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An introduction to picture topological spaces

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Abstract: In this paper we introduce a new class of sets known as picture sets. They can be considered as a crisp version of picture fuzzy sets. We define basic algebraic operations on them. Then we explore topological aspects of this novel framework. Thus, we introduce *picture topological spaces* and related notions (e.g. picture *closure*, picture *interior* and picture *derived set*). We study their properties and relationships between them. Additionally, the concepts of picture *product* and *subspace* topologies are presented along with the demonstration of appropriate theorems. We emphasize the fact that there are some important differences between picture and classical sets. The same can be said about picture and classical points.

Key words: picture set, picture fuzzy set, picture topology

1. Introduction

Traditional logic determines logical values of statements as either true or false. This does not allow for modelling of uncertainty that occurs in many real-life situations. To overcome this, Zadeh (see [18]) created *fuzzy sets theory* where the participation of elements within a set is indicated by a membership grade ranging from 0 to 1. Since its formulation, this theory has undergone numerous developments by different researchers. Among its extensions and modifications we have: *intuitionistic fuzzy sets* of Atanassov (see [1]), *soft* sets of Molodtsov [9], *fuzzy soft sets* [2], *rough* sets of Pawlak [11], *vague* sets (see [16] for their soft version [1]) or even *intuitionistic multi fuzzy sets* (see [10]). Moreover, *neutrosophic* sets introduced by Smarandache (see [14]) can be considered as a very general framework for uncertainty analysis and estimation.

Clearly, the list presented above is not complete and all these concepts have many variations (depending both on their practical applications and theoretical considerations). In particular, in 2014 Cuong [5] has introduced *picture fuzzy* sets (PFS): a modified version of intuitionistic fuzzy sets. They utilize neutral and refusal membership grades together with standard membership and non-membership grades. The idea is to address uncertainty in a more effective manner. PFS has proven to be particularly useful in all those scenarios where individuals hold multiple perspectives such as ”yes”, ”abstain”, ”no”, and ”refusal”.

A notable example of this is are general elections, where a voter may vote for a candidate (”yes”), against a candidate (”no”), abstain from voting or refuse to vote for the available options and choose ”not” (refusal).

The utilization of PFS has increased among researchers for resolving decision-making problems. The authors applied methods of similarity measurement to identify building materials and mineral recognition.

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[1]It seems that vague sets are identical with intuitionistic fuzzy sets. Both terms are frequently used in literature.

There are also purely algebraic applications (see e.g. [15] for interval valued picture fuzzy ideals).

As we can see, fuzzy sets and their extended forms have proven to be effective tools in the field of managing uncertainty and vagueness. However, these non-classical families of sets are important from theoretical point of view too. For example, it is a well-known fact that we can establish fuzzy (and thus also vague, neutrosophic etc.) topological spaces. Generalizations of such topologies (e.g. supra and infra topologies, minimal structures) are possible too.

This paper presents the concept of picture sets, which is an advancement of picture fuzzy sets. We introduce the idea of picture topological space and we examine some of its properties. We show many topological and algebraic results. What is important, is that we analyze the notion of picture points in one of its possible understandings. We prove that the algebra of picture sets is different from the algebra of classical sets.

2. Preliminaries

Let us start from some basic notions. We recall the general idea of fuzzy and intuitionistic fuzzy sets. Then they will be compared with picture fuzzy and picture sets. Moreover, we mention intuitionistic sets too.

First, let us recall the intuition that has been already mentioned. A fuzzy set (FS) is a set in which every element has a degree of membership of belonging in it. This can be formulated in a strict mathematical manner, as it is in the definition below.

Definition 2.1. [7] Let X be a non-empty universal set. A *fuzzy set* M on X is an object of the form:

$M = \{(x, \mu_M(x)); x \in X\}$

where $\mu_M(x) : X \to [0,1]$ is a function that assigns a real number from the interval $[0,1]$ to each element of X. The value $\mu_M(x)$ shows the grade of membership of x in M.

Definition 2.2. [7] An *intuitionistic fuzzy set* (IFS) M on a non empty set X is an object of the form

$$M = \{(x, \mu_M(x), \nu_M(x)); x \in X)\}$$

where $\mu_M(x) \in [0,1]$ is called the degree of membership of x in M, $\nu_M(x) \in [0,1]$ is called the degree of non-membership of x in M and we assume that $\mu_M(x)$ and $\nu_M(x)$ satisfy the following condition [2]:

$$(\forall x \in X)\ (\mu_M(x) + \nu_M(x) \leq 1)$$

Intuitionistic fuzzy sets are constantly studied e.g. in the context of topologies. For example, various subclasses of weakly open sets are analyzed (see [6] for one of the most recent results).

Intuitionistic sets are their crisp version.

Definition 2.3. [7], [3] Let X be a non-empty universal set. We say that $M \subseteq X$ is an *intuitionistic set* (IS) if M is of the form $M = \langle X, M_1, M_2 \rangle$, where M_1 and M_2 are the subsets of X such that $M_1 \cap M_2 = \emptyset$. The sets M_1 and M_2 are interpreted as members and non-members of M respectively.

Definition 2.4. [7], [3] Let X be a non-empty set, $M = \langle X, M_1, M_2 \rangle$ and $N = \langle X, N_1, N_2 \rangle$ be two IS's on X. Let J be a non-empty index set and $\{M_i : i \in J\}$ be an arbitrary family of IS's on X, where each of its elements is of the form [3]:

[2] This condition can be slightly changed if we want to expand the set of possible pairs of values. The reader is encouraged to check Pythagorean and Fermatean fuzzy sets in this context.

[3] There is a little abuse of notation here: we use subscripts to enumerate the elements of the family, while the components of a particular intuitionistic set are enumerated in superscripts.

$M_i = \langle X, M_i^{(1)}, M_i^{(2)} \rangle$ for any $i \in J$.

Then we define:

1. $M \subseteq N$ if and only if $M_1 \subseteq N_1$ and $M_2 \supseteq N_2$ (*inclusion*).
2. $M = N$ if and only if $M \subseteq N$ and $N \subseteq M$ (*equality*).
3. $M^c = \langle X, M_2, M_1 \rangle$ (*complement*).
4. $M \cup N = \langle X, M_1 \cup N_1, M_2 \cap N_2 \rangle$ (*binary union*).
5. $M \cup N = \langle X, M_1 \cap N_1, M_2 \cup N_2 \rangle$ (*binary intersection*).
6. $\bigcup M_i = \langle X, \bigcup M_i^{(1)}, \bigcap M_i^{(2)} \rangle$ (*arbitrary union*).
7. $\bigcap M_i = \langle X, \bigcap M_i^{(1)}, \bigcup M_i^{(2)} \rangle$ (*arbitrary intersection*).
8. $M - N = M \cap N^c$ (*difference of sets*).
9. $\emptyset = \langle X, \emptyset, X \rangle$.
10. $X = \langle X, X, \emptyset \rangle$.
11. $[\,]M = \langle X, M_1, M_1^c \rangle$.
12. $\langle\,\rangle M = \langle X, M_2^c, M_2 \rangle$.

One can easily prove that union, intersection and complement are defined in a coherent way: that is, they produce new intuitionistic sets. Moreover, the following lemma holds.

Lemma 2.1. *[3], [4] Assume that X is a non-empty universe, while M, N and M_i (for arbitrary $i \in J$) are members of some family of intuitionistic sets on X. Then the following identities and relationships hold:*

1. $(\bigcup M_i)^c = \bigcap M_i^c$.
2. $(\bigcap M_i)^c = \bigcup M_i^c$ *(de Morgan laws).*
3. $M \subseteq N \Rightarrow N^c \subseteq M^c$.
4. $(M^c)^c = M$.

Now let us recall the very definition of picture fuzzy set.

Definition 2.5. [5] Let X be a non-empty universe. A *picture fuzzy set* (PFS) M on a non-empty universe X is an object of the form:

$$M = \{(x, \mu_M(x), \eta_M(x), \nu_M(x)); x \in X\}$$

where $\mu_M(x) \in [0,1]$ is called the grade of positive membership of x in M, $\eta_M(x) \in [0,1]$ is called the degree of neutral membership of x in M, $\nu_M(x) \in [0,1]$ is called the degree of negative membership of $x \in M$. We assume that $\mu_M(x) + \eta_M(x) + \nu_M(x) \leq 1$ for any $x \in X$. Then $1 - (\mu_M(x) + \eta_M(x) + \nu_M(x))$ is called the degree of refusal membership of x in M.

Our picture sets (that will be presented in the next section) can be considered as a crisp version of picture fuzzy sets and (at the same time) as a generalization of intuitionistic sets. They are similar to neutrosophic crisp sets invented by Salama et al. (see [13]) but they are not identical with them.

3. Picture sets and picture topological spaces

Let us start from some basic definitions.

3.1. Initial notions

Definition 3.1. Assume that X is a non-empty universe. A *picture set* on X is an object of the form $M = \langle X, M_1, M_2, M_3 \rangle$ where M_1, M_2 and M_3 are the subsets of X such that $M_1 \cap M_3 = \emptyset$. The sets M_1, M_2, M_3 and $(M_1 \cup M_2 \cup M_3)^c$ are the sets of members, neutral members, non-members and refusal members of M respectively.

Note that in neutrosophic crisp sets we have stronger assumptions: that $M_1 \cap M_2 = \emptyset$, $M_1 \cap M_3 = \emptyset$ and $M_2 \cap M_3 = \emptyset$ (see [12], [13]). Hence, each neutrosophic crisp set can be considered as a picture set but the converse of this statement is not true.

Let us define some basic operations on picture sets.

Definition 3.2. Let X be a non empty universe. Assume that M, N are picture sets on X and $\{M_i : i \in J\}$ be an arbitrary family of picture sets on X, where each of its elements is of the form

$$M_i = \langle X, M_i^{(1)}, M_i^{(2)}, M_i^{(3)} \rangle \text{ for any } i \in J.$$

Then we define:

1. $M \subseteq N$ if and only if $M_1 \subseteq N_1$, $M_2 \supseteq N_2$ and $M_3 \supseteq N_3$ (*inclusion*).
2. $M = N$ if and only if $M \subseteq N$ and $N \subseteq M$ (*equality*).
3. $M^c = \langle X, M_3, M_2^c, M_1 \rangle$ (*complement*).
4. $M \cup N = \langle X, M_1 \cup N_1, M_2 \cap N_2, M_3 \cap N_3 \rangle$ (*binary union*).
5. $M \cap N = \langle X, M_1 \cap N_1, M_2 \cup N_2, M_3 \cup N_3 \rangle$ (*binary intersection*).
6. $\bigcup M_i = \langle X, \bigcup M_i^{(1)}, \bigcap M_i^{(2)}, \bigcap M_i^{(3)} \rangle$ (*arbitrary union*) [4].
7. $\bigcap M_i = \langle X, \bigcap M_i^{(1)}, \bigcup M_i^{(2)}, \bigcup M_i^{(3)} \rangle$ (*arbitrary intersection*).
8. $M \setminus N = M \cap N^c$ (*difference*).
9. $X_\delta = \langle X, X, \emptyset, \emptyset \rangle$ (*universal picture set*).
10. $\emptyset_\delta = \langle X, \emptyset, X, X \rangle$ (*empty picture set*).

In the next subsection we shall prove and analyze some basic properties of the operations mentioned above.

3.2. Algebraic aspects of picture sets

First, we should be sure that our unary and binary operations produce new picture sets.

Lemma 3.1. *Let M, N be two picture sets on X. Then $M \cup N$, $M \cap N$ and M^c are picture sets too.*

[4] We omit some subscripts to avoid unnecessary complication. However, it is clear that we mean unions and intersections indexed with $i \in J$. Moreover, just like in the earlier case of intuitionistic sets, we use superscripts to enumerate components for a while.

Proof. In each case the only condition to check is that members and non-members have empty intersection. Take $M\cup N = \langle X, M_1\cup N_1, M_2\cap N_2, M_3\cap N_3\rangle$. Now $(M_1\cup N_1)\cap(M_3\cap N_3) = (M_1\cap M_3\cap N_3)\cup(N_1\cap M_3\cap N_3) = (\emptyset\cap N_3)\cup(M_3\cap\emptyset) = \emptyset\cup\emptyset = \emptyset$.

Similar proof can be performed for intersection. Both these proofs can be adjusted to the concept of arbitrary unions and intersections.

As for the complement, the proof is obvious because of the commutativity of intersection. □

Remark 3.1. *One can point out that other algebraic operations on picture sets are possible too. For example, we can define:*

1. Strong intersection: $M \frown N = \langle M_1\cap N_1, M_2\cup N_2, M_3\cap N_3\rangle$
2. Total intersection: $M\wedge N = \langle M_1\cap N_1, M_2\cap N_2, M_3\cap N_3\rangle$
3. Strong union: $M \smile N = \langle M_1\cup N_1, M_2\cup N_2, M_3\cap N_3\rangle$.

Note that we cannot define something like "total union" $\vee$ as $M\vee N = \langle M_1\cup N_1, M_2\cup N_2, M_3\cup N_3\rangle$. Take e.g. $X=\{a,b,c,d,e,f\}$, $M=\langle X,\{a,b\},\{b\},\{d\}\rangle$, $N=\langle X,\{c,d\},\{c\},\{e\}\rangle$. Now $M\vee N = \langle X,\{a,b,c,d\},\{b,d\},\{d,e\}\rangle$. And now $(M_1\cup N_1)_1\cap(M_3\cup N_3) = \{a,b,c,d\}\cap\{d,e\} = \{d\}\neq\emptyset$. Hence the resulting set is not a properly defined picture set. Clearly, the same can be said about the following function: $M \veebar N = \langle M_1\cup N_1, M_2\cap N_2, M_3\cup N_3\rangle$.

Note that non-standard forms of inclusion are possible too. However, perhaps they would build weaker algebras (at least when considered with $\cap$ and $\cup$).

We think that the properties of $\frown$, $\wedge$ and $\smile$ should be studied in further research. However, we consider $\cup$ and $\cap$ as crucial. Thus, let us prove the following lemma.

Lemma 3.2. *Assume that M, N, L and M_i (for any $i\in J$) are picture sets on a universe X. Then the following properties are true:*

1. $(\bigcup M_i)^c = \bigcap M_i^c$.
2. $(\bigcap M_i)^c = \bigcup M_i^c$.
3. $M\subseteq N \Rightarrow N^c\subseteq M^c$.
4. $(M^c)^c = M$.
5. $M\cap(N\cup L) = (M\cap N)\cup(M\cap L)$.
6. $M\cup(N\cap L) = (M\cup N)\cap(M\cup L)$ (distributivity).
7. $M\cup(N\cup L) = (M\cup N)\cup L$.
8. $M\cap(N\cap L) = (M\cap N)\cap L$ (associativity).
9. $M\cap(M\cup N) = M$.
10. $M\cup(M\cap N) = M$ (absorption laws).
11. $M\subseteq X_\delta$ *and* $\emptyset_\delta\subseteq M$.
12. $M\setminus(N\cap L) = (M\setminus N)\cup(M\setminus L)$.

13. $M \cap \emptyset_\delta = \emptyset_\delta$ *and* $M \cup \emptyset_\delta = M$.

14. $M \cap X_\delta = M$ *and* $M \cup X_\delta = X_\delta$.

Proof. Basically, these proofs are not complicated. However, from the formal point of view, we have some new algebra of sets here (namely, the algebra of picture sets), hence it would be good to prove at least some fragments.

1. We have $(\bigcup M_i)^c = \langle X, \bigcup M_i^{(1)}, \bigcap M_i^{(2)}, \bigcap M_i^{(3)}\rangle^c = \langle X, \bigcap M_i^{(3)}, (\bigcap M_i^{(2)})^c, \bigcup M_i^{(1)}\rangle = \bigcap M_i^c$.
2. Similar to the previous one.
3. Assume that $M \subseteq N$, that is $M_1 \subseteq N_1$, $N_2 \subseteq M_2$ and $N_3 \subseteq M_3$. Now recall the fact (from the basic set theory) that if $N_2 \subseteq M_2$, then $(M_2)^c \subseteq (N_2)^c$. But $N^c \subseteq M^c$ means exactly that $N_3 \subseteq M_3$, $(M_2)^c \subseteq (N_2)^c$ and $M_1 \subseteq N_1$, so we obtain the same conclusion.
4. Obvious.
5. We have $M \cap (N \cup L) = M \cap \langle X, N_1 \cup L_1, N_2 \cap L_2, N_3 \cap L_3\rangle = \langle X, M_1 \cap (N_1 \cup L_1), M_2 \cup (N_2 \cap L_2), M_3 \cup (N_3 \cap L_3)\rangle = \langle X, (M_1 \cap N_1) \cup (M_1 \cap L_1), (M_2 \cup N_2) \cap (M_2 \cup L_2), (M_3 \cup N_3) \cap (M_3 \cup L_3)\rangle = (M \cap N) \cup (M \cap L)$.
6. We have $M \cup (N \cup L) = M \cup \langle X, N_1 \cup L_1, N_2 \cap L_2, N_3 \cap L_3\rangle = \langle X, (M_1 \cup N_1) \cup L_1, (M_2 \cap N_2) \cap L_2, (M_3 \cap N_3) \cap L_3\rangle = (M \cup N) \cup L$. Clearly, we used the fact that associativity laws are true for classical sets.
7. Analogous.
8. We have $M \cap (M \cup N) = M \cap \langle X, M_1 \cup N_1, M_2 \cap N_2, M_3 \cap N_3\rangle = \langle M_1 \cap (M_1 \cup N_1), M_2 \cup (M_2 \cap N_2), M_3 \cup (M_3 \cap N_3)\rangle = \langle X, M_1, M_2, M_3\rangle = M$. Clearly, we used the fact that absorption laws are true for classical sets.
9. Analogous.
10. As for the first part: clearly, $M_1 \subseteq X$ and $\emptyset \subseteq M_2$, $\emptyset \subseteq M_3$ in any case. The second part is similar.
11. $M \setminus (N \cap L) = M \cap \langle X, N_1 \cap L_1, N_2 \cup L_2, N_3 \cup L_3\rangle^c = M \cap \langle X, N_3 \cup L_3, (N_2 \cup L_2)^c, N_1 \cap L_1\rangle = \langle X, M_1 \cap (N_3 \cup L_3), M_2 \cup (N_2 \cup L_2)^c, M_3 \cup (N_1 \cap L_1)\rangle = \langle X, (M_1 \cap N_3) \cup (M_1 \cap L_3), (M_2 \cup N_2^c) \cap (M_2 \cup L_2^c), (M_3 \cup N_1) \cap (M_3 \cup L_1)\rangle$. But at the same time $(M \setminus N) \cup (M \setminus L) = \langle X, M_1 \cap N_3, M_2 \cup N_2^c, M_3 \cup N_1\rangle \cup \langle X, M_1 \cap L_3, M_2 \cup L_2^c, M_3 \cup L_1\rangle = \langle X, (M_1 \cap N_3) \cup (M_1 \cap L_3), (M_2 \cup N_2^c) \cap (M_2 \cup L_2^c), (M_3 \cup N_1) \cap (M_3 \cup L_1)\rangle = M \setminus (N \cap L)$.
12. $M \cap \emptyset_\delta = \langle X, M_1 \cap \emptyset, M_2 \cup X, M_3 \cup X\rangle = \langle X, \emptyset, X, X\rangle = \emptyset_\delta$. The second part is similar.
13. Similar to the former.

□

Clearly, one can ask what about the law of the excluded middle. This has been explained in the remark below.

Remark 3.2. *Note that the union of picture set* M *and its complement* M^c *need not to be identical with the whole universal picture set* $\langle X, X, \emptyset, \emptyset\rangle$. *Take e.g.* $X = \{a, b, c, d, e\}$, $M = \langle X, \{a, b\}, \{c\}, \{d, e\}\}$. *Then* $M^c = \{X, \{d, e\}, \{a, b, d, e\}, \{a, b\}$ *and* $M \cup M^c = \langle X, \{\{a, b, d, e\}, \emptyset, \emptyset\}$. *Hence, the law of the excluded middle does not hold.*

What about the law of consistency (that is, contradiction)?

Remark 3.3. *For any picture set M on X we have:* $M \cap M^c = \langle X, M_1, M_2, M_3 \rangle \cap \langle X, M_3, M_2^c, M_1 \rangle = \langle X, M_1 \cap M_3, M_2 \cup M_2^c, M_3 \cup M_1 \rangle = \langle X, \emptyset, X, M_3 \cup M_1 \rangle$. *Note that the last set need not to be equal with empty picture set. Consider the following counterexample:* $X = \{a, b, c, d, e\}$, $M = \langle X, \{a, b\}, \{c, d\}, \{d, e\} \rangle$. *Thus,* $M^c = \langle X, \{d, e\}, \{a, b, e\}, \{a, b\}$. *Hence,* $M \cap M^c = \langle X, \emptyset, \{a, b, c, d, e\}, \{a, b, d, e\} = \langle X, \emptyset, X, \{a, b, d, e\} \rangle \neq \emptyset_\delta$ *(because $\{a, b, d, e\} \neq X$). Hence, the logic determined by picture sets is (at least in some sense) paraconsistent.*

Lemma 3.3. *Assume that X is a non-empty universe and A, B are picture subsets of X. Suppose that $A \cap B = \emptyset_\delta$. Then $B \subseteq A^c$ (and $A \subseteq B^c$).*

Proof. Let us recall that $A = \langle X, A_1, A_2, A_3 \rangle$, $A^c = \langle X, A_3, A_2^c, A_1 \rangle$, $B = \langle X, B_1, B_2, B_3 \rangle$, $A_1 \cap A_3 = B_1 \cap B_3 = \emptyset$. Moreover, by the very assumption about empty picture intersection of A and B, we have $A_1 \cap B_1 = \emptyset$, $A_2 \cup B_2 = X$ and $A_3 \cup B_3 = X$.

The fact that $B \subseteq A^c$ means that the following three conditions are simultaneously satisfied: $B_1 \subseteq A_3$, $A_2^c \subseteq B_2$ and $A_1 \subseteq B_3$. Suppose that this conjunction is not true. Then:

a) Assume that $B_1 \nsubseteq A_3$. It means that there is some $x \in B_1$ such that $x \notin A_3$. But then $x \notin B_3$ (because $B_1 \cap B_3 = \emptyset$). Hence $x \notin (A_3 \cup B_3) = X$. Contradiction.

b) Assume that $A_2^c \nsubseteq B_2$. Hence, there is $x \in A_2^c$ (which means that $x \notin A_2$) such that $x \notin B_2$. Thus $x \notin (A_2 \cup B_2) = X$. Contradiction.

c) Assume that $A_1 \nsubseteq B_3$. Hence there is $x \in A_1$ (which means tat $x \notin A_3$) such that $x \notin B_3$. But then $x \notin (A_3 \cup B_3) = X$. Contradiction. □

One could say that our definitions of complement operation or empty and universal set are not the only possible ones. In some sense, this is true (in fact, we have already pointed out that alternative versions of union and intersection are possible too). For example, one could think about *total empty set*, that is $\langle X, \emptyset, \emptyset, \emptyset \rangle$. Note, however, that in this case $\emptyset^c = \langle X, \emptyset, X, \emptyset \rangle$. Hence, we cannot say that the whole universe (in its picture interpretation) is a complement of empty set. Moreover, it is always true that $\langle X, \emptyset, X, X \rangle$ is contained in every picture set, while $\langle X, \emptyset, \emptyset, \emptyset \rangle$ is contained only in the sets of the form $\langle X, A, \emptyset, \emptyset \rangle$.

Moreover, we can prove that our inclusion is transitive:

Lemma 3.4. *Let M, N, T be three picture sets on X. If $M \subseteq N$ and $N \subseteq T$, then $M \subseteq T$.*

Proof. If $M \subseteq N$, then $M_1 \subseteq N_1$, $N_2 \subseteq M_2$ and $N_3 \subseteq M_3$. If $N \subseteq T$, then $N_1 \subseteq T_1$, $T_2 \subseteq N_2$ and $T_3 \subseteq N_3$. But then, by transitivity of inclusion of classical sets, $M_1 \subseteq T_1$, $T_2 \subseteq M_2$ and $T_3 \subseteq M_3$. □

Finally, there is a lemma that will be useful later in the context of subspaces.

Lemma 3.5. *Let A and B be two picture sets on X and $A \subseteq B$. Then $A \cap B = A$.*

Proof. $A \cap B = \langle X, A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3 \rangle$. However, $A \subseteq B$, so $A_1 \subseteq B_1$ and thus $A_1 \cap B_1 = A_1$, $B_2 \subseteq A_2$ and thus $A_2 \cup B_2 = A_2$ and $B_3 \subseteq A_3$, so $A_3 \cup B_3 = A_3$. □

The next definition refers to the concept of point.

Definition 3.3. Let X be a non-empty universe. An element $m_\delta \in X$ is called *picture point* if $m_\delta = \langle X, \{m\}, \{m\}^c, \{m\}^c \rangle$, where $m \in X$. If M is a picture set on X, then we say that $m_\delta \in M$ if and only if $m \in M_1$. The set of all picture points that belong to some picture set M will be denoted as $Pt(M)$.

For example, if $M = \langle X, \{a, c, d\}, \{a, b\}, \{b, e\}\rangle$ then $Pt(M) = \{a_\delta, c_\delta, d_\delta\}$. Clearly, $Pt(M)$ is a classical set.

Note that if X contains n elements (that is, n picture points), then the picture power set of X contains 6^n elements. Moreover, note that picture points are defined in a specific manner. As we already know, it is possible that the intersection of M and M^c is not empty in a picture sense. However, belonging of m_δ to M depends only on M_1. Hence, if $m_\delta \in M$, then $m \in M_1$, and thus $m \notin M_3$ (because $M_1 \cap M_3 = \emptyset$). But then $m_\delta \notin M^c$. So the set of common picture points of M and M^c is empty (clearly, this is a classical set: namely, classical empty set).

Besides, we see that the relation of belonging defined in terms of picture points is specific also in the sense that it is always true that $m_\delta \in m_\delta$.

Lemma 3.6. *Suppose that M, N are two picture sets over X and $M \subseteq N$. Now, if $x_\delta \in M$, then $x_\delta \in N$.*

Proof. This is clear. If $M \subseteq N$, then (in particular) $M_1 \subseteq N_1$. If $x_\delta \in M$, then $x \in M_1$, so $x \in N_1$ and thus $x_\delta \in N$. □

Remark 3.4. *On the other hand, it is possible that for each $x_\delta \in M$ we have that $x_\delta \in N$ but $M \nsubseteq N$. Take $X = \{a, b, c, d\}$, $M = \langle X, \{a, b\}, \{c\}, \{d, e\}\rangle$ and $N = \langle X, \{a, b, c\}, \{c, d\}, \{e\}\rangle$. Now, what are the picture points contained in M? These are $a_\delta = \langle X, \{a\}, \{b, c, d\}, \{b, c, d\}\rangle$ and $b_\delta = \langle X, \{b\}, \{a, c, d\}, \{a, c, d\}\rangle$. Clearly, a and b belong to $N_1 = \{a, b, c\}$. Thus, a_δ and b_δ belong to N. However, we cannot say that $M \subseteq N$ because $\{c, d\} = N_2 \nsubseteq M_2 = \{c\}$.*

The last remark suggests a new definition.

Definition 3.4. Let M, N be two picture sets on X. We say that M is *picture point contained* in N if and only if for any x_δ, if $x_\delta \in M$, then $x_\delta \in N$. In such case we write $M \leq N$. If $M \leq N$ and $N \leq M$, then we say that M and N are *picture point similar* and we denote this by $M \approx N$.

In practice, it means that $M \leq N$ if and only $M_1 \subseteq N_1$. Then M and N are picture point similar if they have the same picture points. It means that $M_1 = N_1$.

The next lemma is obvious.

Lemma 3.7. *Assume that M, N are picture sets on X and $M \subseteq N$. Then $M \leq N$.*

Remark 3.5. *Note that the fact that $x_\delta \notin M$ does not mean that $x_\delta \in M^c$. If $x_\delta \notin M$, then $x \notin M_1$. But this does not necessarily mean that $x \in M_3$. This is because $M_3 \subseteq M_1^c$ but it may not be equal with M_1^c.*

We should remember that picture points are on the same ”level of existence” as picture sets. In fact, they *are* picture sets of some specific kind. This is, in general, different than in case of classical sets where we treat elements as being on somewhat ”lower” or ”more elementary” level than sets.

Remark 3.6. *Note that the fact that $x_\delta \in M$ does not necessarily mean that $x_\delta \subseteq M$. Take $X = \{a, b, c\}$ and $M = \langle X, \{a, b\}, \{a\}, \emptyset\rangle$. Now $a_\delta = \langle X, \{a\}, \{b, c\}, \{b, c\}\rangle$ and $a_\delta \in M$ because $a \in M_1 = \{a, b\}$. However, $a_\delta \nsubseteq M$ because $M_2 = \{a\} \nsubseteq \{b, c\}$.*

On the other hand, it is clear that if $x_\delta \subseteq M$, then $x \in M$.

3.3. Topology of picture sets

One of the most natural applications of non-classical sets[5] are topologies and various weaker or just different classes (like supra- and infra-topologies, minimal structures etc.). Due to this fact we define picture topologies.

3.4. Picture topologies

Definition 3.5. Let X be a non-empty universe. A collection τ_δ of picture subsets on X is called *picture topology* on X if and only if the following axioms are satisfied:

1. $X_\delta, \emptyset_\delta \in \tau_\delta$.
2. τ_δ is closed under arbitrary picture unions and finite picture intersections.

An ordered pair (X, τ_δ) is called *picture topological space*. Members of τ_δ are known as *picture open sets* ($P_\delta OS$) in X. Their picture complements are *picture closed sets* ($P_\delta CS$).

Example 3.1. *Assume that* $X = \{a, b, c, d, e, f\}$. *Consider:*

$\tau_\delta = \{\emptyset_\delta, X_\delta, A = \langle X, \{a, b\}, \{c\}, \{d\}\rangle, B = \langle X, \{b, d\}, \{d, e\}, \{e, f\}\rangle,$
$C = \langle X, \{a, b, d\}, \emptyset, \emptyset\rangle, D = \langle X, \{b\}, \{c, d, e\}, \{d, e, f\}\rangle\}$.

One can check by direct inspection that τ_δ *is a picture topology on* X. *For example, we see that* $A \cup B = C$, $A \cap B = D$, $C \cup D = C$, $C \cap D = D$, $A \subseteq C$, $D \subseteq A$ *(so* $A \cup C = C$, $A \cap C = A$, $A \cup D = A$, $A \cap D = D$*) and the intersections and unions with* $\emptyset_\delta$ *and* X_δ *are clear.*

Example 3.2. *Assume that* $X = \mathbb{N}$ *and* τ_δ *consists of* $\emptyset_\delta, \mathbb{N}_\delta$ *and all picture sets of the form* $\langle \mathbb{N}, A, \emptyset, \emptyset\rangle$ *where* $A \subseteq \mathbb{N}$. *Now, if* $M, N \in \tau_\delta$, *then their intersection is of the form* $\langle \mathbb{N}, M \cap N, \emptyset \cup \emptyset, \emptyset \cup \emptyset\rangle = \langle \mathbb{N}, M \cap N, \emptyset, \emptyset\rangle$ *(so it belongs to* τ_δ*). As for the arbitrary union, it will be* $\langle \mathbb{N}, \bigcup M_i, \emptyset, \emptyset\rangle$. *Note that at first glance, it looks like classical discrete topology where* each *set is open. However, this observation is true only if we limit our attention to the first components, that is, to* M_1, N_1 *and so on. In general,* τ_δ *does not contain all the picture sets on* X. *For example, it is clear that e.g.* $\langle \mathbb{N}, \{1, 2, 3\}, \{3, 4, 5\}, \{8, 9\}\rangle$ *does not belong to* τ_δ.

Of course, as always in such cases, it is possible to assume that every picture set is open and to formulate discrete picture topology.

Example 3.3. *Let* $X = \mathbb{R}$. *Assume that* $\tau_\delta(Z)$ *consists of* $\emptyset_\delta$, $\mathbb{R}_\delta$ *and all the picture sets of the form* $\langle X, A, B, C\rangle$ *where* A *can be written as a union of open intervals, while* B *and* C *are arbitrary subsets of some determined finite* $Z \subseteq \mathbb{R}$. *Note that if we limit our attention only to the first components, then this is just like ordinary topology on* $\mathbb{R}$. *On the other hand, when we consider only second and third components, then we may think about discrete topologies on* Z.

In general, exemplary elements of $\tau_\delta(Z)$ *are (if* $Z = \{10, 11, 12, 13\}$*):* $M = \langle \mathbb{R}, (0, 1), \{10, 11\}, \{11, 12\}\rangle$ *or* $N = \langle \mathbb{R}, (-5, -1) \cup (1, 5), \{12\}, \{13\}\rangle$.

Definition 3.6. We say that a *basis* $\mathcal{B}$ for a picture topology τ_δ on X is any collection B_δ of picture subsets of X such that;

1. For each $x_\delta \in X_\delta$ there is at least one basis element B containing x_δ.

[5] By "non-classical" sets we understand all those classes that are used to model uncertainty in non-probabilistic sense: that is, fuzzy, neutrosophic, rough, soft sets etc.

2. If x_δ belongs to $B_1 \cap B_2$ (where $B_1, B_2 \in \mathcal{B}$), then there is $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Definition 3.7. A *subbasis* $\mathcal{S}$ for a picture topology τ_δ on X is a collection of picture subsets of X whose union equals X. The picture topology *generated* by the subbasis $\mathcal{S}$ is defined as a collection of all unions of finite intersections of the elements of S.

Definition 3.8. Assume that (X, τ_δ) is a picture topological space and $x_\delta \in X_\delta$. We say that a picture set G on X is a *picture neighborhood* of x_δ if there exists a picture open set U in X such that $x_\delta \in U \subseteq G$.

3.5. Picture interior

The notion of interior is crucial in topology. Here we have its two interpretations in picture framework.

Definition 3.9. Assume that (X, τ_δ) is a picture topological space, $M \in Pic(X)$ and $x_\delta \in X_\delta$. We say that x_δ is a *picture interior point* of M if there exists picture open set U such that $x_\delta \in U \subseteq M$. The union of picture interior points [6] of M is called *picture point interior* of M, that is $P^p_\delta int(M))$. We say that M is *picture point open* if and only if $M = P^p_\delta int(M)$.

Remark 3.7. *Note that we defined $P^p_\delta int(M)$ not as a* collection *of picture interior points of M but as their* union. *A collection would be classical while we want this kind of interior to be a picture set too. After some additional investigation of sources, we recognized that a similar concept has been introduced in the context of intuitionistic topological spaces by Kim et al. in [8].*

Definition 3.10. Assume that (X, τ_δ) is a picture topological space and $M \in Pic(X)$. Then we say that the union of all picture open sets contained in M is a *picture interior* of M and it is denoted by $P_\delta int(M)$.

These two definitions are not equivalent. This is clear from the remark below.

Remark 3.8. *Take topology from Example 3.1. Take $M = \langle X, \{a, b, c\}, \{c\}, \{d\}\rangle$. Let us compute its picture interior. The only picture open set contained in M is $A = \langle X, \{a, b\}, \{c\}, \{d\}\rangle$, hence $A = P_\delta int(M)$. However, both a_δ and b_δ (but not c_δ) are of this kind that there is $U \in \tau_\delta$ such that $a_\delta \in U$, $b_\delta \in U$ and $U \subseteq E$. In fact, this U can be just A. However, $a_\delta \cup b_\delta = \langle X, \{a\}, \{b, c, d, e, f\}, \{b, c, d, e, f\}\rangle \cup \langle X, \{b\}, \{a, c, d, e, f\}, \{a, c, d, e, f\}\rangle = \langle X, \{a, b\}, \{c, d, e, f\}, \{c, d, e, f\}\rangle$. This set, namely $P^p_\delta int(M)$ is different than $P_\delta int(M)$.*

However, we have the following relationship:

Lemma 3.8. *Let (X, τ_δ) be a picture topological space and $M \in Pic(X)$. Then $P^p_\delta int(M) \approx P_\delta int(M)$.*

Proof. $(\leq)$.

Suppose that there is $y_\delta \in P_\delta int(M)$ such that $y_\delta \notin P^p_\delta int(M)$. Then for any $U \in \tau_\delta$ such that $y_\delta \in U$, we have $U \nsubseteq M$. However, we assumed that $y_\delta \in P_\delta int(M)$, hence it belongs to the union of all picture open sets contained in M. Thus, it must belong to at least one such set. Contradiction.

$(\geq)$.

Suppose that there is $y_\delta \in P^p_\delta int(M)$ such that $y_\delta \notin P_\delta int(M)$. Hence y_δ is beyond the union of all picture open sets contained in M. But because of the first part of our assumption there must be at least one such set. Contradiction. □

[6] Again, recall the fact that picture points are, in particular, picture sets, so here we mean their picture union.

In Remark 3.8 we have the following situation: that $P^p_\delta int(M) \subseteq P_\delta int(M)$. But in general, it is not true. Consider the same topology and the set $B \in \tau_\delta$. Recall that it was $\langle X, \{b,d\}, \{d,e\}, \{e,f\}\rangle$. This set is picture open in this topology, so $P_\delta int(B) = B$ [7]. But $P^p_\delta int(B) = b_\delta \cup d_\delta = \langle X, \{b,d\}, \{a,c,e,f\}, \{a,c,e,f\}\rangle$. However, $B_2 = \{d,e\} \nsubseteq \{a,c,e,f\}$, so we cannot say that $P^p_\delta int(B) \subseteq P_\delta int(B)$.

Lemma 3.9. *Let (X, τ_δ) be the picture topological space, then for any M in $Pic(X)$ the following statements hold:*

1. $P_\delta\ int(M) \subseteq M$.
2. *M is picture open set iff $P_\delta\ int(M) = M$.*
3. *If M is picture open and $N \subseteq M$, then $N \subseteq P_\delta\ int(M)$.*
4. *$P_\delta\ int(M)$ is a picture open set.*
5. *If $M \subseteq N$, then $P_\delta\ int(M) \subseteq P_\delta\ int(N)$.*
6. $P_\delta\ int(P_\delta\ int(M)) = P_\delta\ int(M)$.
7. $P_\delta\ int(X_\delta) = X_\delta$;

   $P_\delta\ int(\phi_\delta) = \phi_\delta$.
8. $P_\delta\ int(M) \cup P_\delta\ int(N) \subseteq P_\delta\ int(M \cup N)$.
9. $P_\delta\ int(M) \cap P_\delta\ int(N) = P_\delta\ int(M \cap N)$.

Proof. At first glance, these statements seem to be very natural and the reader can suppose that all the proofs are standard. However, we know that the algebra of picture sets is not identical with the classical Boolean algebra of $P(X)$. Hence, we should ensure that picture operations behave in a manner that makes all the statements above true. Some cases are obvious while the others require more attention.

1. This is obvious. Note that analogous statement is not necessarily true in case of $P^p_\delta int(M)$, as it was shown in Remark 3.8.
2. $(\Rightarrow)$. This is clear because $M \subseteq M$. $(\Leftarrow)$. Assume that $P_\delta\ int(M) = M$. It means that M is identical with the union of all picture open sets contained in M. But τ_δ is closed under picture unions, so this union (and thus, M itself) must be picture open.
3. This is clear in the light of (2), taking into account the fact that M is just identical with its picture interior.
4. This is obvious in the light of the definition and because of the closure of τ_δ under picture unions.
5. $P_\delta int(M)$ is of the form $\bigcup\{U \in \tau_\delta : U \subseteq M\}$ and $P_\delta int(N)$ is of the form $\bigcup\{V \in \tau_\delta : V \subseteq N\}$. But $M \subseteq N$, so for any U taken into account in $P_\delta int(M)$ we have $U \subseteq N$. But then U is included in $P_\delta int(N)$.
6. $(\subseteq)$. It is clear from (1). $(\supseteq)$. From (1) $P_\delta int(M) \subseteq M$. But then $P_\delta int(P_\delta int(M)) \subseteq P_\delta int(M)$ (from (5)).
7. Both cases are clear if one remembers that $X_\delta, \emptyset_\delta \in \tau_\delta$. Then it is enough to use (2).
8. Clearly, $M \subseteq M \cup N$ and $N \subseteq M \cup N$. Thus, $P_\delta int(M) \subseteq P_\delta int(M \cup N)$ and $P_\delta int(N) \subseteq P_\delta int(M \cup N)$. Hence the conclusion is clear.

[7] To be precise, this property will be mentioned in the next lemma.

9. ($\supseteq$). Clearly, $M \cap N \subseteq M$ and $M \cap N \subseteq N$, so $P_\delta int(M \cap N) \subseteq P_\delta int(M)$ and $P_\delta int(M \cap N) \subseteq P_\delta int(N)$. Hence the conclusion is natural. ($\subseteq$). First, $P_\delta int(M) \cap P_\delta int(N) \subseteq M \cap N$. Now, $P_\delta int(P_\delta int(M) \cap P_\delta int(N)) \subseteq P_\delta int(M \cap N)$. But both $P_\delta int(M)$ and $P_\delta int(N)$ are picture open, so their intersection is open too. Thus, from (2), $P_\delta int(P_\delta int(M) \cap P_\delta int(N)) = P_\delta int(M) \cap P_\delta int(N)$ and now we are ready.

□

3.6. The notion of closure

The notion of closure is another term typical for topology. It can be considered as a counterpart of the notion of interior.

Definition 3.11. Let (X, τ_δ) be a picture topological space and M be a picture set on X. We say that $P_\delta cl(M)$ is a *picture closure* of M if it is the intersection of all picture closed sets containing M.

Lemma 3.10. *Let (X, τ_δ) be a picture topological space. Then for any picture subset M of X the following statements hold:*

1. *$P_\delta\ cl(M)$ is a picture closed set.*
2. *M is picture closed set iff $P_\delta\ cl(M) = M$.*
3. *If M is picture closed set and $M \subseteq N$, then $P_\delta\ cl(M) \subseteq N$.*
4. *$P_\delta\ cl(X_\delta) = X_\delta$;*

   $P_\delta\ cl(\phi_\delta) = \phi_\delta$.
5. *$P_\delta\ cl(P_\delta\ cl(M)) = P_\delta\ cl(M)$.*
6. *If $M \subseteq N$, then $P_\delta\ cl(M) \subseteq P_\delta\ cl(N)$.*
7. *$P_\delta\ cl(M \cup N) = P_\delta\ cl(M) \cup P_\delta\ cl(N)$.*
8. *$P_\delta\ cl(M \cap N) \subseteq P_\delta\ cl(M) \cap P_\delta\ cl(N)$.*
9. *For any $x_\delta \in X$; if $x_\delta \in P_\delta\ cl(M)$ then $V \cap M \neq \phi_\delta$ for every picture open set V containing x_δ.*
10. *$X \backslash P_\delta\ cl(M) = P_\delta\ int(X \backslash M)$;*
    $X \backslash P_\delta\ int(M) = P_\delta\ cl(X \backslash M)$.
11. *$P_\delta\ int(M) = X \backslash P_\delta\ cl(X \backslash M)$*
    $P_\delta\ cl(M) = X \backslash P_\delta\ int(X \backslash M)$.

Proof. We leave the majority of proofs for the reader. Basically, the statements presented in this lemma are counterparts of analogous statements for (picture) interior. However, think about (9).

Let $x_\delta \in P_\delta cl(M)$. It means that for any picture closed set B (that is, for any picture set B such that B^c is picture open) such that $M \subseteq B$, we have $x_\delta \in B$ (hence, $x \in B_1$). Now suppose that there is some $V \in \tau_\delta$ such that $x_\delta \in V$ and $V \cap M = \emptyset_\delta = \langle X, \emptyset, X, X \rangle$. Now, $x \in V_1$, so $x \notin V_3$. Thus, $x_\delta \notin V^c$ and V^c is picture closed as a complement of a picture open set. Moreover, by virtue of Lemma 3.3, we know that $M \subseteq V^c$. Thus, we obtain a contradiction. □

Remark 3.9. *Note that the converse of (9) may not be true. Basically, in classical topological spaces, the analogue of this converse would be proven in the following way. Assume that* $U \cap M \neq \emptyset_\delta$ *for some* x_δ *such that* $x_\delta \in U \in \tau_\delta$. *Now suppose that* $x_\delta \notin cl(M)$. *Then there must be some closed* F *such that* $M \subseteq F$ *and* $x_\delta \notin F$. *But then* $x_\delta \in F^c$. *However,* F^c *is open (as a complement of closed set) and* F^c *has empty intersection with* M. *Thus we have an example of open set containing* x_δ *and having empty intersection with* M. *Contradiction.*

However, in case of picture sets and points the fact that $x_\delta \notin F$ *does not necessarily imply that* $x_\delta \in F^c$. *The only necessary conclusion is that* $x \notin F_1$, *so* $x \in F_1^c$. *However,* F_1^c *may be different than* F_3 *(even if* $F_3 \subseteq F_1^c$*). We cannot say that* $x \in F_3$.

To be precise, think about the following picture topology. Let $X = \{a, b, c, d, e, f\}$. *Then let* $\tau_\delta = \{\emptyset, X, A = \langle X, \{a, b\}, \{b, c\}, \{c, d, e, f\}\rangle, B = \langle X, \{c, d\}, \{a, b\}, \{e, f\}\rangle, C = \langle X, \emptyset, \{a, b, c\}, \{c, d, e, f\}\rangle, D = \langle X, \{a, b, c, d\}, \{b\}, \{e, f\}\rangle\}$.

One can easily check that τ_δ *is a picture topology on* X. *Both* A *and* B *are just arbitrary picture sets, while* C *is their intersection and* D *is their union.*

Then $\tau_\delta^c = \{\emptyset, X, E = \langle X, \{c, d, e, f\}, \{a, d, e, f\}, \{a, b\}\rangle, F = \langle X, \{e, f\}, \{c, d, e, f\}, \{c, d\}\rangle,$
$G = \langle X, \{c, d, e, f\}, \{d, e, f\}, \emptyset\rangle, H = \langle X, \{e, f\}, \{a, c, d, e, f\}, \{a, b, c, d\}\rangle\}$.

Now take picture point $b_\delta = \langle X, \{b\}, \{a, c, d, e, f\}, \{a, c, d, e, f\}\rangle$ *and consider* $M = \langle X, \{c, d, e, f\}, \{d, e, f\}, \emptyset\rangle$. *In fact,* $M = G$ *(so it is picture closed, hence* $P_\delta cl(M) = M$*) and we see that* $b_\delta \notin M$ *(because* $b \notin \{c, d, e, f\}$*). Thus* $b_\delta \notin P_\delta cl(M)$.

Now let us find all those picture open sets that contain b_δ. *These are* X, A *and* D. *Clearly,* $M \cap X \neq \emptyset$. *Now let us calculate:*

$M \cap A = \langle X, \emptyset, \{b, c, d, e, f\}, \{c, d, e, f\}\rangle$. *This set is different than* $\emptyset = \langle \emptyset, X, X\rangle$.

$M \cap D = \langle X, \{c, d\}, \{b, d, e, f\}, \{e, f\}\rangle$. *Again, this is not* $\emptyset$.

Hence, we have just shown that it is possible that some picture point (in our case, b_δ*) does not belong to the closure of some picture set (in our case,* M*) even if the intersection of any picture open set containing this point with* M *is non-empty in a picture sense.*

3.7. Picture limit points

In this subsection we define and discuss the idea of convergence. We start from the initial definition.

Definition 3.12. A picture point p_δ is said to be a *picture limit point* of a picture set M if for each picture neighborhood N_δ of p_δ we have $N_\delta \cap (M \setminus p_\delta) \neq \emptyset_\delta$. The set of all picture limit points of a set M is known as a *picture derived set* ($D_\delta(M)$).

Note that (contrary to its name) this picture derived set is a *classical* set. Picture points are, as we already know, just picture sets of some specific type. Hence, any set of them is just a classical set of such objects: e.g. of the form $D(A) = \{a_\delta, c_\delta d_\delta\}$ and so on. Due to the same reason in the definition of limit points we do not substract $\{p_\delta\}$ from M (that is, singleton of p_δ) but p_δ itself (to make this substraction possible).

Example 3.4. *Consider picture topology presented in Remark 3.9. In fact, we proved that* b_δ *is a picture limit point of* M. *Let us calculate* $M' = M \setminus b_\delta = M \cap b_\delta^c = \langle X, \{c, d, e, f\}, \{d, e, f\}, \emptyset\rangle \cap \langle X, \{a, c, d, e, f\}, \{b\}, \{a, c, d, e, f\}\rangle = \langle X, \{c, d, e, f\}, \{b, d, e, f\}, \{a, c, d, e, f\}\rangle$.

Now (recall that A *and* D *are the only open neighborhoods of* b_δ*):*

$M' \cap A = \langle X, \emptyset, \{b,c,d,e,f\}, \{a,c,d,e,f\} \rangle \neq \emptyset_\delta$.

$M' \cap D = \langle X, \{c,d\}, \{b,d,e,f\}, \{a,c,d,e,f\} \rangle \neq \emptyset_\delta$.

Example 3.5. *Take topology from Example 3.1. Consider $M = \langle X, \{a,b,c\}, \{c\}, \{d\} \rangle$ and picture point b_δ. We calculate:*

$M' = M \setminus b_\delta = M \cap b_\delta^c = M \cap \langle X, \{a,c,d,e,f\}, \{b\}, \{b\} \rangle = \langle X, \{a,c\}, \{b,c\}, \{b,d\} \rangle$. *As for the picture open neighborhoods of b_δ in this topology, these are A, B, C and D. So we calculate:*

$M' \cap A = \langle X, \{a\}, \{b,c\}, \{b,d\} \rangle \neq \emptyset_\delta$.

$M' \cap B = \langle X, \emptyset, \{b,c,d,e\}, \{b,d,f\} \rangle \neq \emptyset_\delta$.

$M' \cap C = \langle X, \{a\}, \{b,c\}, \{b,d\} \rangle \neq \emptyset_\delta$.

$M' \cap D = \langle X, \emptyset, \{b,c,d,e\}, \{b,d,e,f\} \rangle \neq \emptyset_\delta$.

Obviously, $M' \cap X \neq \emptyset_\delta$, so b_δ is a picture limit point of M.

Remark 3.10. *There is a natural temptation to think that if x_δ is a picture limit point of some picture set M, then it means that each picture open neighborhood of x_δ contains some y_δ that belongs to M and is different than x_δ. However, Example 3.5 shows that this is not true. We have proven that b_δ is a limit point of M but now let us think about D. This set is one of the open neighborhoods of b_δ. However, it contains only one picture point, namely b_δ. It does not change the fact that $D \cap M'$ is not empty in a picture sense (albeit its first component is empty classical set).*

We shall prove the following lemma.

Lemma 3.11. *Let M and N be the picture subsets of X. Let $D_\delta(M)$ and $D_\delta(N)$ denote picture derived sets of M and N respectively. Then the following statement is true: if $M \subseteq N$, then $D_\delta(M) \subseteq D_\delta(N)$.*

Proof. Assume that $D_\delta(M) \nsubseteq D_\delta(N)$. It means that there is some picture point x_δ such that x_δ is a limit point of M but not a limit point of N. It means that there is some picture open neighborhood A of x_δ such that $A \cap (N \cap x_\delta^c) = \emptyset_\delta$. On the other hand, we have that $A \cap (M \cap x_\delta^c) \neq \emptyset_\delta$. But $M \cap x_\delta^c \subseteq N \cap x_\delta^c$, so $A \cap (M \cap x_\delta^c) \subseteq A \cap (N \cap x_\delta^c)$ and this is contradiction. $\square$

In the next lemma we should distinguish between two situations. When we say that $x_\delta \in D_\delta(M)$ then we mean that x_δ is a member of a classical set $D_\delta(M)$ (a collection of picture limit points of M). But when we say that $x_\delta \in B$ (where B is a picture set) then we mean belonging defined as in Definition 3.3.

Lemma 3.12. *Let M and N be the picture subsets of X. Then $D_\delta(M \cup N) = D_\delta(M) \cup D_\delta(N)$.*

Proof. $(\subseteq)$.

We want to show that if $x_\delta \in D_\delta(M \cup N)$ then x_δ must be in $D_\delta(M)$ or $D_\delta(N)$. By the assumption: for any $A \in \tau_\delta$ such that $x_\delta \in A$ we have $A \cap ((M \cup N) \cap x_\delta^c) \neq \emptyset_\delta$. Now assume that $x_\delta \notin D_\delta(M) \cup D_\delta(N)$. Hence, $x_\delta \notin D_\delta(M)$ and $x_\delta \notin D_\delta(N)$. The first part of this conjunction says that there is some $B \in \tau_\delta$ such that $x_\delta \in B$ and $B \cap (M \cap x_\delta^c) = \emptyset_\delta$. Analogously, the second part says that there is some $C \in \tau_\delta$ such that $x_\delta \in C$ and $C \cap (M \cap x_\delta^c) = \emptyset_\delta$.

But for any $A \in \tau_\delta$ such that $x_\delta \in A$ we have $A \cap ((M \cup N) \cap x_\delta^c) = A \cap ((M \cap x_\delta^c) \cup (N \cap x_\delta^c)) = (A \cap (M \cap x_\delta^c)) \cup (A \cap (N \cap x_\delta^c))$. But both components of this final disjunction are non-empty in picture sense (for any $A \in \tau_\delta$). Hence, there is no place for B and C.

$(\supseteq)$. Assume that $x_\delta \in D_\delta(M) \cup D_\delta(N)$. Without loss of generality, assume that $x_\delta \in D_\delta(M)$. This means that for any $A \in \tau_\delta$ such that $x_\delta \in A$, we have $A \cap (M \cap x_\delta^c) \neq \emptyset_\delta$. Now assume that $x_\delta \notin D_\delta(M \cup N)$. But this means that there is some $B \in \tau_\delta$ such that $x_\delta \in B$ and $B \cap ((M \cup N) \cap x_\delta^c) = (B \cap (M \cap x_\delta^c)) \cup (B \cap (N \cap x_\delta^c)) = \emptyset_\delta$. However, at least the first component of this last union is non-empty in picture sense. Thus, the whole union cannot be empty. Contradiction. $\square$

The reader is encouraged to prove the next lemma in a similar manner:

Lemma 3.13. *Let M and N be the picture subsets of X. Then $D_\delta(M \cap N) \subseteq D_\delta(M) \cap D_\delta(N)$.*

3.8. Picture contact points

No we define another notion, slightly different that the notion of picture limit point (but similar).

Definition 3.13. A picture point p_δ is said to be a *picture contact point* of M if and only if for each picture neighborhood N_δ of p_δ we have $N_\delta \cap M \neq \emptyset_\delta$. Picture union of all the picture contact points of M is called the *picture point closure* of M, that is $P_\delta^p cl(M)$. We say that M is *picture point closed* if $M = P_\delta^p cl(M)$.

This notion is a counterpart of the notion of picture point interior (not necessarily of picture interior). The latter two are not identical, as we already know. However, is we say "counterpart", then we should be aware of some limitations of this term. The next remark is important:

Remark 3.11. *The fact that a picture set is picture point open does not necessarily mean that its complement is picture point closed. Take topology from Example 3.1. Take $N = \langle X, \{a,b\}, \{c,d,e,f\}, \{c,d,e,f\} \rangle$. One can easily check that $P_\delta^p Int(N) = N$ (just sum up a_δ and b_δ). Now take $N^c = \langle X, \{c,d,e,f\}, \{a,b\}, \{a,b\} \rangle$. Consider picture point a_δ. On the one hand, it does not belong to N^c (because $a \notin \{c,d,e,f\}$). On the other, let us calculate the intersections of N^c with all non-trivial [8] open neighborhoods of a_δ. These neighborhoods are $A = \langle X, \{a,b\}, \{c\}, \{d\} \rangle$ and $C = \langle X, \{a,b,d\}, \emptyset, \emptyset \rangle$. Thus:*

$N^c \cap A = \langle X, \emptyset, \{a,b,c\}, \{a,b,d\} \rangle \neq \emptyset_\delta$.

$N^c \cap C = \langle X, \{d\}, \{a,b\}, \{a,b\} \rangle \neq \emptyset_\delta$.

It means that a_δ is a picture contact point of N^c but it does not belong to N^c. So N^c is not picture point closed.

Besides, think about $A^c = \langle X, \{d\}, \{a,b,d,e,f\}, \{a,b\} \rangle$. Take (again) picture point a_δ. It does not belong to A^c. However, it is a picture contact point of A^c because $A^c \cap A = \langle X, \emptyset, X, \{a,b,d\} \rangle \neq \emptyset_\delta$ and $A^c \cap C = \langle X, \{d\}, \{a,b,d,e,f\}, \{a,b\} \rangle \neq \emptyset_\delta$. This means that a_δ is a picture contact point of A^c, so A^c is not picture point closed. However, as a complement of picture point open A, it is picture closed. Hence, picture closed set need not to be picture point closed.

Now let us calculate $A \cap (A^c \setminus a_\delta) = A \cap (A^c \cap a_\delta^c) = (A \cap A^c) \cap a_\delta^c$. This will be: $\langle X, \emptyset, X, \{a,b,d\} \rangle \cup \langle X, \{a\}^c, \{a\}, \{a\} \rangle = \langle X, \emptyset, X, \{a,b,d\} \rangle \cup \langle X, \{b,c,d,e,f\}, \{a\}, \{a\} \rangle = \langle X, \emptyset, X, \{a,b,d\} \rangle \neq \emptyset_\delta$. This shows that a_δ is a picture limit point of A^c.

Remark 3.12. *One could think that any picture contact point is a picture point of M or it's picture limit point, e.g. that $Pt(P_\delta^p cl(M)) = Pt(M) \cup D_\delta(M)$. However, we already know that picture points do not always*

[8] We consider X as trivial.

behave as classical points. We leave this question as a small open problem for the reader and for our future work. Let us give some initial clues. Assume that $a_\delta \in Pt(P^p_\delta cl(M))$ *but is does not belong to* $Pt(M)$ *nor* $D_\delta(M)$*. The last assumption implies that there is some* $K \in \tau_\delta$ *such that* $a \in k$ *and* $K \cap (M \cap a^c_\delta) = \emptyset_\delta$*. However, the fact that* $a_\delta \in Pt(P^p_\delta cl(M))$ *implies that* $K \cap M \neq \emptyset_\delta$*. Clearly,* $a_\delta \notin M$ *(by the assumption), so* $K \cap M = \langle X, \emptyset, (K \cap M)_2, (K \cap M)_3 \rangle$ *where both* $(K \cap M)_2$ *and* $(K \cap M)_3$ *are different than* X*. However, their union with* $\{a\}$ *must give* X *because* $K \cap M \neq \emptyset_\delta$ *(we know that the second and the third component of intersection of picture sets are calculated as the unions of corresponding components of the sets in question). Hence, they are both of the form* $X \setminus \{a\}$*.*

Thus, the task is to find such universe X*, such picture topology* τ_δ*, such picture point* a_δ*, such picture set* M *and such picture open neighborhood* K *(of* a_δ*) that* $K \cap M = \langle X, \emptyset, X \setminus \{a\}, X \setminus \{a\} \rangle$ *while* $(K \cap M) \cap a^c_\delta = \langle X, \emptyset, X, X \rangle$*. Alternatively, one can try to prove that it is not possible.*

4. Picture product and subspace topology

In the last section we briefly introduce the product and subspace topologies in picture environment.

4.1. Picture product topologies

Definition 4.1. Assume that $M = \langle X, M_1, M_2, M_3 \rangle$ and $N = \langle Y, N_1, N_2, N_3 \rangle$ are picture subsets of universes X and Y respectively. Then the picture Cartesian product of these sets is defined as: $M \times_\delta N = \langle X \times Y, M_1 \times N_1, M_2 \times N_2, M_3 \times N_3 \rangle$.

As we can see, we use classical products to define picture product. Now let us think about the topological aspect.

Definition 4.2. Assume that (X, τ_δ) and (Y, μ_δ) are picture topological spaces. A *picture product topology* on $X \times_\delta Y$ is a picture topology whose basis is a collection $\mathcal{D}$ of all picture sets of the form $U \times_\delta V$ with the assumption that U and V are picture open subsets of X and Y respectively.

Theorem 4.1. *Assume that* $\mathcal{B}$ *is a basis for the picture topology* τ_δ *on* X *while* $\mathcal{C}$ *is a basis for the picture topology* μ_δ *on* Y*. Then the following collection* $\mathcal{D} = \{B \times_\delta C; B \in \mathcal{B}, C \in \mathcal{C}\}$ *is a basis for the picture topology* $X \times_\delta Y$*.*

Proof. Picture elements of the form $B \times_\delta C$ from the collection $\mathcal{D}$ are picture open in the picture product topology (since each $B \in \mathcal{B}$ and $C \in \mathcal{C}$ are picture open in Y).

Now let $(x_\delta, y_\delta) \in W \subseteq X \times_\delta Y$ (where W is picture open in the picture product topology). By the very definition of picture product topology generated by a basis, there exists a basis element $U \times_\delta V$ such that $(x_\delta, y_\delta) \in U \times_\delta V \subseteq W$. Since $x_\delta \in U$, U is picture open in X and $\mathcal{B}$ is a basis for the picture topology on X, there exists a basis element $B \in \mathcal{B}$ such that $x_\delta \in B \subseteq U$. Analogously, there exists a basis element $C \in \mathcal{C}$ such that $y_\delta \in C \subseteq V$. Thus, $B \times_\delta C$ is in the collection $\mathcal{D}$ and $(x_\delta, y_\delta) \in B \times_\delta C \subseteq U \times_\delta v \subseteq W$. □

4.2. Picture subspaces: initial remarks

Definition 4.3. Assume that (X, τ_δ) is a picture topological space and Y is a picture subset of X (that is, $Y \subseteq X_\delta$). Then the following collection $\tau^Y_\delta = \{G \cap Y; G \in \tau_\delta\}$ is called *picture topology* on Y. The picture topological space (Y, τ_δ^Y) is called *picture subspace* of (X, τ_δ). Its elements are *picture open in* Y, while their picture complements with respect to Y (that is, picture sets of the form $Y \setminus M$) are *picture closed in* Y.

The following remark seems to be important in our opinion. It contains certain problem to solve.

Remark 4.1. *Note that in classical topological spaces we are able to prove the following theorem: a subset* M *of* Y *is closed in* Y *if and only if there exists a set* F *which is closed in* X *such that* $M = F \cap Y$.

The proof goes as follows: M *is closed in* Y $\Leftrightarrow$ $Y \setminus M$ *is open in* Y $\Leftrightarrow$ $Y \setminus M = G \cap Y$ *for some set* G *that is open in* X $\Leftrightarrow$ $M = Y \setminus (G \cap Y) = (Y \setminus G) \cup (Y \setminus Y)$ $\Leftrightarrow$ $M = Y \setminus G$ $\Leftrightarrow$ $M = Y \cap G^c$ $\Leftrightarrow$ $M = Y \cap F$, *where* $F = G^c$ *is closed in* X.

However, in our picture environment there is a problem with this sequence of meta-equivalences. Note that we used the fact that $Y \setminus Y = \emptyset_\delta$, *hence we were able to write that* $Y \setminus (G \cap Y) = Y \setminus G$. *However, we already know that in the algebra of picture sets* $Y \setminus Y = Y \cap Y^c$ *can be different than* $\emptyset_\delta$.

Alternatively, we we can compute $Y \setminus G = Y \cap G^c = \langle X, Y_1 \cap G_3, Y_2 \cup G_2^c, Y_3 \cup G_1 \rangle$ *and compare this with* $Y \setminus (G \cap Y) = Y \cap \langle X, G_1 \cap Y_1, G_2 \cup Y_2, G_3 \cup Y_3 \rangle^c = \langle X, Y_1 \cap (G_3 \cup Y_3), Y_2 \cup (G_2 \cup Y_2)^c, Y_3 \cup (G_1 \cap Y_1) \rangle = \langle X, (Y_1 \cap G_3) \cup (Y_1 \cap Y_3), (Y_2 \cup G_2^c) \cap (Y_2 \cup Y_2^c), (Y_3 \cup G_1) \cap (Y_3 \cup Y_1) \rangle = \langle X, (Y_1 \cap G_3) \cup \emptyset, (Y_2 \cup G_2^c) \cap X, (Y_3 \cup G_1) \cap (Y_3 \cup Y_1) \rangle = \langle X, Y_1 \cap G_3, Y_2 \cup G_2^c, (Y_3 \cup G_1) \cap (Y_3 \cup Y_1) \rangle$. *Now the difference between these two final picture sets is visible: in general, we do not have any guarantee that* $Y_3 \cup Y_1 = X$. *Only in this case it would be that* $Y \setminus G = Y \setminus (G \cap Y)$.

Our conjecture is that the theorem in question (the one about closed sets) cannot be repeated in the framework of picture sets. The reader is encouraged to check this hypothesis. However, it seems that it would be enough to find a simple counter-example.

However, we are able to prove the following two lemmas. By $P_\delta^p int_Y(M)$ we mean picture point interior of M with respect to Y, while $P_\delta^p int_X(M)$ means picture point interior of M with respect to the initial topology on X.

Lemma 4.1. *A picture subset* M *of* Y *is a* τ_δ^Y*-picture neighborhood of a picture point* $y_\delta \in Y$ *if and only if* $M = N \cap Y$ *for some* τ_δ*-picture neighborhood* N *of* y_δ.

Proof. $(\Rightarrow)$.

Let M be a τ_δ^Y-picture neighborhood of y_δ. Then there is some τ_δ^Y-picture open set H such that $y_\delta \in H \subseteq M$. But then there is a τ_δ-picture open set G such that $H = G \cap Y$. Let $N = M \cup G$. Now N is a τ_δ-picture neighborhood of y_δ since G is a τ_δ-picture open set such that $y_\delta \in G \subseteq N$. Hence: $N \cap Y = (M \cup G) \cap Y = (M \cap Y) \cup (G \cap Y) = M \cup (G \cap Y) = M$. This is true because $M \subseteq Y$ and $G \cap Y \subseteq M$.

$(\Leftarrow)$.

Conversely, assume that $M = N \cap Y$ for some τ_δ-picture neighborhood N of y_δ. Then there exists a τ_δ-picture open set G such that $y_\delta \in G \subseteq N$ which implies that $y_\delta \in G \cap Y \subseteq N \cap Y = M$. Since $G \cap Y$ is τ_δ^Y-picture open, M can be treated as a τ_δ^Y-picture neighborhood of y_δ. □

Lemma 4.2. *Let* (Y, τ_δ^Y) *be a picture subspace of* (X, τ_δ) *and* M *be a picture set in* X. *Then* $P_\delta^p int_X(M) \leq P_\delta^p int_Y(M)$.

Proof. Let $x_\delta \in P_\delta^p int_X(M)$. Then there exists some picture set $U \in \tau_\delta$ such that $U \subseteq M$ and $x_\delta \in U$. Thus, M is a τ_δ-picture neighborhood of x_δ. Consequently, $U \cap Y \in \tau_\delta^Y$ (by the very definition of subspace topology) and (taking into account the fact that $U \cap Y \subseteq M \cap Y$) we have that $M \cap Y$ is a τ_δ-picture neighborhood of x_δ. However, $M \subseteq Y$, so $M \cap Y = M$ (by virtue of Lemma 3.5). But then we can say that $x_\delta \in P_\delta^p int_Y(M)$. Hence, every picture point belonging to $P_\delta^p int_X(M)$ is a picture point of $P_\delta^p int_Y(M)$. Thus $P_\delta^p int_X(M) \leq P_\delta^p int_Y(M)$. □

Lemma 4.3. *A picture point $y_\delta \in Y$ is a τ_δ^Y-picture limit point of $M \subseteq Y$ if and only if y_δ is a picture limit point of M.*

Proof. Suppose that y_δ is a τ_δ^Y-picture limit point of M. Equivalently, it means that $(M \setminus y_\delta) \cap N \neq \emptyset_\delta$ for every τ_δ^Y-picture neighborhood N of y_δ. But any such N can be rewritten as $L \cap Y$ for some τ_δ-picture neighborhood L of y_δ. But $M \subseteq Y$, so we can write that $(M \setminus y_\delta) \cap L \neq \emptyset_\delta$. But then our thesis becomes true. □

Lemma 4.4. *Let Y be a picture subspace of a picture topological space (X, τ_δ). If $M \subseteq Y$ is picture open in X, then M is picture open in Y too.*

Proof. Since $M \subseteq Y$, we have $M = M \cap Y$. But then M is exactly of the form that is expected from picture sets that are open in Y. □

Theorem 4.2. *Let (Y, τ_δ^Y) be a picture subspace of a picture topological space X. Then every τ_δ^Y-picture open subset is picture open in X if and only if Y is picture open in X.*

Proof. (⇒)

Assume that every picture subset M of Y that is picture open in Y is also picture open in X. But $Y \subseteq Y$ and $Y = Y \cap X$, so Y is picture open in Y. It follows that Y is picture open in X.

(⇐)

Conversely, let M be a subset of Y that is picture open in Y and assume that Y is picture open in X. There must be a picture subset G, picture open in X, such that $M = G \cap Y$. But then M is an intersection of two sets that are picture open in X. □

The last theorem is about subspace basis.

Theorem 4.3. *Let (Y, τ_δ) be a picture subspace of a picture topological space X and assume that $\mathcal{B}$ is a basis for the picture topology τ_δ on X. Then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a base for τ_δ^Y.*

Proof. Let H be a τ_δ^Y-picture open subset of Y and let $x_\delta \in H$. Then there exists (by the very definition of subspace) a τ_δ-picture open subset G of X such that $H = G \cap Y$. Since $\mathcal{B}$ is a basis for τ_δ, there is a set $B \in \mathcal{B}$ such that $x_\delta \in B \subseteq G$. But $x \in Y$ (because $H \subseteq Y$), so $x \in B \cap Y \subseteq G \cap Y = H$. Hence there is $B \cap Y \in \mathcal{B}_Y$ such that $x \in B \cap Y \subseteq H$.

Now, if $x_\delta \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{B}_Y$, then $C_1 = B_1 \cap Y$ and $C_2 = B_2 \cap Y$ for some $B_1, B_2 \in \mathcal{B}$. Clearly, $x_\delta \in B_1 \cap B_2$. But then there is $B_3 \in \mathcal{B}$ such that $x_\delta \in B_3 \subseteq B_1 \cap B_2$ and thus $B_3 \cap Y \subseteq (B_1 \cap B_2) \cap Y$. Thus the second condition of basis is satisfied. □

5. Conclusion

This paper introduces a novel type of sets known as picture sets. They can be considered as a crisp version of picture fuzzy sets. On the other hand, they can be viewed as a generalization of intuitionistic sets. They have some connections with neutrosophic crisp sets too. The reader is encouraged to compare picture sets with triple sets (see [17]) that are similar but not identical.

Utilizing these sets, the authors defined the concept of picture topological spaces (together with corresponding product and subset spaces). Many algebraic and topological properties have been checked and proved. In some cases it appeared that some classical theorems are not true in picture space. The authors gave appropriate counter-examples. Some conjectures and suggestions for the readers have been stated too.

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