Asia Mathematatika
Volume: 7 Issue: 3 , (2023) Pages: 12 – 14
Available online at **www.asiamath.org**

Application of α-open sets in generalizations of some new notions

Saeid Jafari[1*]
[1]Department of Mathematics, College of Vestsjaelland South,
Slagelse, Denmark. Orchid iD: 0000-0001-5744-7354

Received: 15 Sep 2023 • **Accepted:** 23 Dec 2023 • **Published Online:** 15 Jan 2024

Abstract: We introduce generalizations of the notions of sheaves, Etale spaces and torsors by applying the notion of α-open sets.

Key words: α-open set, α-sheave, α-Etale space and α-torsor

1. α-Sheaves

For the basic facts on categories and functors, we refer the reader to [6].

Definition 1.1. A subset A of a topological space (X, τ) is called:
1) semi-open [2] if $A \subset Cl(Int(A))$.
2) α-open [5] if $A \subset Int(Cl(Int(A)))$.
3) preopen [3]if $A \subset Int(Cl(A))$.

The family of all α-open (resp. semi-open, pre-open) subsets of a topological space (X, τ) is denoted by $\alpha O(X)$ (resp. $SO(X), PO(X)$.

Any open set is α-open but the converse is not true (see [5]). It is well-known that any α-open set is semiopen and preopen.The family of all α-open subsets of a topological space (X, τ) is denoted by $\alpha O(X)$ and Njåsted showed that it is a topology on X.

It is obvious that the poset of α-open sets of a topological space X can be considered as a category by taking as arrows, $\xi : D \to E$ (inclusion) such that $D \subseteq E$.

Definition 1.2. A α-presheaf on a space X is a covariant functor $\mathcal{F} : \alpha O(X)^{op} \to Sets$.

Observe that every presheaf is α-presheaf but the converse is not true since there is α-open sets which are not open. It is well-known that a presheaf on a space X is a covariant functor $\mathcal{F} : O(X)^{op} \to Sets$, where O(X) is the family of all open subsets of a topological space (X, τ).

For each α-open set D of X, we have a set $\mathcal{F}(D)$. In addition, we have the operation $\mathcal{F}(\xi) : \mathcal{F}(E) \to \mathcal{F}(D)$ if $\xi : D \to E$ is an inclusion. It is obvious that morphism between α-presheaves defines the category of α-presheaves on X. Mind that in this paper we write $e \mid D$ instead of $\mathcal{F}(\xi)(e)$. To find the compatible family in the α-presheaf $\mathcal{F}$, we take $\cup D_i$ to be an α-open cover of D. Now the compatible family refers to those elements e_i in $\mathcal{F}(D_i)$ such that $e_i \mid D_i j = e_j \mid D_i j$, where $D_i j = E_i \cap D_j$.

*Correspondence: jafaripersia@gmail.com

An α-presheaf is called an α-sheaf if for any α-open cover of $D = \cup D_i$ and for any compatible family $\{e_j\}$, there exists an element e in $\mathcal{F}(D)$ such that $e \mid D_i = e_i$, $\forall\ i$. If there is only one such e, the α-presheaf is called separated α-presheaf.

Recall that a function $f : X \to Y$ is called α-continuous [4] if for each $x \in X$ and each open set V containing $f(x)$, there exists an α-open set U in X containing x such that $f(U) \subset V$.

Example 1.1. *Take the maps $f : Y \to X$ and $g : D \to Y$, where g is α-continuous and $f \circ g = I$, I is identity. Then the α-sheaf of sections of f is $\mathcal{S}_\alpha(f) = \{g : D \to Y \mid f \circ g = I\}$.*

Observe that α-sheaves $\hookrightarrow$ separated α-presheaves $\hookrightarrow$ α-presheaves have left adjoints, where $\hookrightarrow$ denotes the inclusion functor.

2. α-Etale spaces

We say that a topological space X is locally α-homeomorphic to Y if every point of X has a neighborhood that is α-homeomorphic to an α-open subset of Y. Recall that two topological spaces are called α-homeomorphic if there exists a bijective function $f : X \to Y$ such that f and its inverse are α-continuous.

Here we introduce the notion of α-Etale space by involving α-open sets. It is obvious that an Etale space is always α-Etale but the converse is not true.

Definition 2.1. A space $\mathcal{E}$ provided with an α-continuous and local α-homeomorphism $\psi : \mathcal{E} \to X$ is called an α-Etale space over X.

Take α-Etale spaces $\phi_1 : \mathcal{E}_1 \to X$ and $\phi_2 : \mathcal{E}_2 \to X$ and consider α-continuous map $\mu : \mathcal{E}_1 \to \mathcal{E}_2$ for which $\phi_2\mu = \phi_1$. This suggests a definition of a category of α-Etale spaces over X. Observe that $\mathcal{S}_\alpha$: α-Etale spaces $\to$ α-sheaves takes $\phi_1 : \mathcal{E}_1 \to X$ and send it to the α-sheaf of sections $\mathcal{S}_\alpha(\phi_1)$. It is easy to show that this functor is an equivalence of categories.
Indeed if $f : X \to Y$ is a local α-homeomorphism, then the topological space X is said to be an Etale space over Y.

3. α-Torsors

Suppose $\mathcal{G}$ is an sheaf of groups on X and $\mathcal{S}_\alpha$ an α-sheaf on X. Observe the following map of α-sheaves $\eta : \mathcal{G} \times \mathcal{S}_\alpha \to \mathcal{S}_\alpha$

We say $\mathcal{S}_\alpha$ is a $\mathcal{G}$-α-torsor if

(a) $X = \cup U$ such that $\mathcal{S}_\alpha$(U) is not empty.
(b) For each α-open set $U \subseteq X$, the action of $\mathcal{G}(U)$ on $\mathcal{S}_\alpha$(U) is free and transitive.

It is true that torsor implies α-torsor but not conversely. Also it is obvious that a morphism between α-torsors is a morphsim of sheaves which is also an isomorphism [1], commutes with the action.
The first Čech cohomology classes of an α-sheaf of groups can be given a geometric interpretation in terms of $\mathcal{G}$-α-torsor similar to the usual one in the theory of torsors. If we tak $\mathcal{U} = U_\alpha$ to be an α-open cover of X, then we have:

$H^1(X, \mathcal{G}) = \lim_{\to \mathcal{U}} H^1(\mathcal{U}, \mathcal{G})$.

By the same manner one can define *semi-sheaves, semi-Etale space and semi-torsor, pre-sheaves, pre-Etale space and pre-torsor* by involving semi-open and preopen sets. Since every α-open set is semi-open and preopen then we have new and weaker generalizations. In fact we can use other types of generalized open sets to define new notions by the same token.

Remark 3.1. *It should be noted that a $\mathcal{G}$-α-torsor is an α-Etale space $\psi : \mathcal{E} \to X$ by considering the action $\eta : \mathcal{G} \times_X \mathcal{E} \to \mathcal{E}$ such that ψ is surjective and $\mathcal{G} \times_X \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ is α-homeomorphism. One can also define α-prestack and α-stack by utilizing α-fibered category and the notions of α-presheaf and α-sheaf. It is well-known that stacks are important not only as a research area itself but also with respect to the theory of gerbs which are important and have applications in Geometry, Field theory and quantisation.*

Questions

How the new notions defined in this paper can be used when one uses ideal topological spaces and the ideal poset of I-open sets?

Can the introduction of ideals help in constructing more proper examples which otherwise could have been difficult to tackle as it was the case with topological spaces?

References

[1] Gallego, G. Torsors and cocycles, (see: https://blog.guillegallego.xyz/Torsors/torsors.html)

[2] Levine, N. Semi-open sets and semi-continuity in topological spaces, Amer. Math. Montrhly, 1963; 70: 36-41.

[3] Mashhour, A. S, Hasanein, I. A., El-Deeb, S. N. A note on semi-continuity and precontinuity, Indian J. Pure Appl. Math., 1982; 13(10): 1119-1123.

[4] Mashhour, A. S., Hasanein, I. A., El-Deeb, S. N. α-continuous and α-open mappings, Acta Math. Hung., 1983; 41: 213-218.

[5] Njåsted, O. On some classes of nearly open sets, Pacific J. Math. 1965; 961-070.

[6] Riehl, E. Category theory in context, Dover Publications, INC, Mineola, New York, 2016.