng^\#$-open set F containing E, there exists an nano open set P of U such that $E\subseteq P\subseteq ncl(P)\subseteq F$.*

Theorem 2.6. *In an ideal nano topological space $(U,\tau_R(X),\mathcal{I})$, the following are equivalent.*

1. *U is ${}_{g^\#}n\mathcal{I}$-normal.*
2. *For each pair of disjoint $n\mathcal{I}_{g^\#}$-closed subsets E and F of U, there exists an nano open set P of U containing E such that $cl(P)\cap F=\phi$.*
3. *For each pair of disjoint $n\mathcal{I}_{g^\#}$-closed subsets E and F of U, there exist an nano open set P containing E and an nano open set Q containing F such that $ncl(P)\cap ncl(Q)=\phi$.*

Proof. $(1)\Rightarrow(2)$. Suppose that E and F are disjoint $n\mathcal{I}_{g^\#}$-closed subsets of U. Then the $n\mathcal{I}_{g^\#}$-closed set E is contained in the $n\mathcal{I}_{g^\#}$-open set $U-F$. By Theorem 2.5, there exists an nano open set P such that $E\subseteq P\subseteq ncl(P)\subseteq U-F$. Therefore, P is the required nano open set containing E such that $ncl(P)\cap F=\phi$.

$(2)\Rightarrow(3)$. Let E and F be two disjoint $n\mathcal{I}_{g^\#}$-closed subsets of U. By hypothesis, there exists an nano open set P of U containing E such that $ncl(P)\cap F=\phi$. Also, $ncl(P)$ and F are disjoint $n\mathcal{I}_{g^\#}$-closed sets of U. By hypothesis, there exists an nano open set Q of U containing F such that $ncl(P)\cap ncl(Q)=\phi$.

$(3)\Rightarrow(1)$. The proof is clear.

If $\mathcal{I}=\{\phi\}$, in Theorem 2.6, then we have the following characterizations for $ng^\#$-normal spaces. □

Corollary 2.6. *Let $(U, \tau_R(X))$ be a nano topological space. Then the following are equivalent.*

1. *U is $ng^{\#}$-normal.*
2. *For each pair of disjoint $ng^{\#}$-closed subsets E and F of U, there exists an nano open set P of U containing E such that $ncl(P) \cap F = \phi$.*
3. *For each pair of disjoint $ng^{\#}$-closed subsets E and F of U, there exist an nano open set P containing E and an nano open set Q containing F such that $ncl(P) \cap ncl(Q) = \phi$.*

Theorem 2.7. *Let $(U, \tau_R(X), \mathcal{I})$ be an ${}_{g\#}n\mathcal{I}$-normal space. If E and F are disjoint $n\mathcal{I}_{g\#}$-closed subsets of U, then there exist disjoint nano open sets P and Q such that $ncl^{\star}(E) \subseteq P$ and $ncl^{\star}(F) \subseteq Q$.*

Proof. Suppose that E and F are disjoint $n\mathcal{I}_{g\#}$-closed sets. By Theorem 2.6(3), there exist an nano open set P containing E and an nano open set Q containing F such that $ncl(P) \cap ncl(Q) = \phi$. Since E is $n\mathcal{I}_{g\#}$-closed, $E \subseteq P$ implies that $ncl^{\star}(E) \subseteq P$. Similarly $ncl^{\star}(F) \subseteq Q$.

If $\mathcal{I}$={ϕ}, in Theorem 2.7, then we have the following property of disjoint $ng^{\#}$-closed sets in $ng^{\#}$-normal spaces. □

Corollary 2.7. *Let $(U, \tau_R(X))$ be a $ng^{\#}$-normal space. If E and F are disjoint $ng^{\#}$-closed subsets of U, then there exist disjoint open sets P and Q such that $ncl(E) \subseteq P$ and $cl(F) \subseteq Q$.*

Theorem 2.8. *Let $(U, \tau_R(X), \mathcal{I})$ be an ${}_{g\#}n\mathcal{I}$-normal space. If E is an $n\mathcal{I}_{g\#}$-closed set and F is an $n\mathcal{I}_{g\#}$-open set containing E, then there exists an nano open set P such that $E \subseteq ncl^{\star}(E) \subseteq P \subseteq nint^{\star}(F) \subseteq F$.*

Proof. Suppose E is an $n\mathcal{I}_{g\#}$-closed set and F is an $n\mathcal{I}_{g\#}$-open set containing E. Since E and $U - F$ are disjoint $n\mathcal{I}_{g\#}$-closed sets, by Theorem 2.7, there exist disjoint nano open sets P and Q such that $ncl^{\star}(E) \subseteq P$ and $ncl^{\star}(U - F) \subseteq Q$. Now, $U - nint^{\star}(F) = ncl^{\star}(U - F) \subseteq Q$ implies that $U - Q \subseteq nint^{\star}(F)$. Again, $P \cap Q = \phi$ implies $P \subseteq U - Q$ and so $E \subseteq ncl^{\star}(E) \subseteq P \subseteq U - Q \subseteq nint^{\star}(F) \subseteq F$.

If $\mathcal{I}$={ϕ}, in Theorem 2.8, then we have the following Corollary 2.8. □

Corollary 2.8. *Let $(U, \tau_R(X))$ be a $ng^{\#}$-normal space. If E is a $ng^{\#}$-closed set and F is a $ng^{\#}$-open set containing E, then there exists an nano open set P such that $E \subseteq ncl(E) \subseteq P \subseteq nint(F) \subseteq F$.*

The following Theorem 2.9 gives a characterization of nano normal spaces in terms of $ng^{\#}$-open sets which follows from Lemma 1.1 if $\mathcal{I}$={ϕ}.

Theorem 2.9. *Let $(U, \tau_R(X))$ be a nano topological space. Then the following are equivalent.*

1. *U is normal.*
2. *For any disjoint nano closed sets E and F, there exist disjoint $ng^{\#}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
3. *For any nano closed set E and nano open set Q containing E, there exists a $ng^{\#}$-open set P such that $E \subseteq P \subseteq ncl(P) \subseteq Q$.*

The rest of the section is devoted to the study of mildly nano normal spaces in terms of $n\mathcal{I}_{g\#}$-open sets, $n\mathcal{I}_g$-open sets and $n\mathcal{I}_{rg}$-open sets.

Theorem 2.10. *Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nano topological space where $\mathcal{I}$ is completely nano codense. Then the following are equivalent.*

1. *U is mildly nano normal.*
2. *For disjoint nano regular closed sets E and F, there exist disjoint $n\mathcal{I}_{g^\#}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
3. *For disjoint nano regular closed sets E and F, there exist disjoint $n\mathcal{I}_g$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
4. *For disjoint nano regular closed sets E and F, there exist disjoint $n\mathcal{I}_{rg}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
5. *For a nano regular closed set E and a nano regular open set Q containing E, there exists an $n\mathcal{I}_{rg}$-open set P of U such that $E \subseteq P \subseteq ncl^\star(P) \subseteq Q$.*
6. *For a nano regular closed set E and a nano regular open set Q containing E, there exists an $n\star$-open set P of U such that $E \subseteq P \subseteq ncl^\star(P) \subseteq Q$.*
7. *For disjoint nano regular closed sets E and F, there exist disjoint $n\star$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*

Proof. $(1) \Rightarrow (2)$. Suppose that E and F are disjoint nano regular closed sets. Since U is mildly nano normal, there exist disjoint nano open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$. But every nano open set is an $n\mathcal{I}_{g^\#}$-open set. This proves (2).

$(2) \Rightarrow (3)$. The proof follows from the fact that every $n\mathcal{I}_{g^\#}$-open set is an $n\mathcal{I}_g$-open set.

$(3) \Rightarrow (4)$. The proof follows from the fact that every $n\mathcal{I}_g$-open set is an $n\mathcal{I}_{rg}$-open set.

$(4) \Rightarrow (5)$. Suppose E is a nano regular closed and F is a nano regular open set containing E. Then E and $U - F$ are disjoint nano regular closed sets. By hypothesis, there exist disjoint $n\mathcal{I}_{rg}$-open sets P and Q such that $E \subseteq P$ and $U - F \subseteq Q$. Since $U - F$ is nano regular closed and Q is $n\mathcal{I}_{rg}$-open, by Lemma 1.4, $U - F \subseteq nint^\star(Q)$ and so $U - nint^\star(Q) \subseteq F$. Again, $P \cap Q = \phi$ implies that $P \cap nint^\star(Q) = \phi$ and so $ncl^\star(P) \subseteq U - nint^\star(Q) \subseteq F$. Hence P is the required $n\mathcal{I}_{rg}$-open set such that $E \subseteq P \subseteq ncl^\star(P) \subseteq F$.

$(5) \Rightarrow (6)$. Let E be a nano regular closed set and Q be a nano regular open set containing E. Then there exists an $n\mathcal{I}_{rg}$-open set H of U such that $E \subseteq H \subseteq ncl^\star(H) \subseteq Q$. By Lemma 1.4, $E \subseteq nint^\star(H)$. If $P = nint^\star(H)$, then P is an $n\star$-open set and $E \subseteq P \subseteq ncl^\star(P) \subseteq ncl^\star(H) \subseteq Q$. Therefore, $E \subseteq P \subseteq ncl^\star(P) \subseteq Q$.

$(6) \Rightarrow (7)$. Let E and F be disjoint nano regular closed subsets of U. Then $U - F$ is a nano regular open set containing E. By hypothesis, there exists an $n\star$-open set P of U such that $E \subseteq P \subseteq ncl^\star(P) \subseteq U - F$. If $Q = U - ncl^\star(P)$, then P and Q are disjoint $n\star$-open sets of U such that $E \subseteq P$ and $F \subseteq Q$.

$(7) \Rightarrow (1)$. Let E and F be disjoint nano regular closed sets of U. Then there exist disjoint $n\star$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$. Since $\mathcal{I}$ is completely nano codense, by Lemma 1.1, $n\tau^\star \subseteq n\tau^\alpha$ and so P, $Q \in n\tau^\alpha$. Hence $E \subseteq P \subseteq nint(ncl(nint(P))) = G$ and $F \subseteq Q \subseteq nint(ncl(nint(Q))) = H$. G and H are the required disjoint nano open sets containing E and F respectively. This proves (1).

If $\mathcal{I}=\mathcal{N}$, in the above Theorem 2.10, then $n\mathcal{I}_{rg}$-closed sets coincide with $nr\alpha g$-closed sets and so we have the following Corollary 2.9. □

Corollary 2.9. *Let $(U, \tau_R(X))$ be a nano topological space. Then the following are equivalent.*

1. *U is mildly nano normal.*
2. *For disjoint nano regular closed sets E and F, there exist disjoint $n\alpha g^{\#}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
3. *For disjoint nano regular closed sets E and F, there exist disjoint $n\alpha g$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
4. *For disjoint nano regular closed sets E and F, there exist disjoint $nr\alpha g$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
5. *For a nano regular closed set E and a nano regular open set Q containing E, there exists an $nr\alpha g$-open set P of U such that $E \subseteq P \subseteq n\alpha\text{-}cl(P) \subseteq Q$.*
6. *For a nano regular closed set E and a nano regular open set Q containing E, there exists an $n\alpha$-open set P of U such that $E \subseteq P \subseteq n\alpha\text{-}cl(P) \subseteq Q$.*
7. *For disjoint nano regular closed sets E and F, there exist disjoint $n\alpha$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*

If $\mathcal{I}=\{\phi\}$ in the above Theorem 2.10, we get the following Corollary 2.10.

Corollary 2.10. *Let $(X, \tau_R(X))$ be a nano topological space. Then the following are equivalent.*

1. *U is mildly nano normal.*
2. *For disjoint nano regular closed sets E and F, there exist disjoint $ng^{\#}$-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
3. *For disjoint nano regular closed sets E and F, there exist disjoint ng-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
4. *For disjoint nano regular closed sets E and F, there exist disjoint nrg-open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*
5. *For a nano regular closed set E and a nano regular open set Q containing E, there exists an nrg-open set P of U such that $E \subseteq P \subseteq ncl(P) \subseteq Q$.*
6. *For a nano regular closed set E and a nano regular open set Q containing E, there exists an nano open set P of U such that $E \subseteq P \subseteq ncl(P) \subseteq Q$.*
7. *For disjoint nano regular closed sets E and F, there exist disjoint nano open sets P and Q such that $E \subseteq P$ and $F \subseteq Q$.*

3. Nano $\mathcal{I}_{g^{\#}}$-regular spaces

Definition 3.1. An ideal nano topological space $(U, \tau_R(X), \mathcal{I})$ is said to be an $n\mathcal{I}_{g^{\#}}$-regular space if for each pair consisting of a point x and a nano closed set F not containing x, there exist disjoint $n\mathcal{I}_{g^{\#}}$-open sets P and Q such that $x \in P$ and $F \subseteq Q$.

Remark 3.1. *Every nano regular space is $n\mathcal{I}_{g^{\#}}$-regular.*

The following Example 3.1 shows that an $n\mathcal{I}_{g\#}$-regular space need not be nano regular. Theorem 3.1 gives a characterization of $n\mathcal{I}_{g\#}$-regular spaces.

Example 3.1. *Consider the ideal topological space* $(U, \tau_R(X), \mathcal{I})$ *of Example 2.1. Then* $\phi^* = \phi$, $(\{a_4\})^* = \{\phi\}$, $(\{a_1, a_3\})^* = \{a_1, a_2, a_3\}$, $(\{a_1, a_3, a_4\})^* = \{a_1, a_2, a_3\}$ *and* $U^* = \{a_1, a_2, a_3\}$. *Since every* $n\alpha g$*-open set is* $n\star$*-closed, every subset of* U *is* $n\mathcal{I}_{g\#}$*-closed and so every subset of* U *is* $n\mathcal{I}_{g\#}$*-open. This implies that* $(U, \tau_R(X), \mathcal{I})$ *is* $n\mathcal{I}_{g\#}$*-regular. Now,* $\{a_2\}$ *is a nano closed set not containing* $a_4 \in U$, $\{a_2\}$ *and a are not separated by disjoint nano open sets. So* $(U, \tau_R(X), \mathcal{I})$ *is not nano regular.*

Theorem 3.1. *In an ideal nano topological space* $(U, \tau_R(X), \mathcal{I})$, *the following are equivalent.*

1. U *is* $n\mathcal{I}_{g\#}$*-regular.*
2. *For every nano open set* Q *containing* $x \in U$, *there exists an* $n\mathcal{I}_{g\#}$*-open set* P *of* U *such that* $x \in P \subseteq ncl^\star(P) \subseteq Q$.

Proof. (1) $\Rightarrow$ (2). Let Q be an nano open subset such that $x \in Q$. Then $U - Q$ is a nano closed set not containing x. Therefore, there exist disjoint $n\mathcal{I}_{g\#}$-open sets P and S such that $x \in P$ and $U - Q \subseteq S$. Now, $U - Q \subseteq S$ implies that $U - Q \subseteq nint^\star(S)$ and so $U - nint^\star(S) \subseteq Q$. Again, $P \cap S = \phi$ implies that $P \cap nint^\star(S) = \phi$ and so $ncl^\star(P) \subseteq U - nint^\star(S)$. Therefore, $x \in P \subseteq ncl^\star(P) \subseteq Q$. This proves (2).

(2) $\Rightarrow$ (1). Let F be a nano closed set not containing x. By hypothesis, there exists an $n\mathcal{I}_{g\#}$-open set P such that $x \in P \subseteq ncl^\star(P) \subseteq U - F$. If $S = U - ncl^\star(P)$, then P and S are disjoint $n\mathcal{I}_{g\#}$-open sets such that $x \in P$ and $F \subseteq S$. This proves (1). □

Theorem 3.2. *If* $(U, \tau_R(X), \mathcal{I})$ *is an* $n\mathcal{I}_{g\#}$*-regular,* $nT_{\mathcal{I}}$*-space where* $\mathcal{I}$ *is completely nano codense, then* U *is nano regular.*

Proof. Let F be a nano closed set not containing $x \in U$. By Theorem 3.1, there exists an $n\mathcal{I}_{g\#}$-open set P of U such that $x \in P \subseteq ncl^\star(P) \subseteq U - F$. Since U is a $nT_{\mathcal{I}}$-space, $\{x\}$ is $n\alpha g$-closed and so $\{x\} \subseteq nint^\star(P)$, by Lemma 1.2. Since $\mathcal{I}$ is completely nano codense, $n\tau^\star \subseteq n\tau^\alpha$ and so $nint^\star(P)$ and $U - ncl^\star(P)$ are $n\alpha$-open sets. Now, $x \in nint^\star(P) \subseteq nint(ncl(nint(nint^\star(P)))) = G$ and $F \subseteq U - ncl^\star(P) \subseteq nint(ncl(nint(U - ncl^\star(P)))) = H$. Then G and H are disjoint nano open sets containing x and F respectively. Therefore, U is nano regular.

If $\mathcal{I}=\mathcal{N}$ in Theorem 3.1, then we have the following Corollary 3.1 which gives characterizations of nano regular spaces, the proof of which follows from Theorem 3.2. □

Corollary 3.1. *If* $(U, \tau_R(X))$ *is a* $nT_{\mathcal{I}}$*-space, then the following are equivalent.*

1. U *is nano regular.*
2. *For every nano open set* Q *containing* $x \in U$, *there exists an* $n\alpha g^\#$*-open set* P *of* U *such that* $x \in P \subseteq n\alpha$*-*$ncl(P) \subseteq Q$.

If $\mathcal{I}=\{\phi\}$ in Theorem 3.1, then we have the following Corollary 3.2 which gives characterizations of nano regular spaces, the proof of which follows from Theorem 3.2.

Corollary 3.2. *If* $(U, \tau_R(X))$ *is a* $nT_{\mathcal{I}}$*-space, then the following are equivalent.*

1. *U is nano regular.*
2. *For every nano open set Q containing $x \in U$, there exists a $ng^{\#}$-open set P of U such that $x \in P \subseteq ncl(P) \subseteq Q$.*

Theorem 3.3. *If every $n\alpha g$-open subset of an ideal nano topological space $(U, \tau_R(X), \mathcal{I})$ is $n\star$-closed, then $(U, \tau_R(X), \mathcal{I})$ is $n\mathcal{I}_{g^{\#}}$-regular.*

Proof. Suppose every $n\alpha g$-open subset of U is $n\star$-closed. Then by Lemma 1.3, every subset of U is $n\mathcal{I}_{g^{\#}}$-closed and hence every subset of U is $n\mathcal{I}_{g^{\#}}$-open. If F is a nano closed set not containing x, then $\{x\}$ and F are the required disjoint $n\mathcal{I}_{g^{\#}}$-open sets containing x and F respectively. Therefore, $(U, \tau_R(X), \mathcal{I})$ is $n\mathcal{I}_{g^{\#}}$-regular. □

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