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The geometric condition related to existence of the singular prescribed scalar curvature

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Abstract: Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and f be a function. In this paper, we recall the singular prescribed scalar curvature problem in [16] and we establish the geometric condition on (M, g) and f that guarantees the existence of solutions to the nonlinear singular scalar curvature equation introduced in [16].

Key words: Yamabe problem, second order elliptic equation, geometric test functions, singular term.

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. In 1960, Yamabe [30] showed that there exists a metric $\overline{g}$, conformal to g, such that its scalar curvature $S_{\overline{g}}$ is constant. Unfortunately eight years later Trudinger [29] found a gap in the Yamabe's proof when the scalar curvature $S_g \geq 0$. Nowadays, the problem is completely solved and in the literature it is known as the Yamabe problem. In fact, Aubin [4] in 1976 solved the problem for any non locally conformally flat manifolds of dimension $n \geq 6$ and Schoen [26] in 1984 achieved the proof. The reader can be refereed to [22] or [21] for more details. Let us talk about how this problem was solved :

Let $u \in C^\infty(M)$, $u > 0$ be a function. Obviously the metric $\overline{g} = u^{N-2}g$ is a conformal metric to g (N is chosen such that $N = \frac{2n}{n-2}$) and we can easily check out that the scalar curvatures S_g and $S_{\overline{g}}$ are linked as follows [4]:

$$\Delta_g u + C_n S_g u = C_n S_{\overline{g}} |u|^{N-2} u \tag{1}$$

where $\Delta_g = -div_g(\nabla_g)$ is the Laplacian-Beltrami operator and $C_n = \frac{n-2}{4(n-1)}$. let

$$P_g = \Delta_g + C_n S_g,$$

Solving the Yamabe problem is equivalent to find a smooth positive function u solution of the following equation

$$P_g u = C|u|^{N-2}u \tag{2}$$

where C is a constant. In order to obtain solutions of (2), Yamabe defined the quantity

$$\mu(M, g) = \inf_{u \in H_1^2(M), u \neq 0} Y(u) \tag{3}$$

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where the Sobolev space $H_1^2(M)$ is the completion of the space $C^\infty(M)$ with respect to the norm

$$\|u\|_{H_1^2(M)} = (\int_M |\nabla_g u|^2 + u^2 dv_g)^{\frac{1}{2}} \tag{4}$$

and

$$Y(u) = \frac{\int_M uP_g u dv_g}{(\int_M u^N dv_g)^{\frac{2}{N}}}.$$

The constant $\mu(M,g)$ is known as the Yamabe constant, Y is the Yamabe functional, (2) is just the Euler-Lagrange equation associated to this functional which its critical points are exactly solutions of (2). In particular, if $u > 0$, smooth and satisfy $Y(u) = \mu(M,g)$, u is solution of (2) and $\overline{g} = u^{N-2}\ g$ is the desired metric (its scalar curvature is constant). To solve the problem, Aubin [4] and Schoen [26] showed that it is sufficient to prove the following theorem :

Theorem 1.1.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$.
(1) Assume that the Yamabe invariant

$$\mu(M,g) < K_0^{-2}(n,1),$$

then there exists a positive smooth function u such that $Y(u) = \mu(M,g)$.
(2) The following inequality is always satisfied :

$$\mu(M,g) \leq K_0^{-2}(n,1) \tag{5}$$

and we only have equality in this inequality if and only if (M,g) is conformally diffeomorphic to the sphere $\mathbb{S}^n$.

Here the constant

$$K_0^2(n,1) = \frac{4}{n(n-2)\omega_n^{\frac{2}{n}}}$$

where ω_n stands for the volume of the unit n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

In [24], the authors introduced a kind of singular term to the Yamabe problem by assuming that the metric g satisfied the following assumption :

(H) : the metric $g \in H_2^p(M, T^*M \otimes T^*M)$ where $p > n$ and there exists a point P such that g is smooth in the ball $B(P,\delta)$,

where the space T^*M is the cotangent space of M and $B(P,\delta)$ is the geodesic ball of center P and of radius δ with $0 < \delta < \frac{r_g(M)}{2}$ and $r_g(M)$ is the injectivity radius. The space $H_2^p(M, T^*M \otimes T^*M)$ is the space of all sections g ($2-$ covariant tensors) such that in normal coordinates the components g_{ij} of g are in $H_2^p(M)$ where $H_2^p(M)$ is the completion of the space $C^\infty(M)$ with respect to the norm

$$\|u\|_{H_2^p(M)} = \left(\int_M |\nabla_g^2 u|^p + |\nabla_g u|^p + |u|^p dv_g\right)^{\frac{1}{p}}. \tag{6}$$

By Sobolev's embedding, we get that for all $p > n$:

$$H_2^p(M, T^*M \otimes T^*M) \subset C^1(M, T^*M \otimes T^*M) \tag{7}$$

then the Christoffels symbols belong to the space $H_1^p(M) \subset C^0(M)$, the components of the Riemannian curvature tensor Rm_g, Ricci tensor Ric_g and the scalar curvature S_g are in $L^p(M)$. The assumption (H) allowed them to introduce the singular Yamabe problem. Moreover, $\mu(M,g)$ is called the singular Yamabe invariant and P_g is the singular Yamabe operator.

The authors in [24] have proved the following result :

Theorem 1.2.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. The operator P_g is weakly conformally invariant and if the singular Yamabe invariant $\mu(M,g) > 0$, P_g is coercive and invertible. In particular if (M,g) is not conformal to the n-sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$, then there exists a metric $\overline{g} = u^{N-2}g$ conformal to g such that $u \in H_2^p(M)$, $u > 0$ and the scalar curvature $S_{\overline{g}}$ of $\overline{g}$ is constant.

In a recent paper [16], under the same assumption (H) we have studied the following equation :

$$\Delta_g u + C_n S_g u = f|u|^{N-2}u \tag{8}$$

where f a positive $C^\infty(M)$ function on M. The above equation (8) is elliptic, nonlinear with critical Sobolev growth and its second coefficient is singular (it does not have the usual regularity) which has allowed us to introduce exactly this singular prescribed scalar curvature equation. We pointed out that we have obtained the important following theorem :

Theorem 1.3.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $S_g \in L^p(M)$ where $p > n$, f a positive $C^\infty(M)$ function on M and $P \in M$ such that $f(P) = \sup_{x \in M} f(x)$. If

$$\mu(M,g) < 2(K_0^{-2}(n,1))(f(P))^{-\frac{2}{N}} \tag{9}$$

then, the equation (8) has a nontrivial solution $u \in C^1(M)$, $u > 0$ such that $E(u) = \mu(M,g)$ and $\int_M f|u|^N dv_g = 1$.

In this paper, we are going to investigate and seek for the general conditions where the crucial inequality (9) can be satisfied. In fact we wanted to stand out the geometric condition and the kind of manifolds where (9) holds. We also notice that there has been many results for second-order elliptic equations, see [1-10,12,15,16,19-30] for more details. Many techniques have been used to solve second-order equations, and we think that variational methods are the most suitable, we invite the reader to see [21],[22] and the references therein. [11-14],[16] and [28] concern fourth order elliptic equations.

2. Notations and preliminaries

In this section, we recall some basic facts and definitions which were used in [16] and will be used in this paper. Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. By Sobolev's embedding [4], one gets that

$$H_1^2(M) \subset L^q(M)$$

where $1 < q \leq N$, and this embedding is compact when $q < N$. The number $N = \frac{2n}{n-2}$ is known as the critical exponent of the Sobolev embedding.
The constant $K_0(n,1)$ introduced above is just the best constant in the following Sobolev inequality that guarantees that there exists a constant $B > 0$ such that for any $u \in H_1^2(M)$,

$$(\int_M |u|^N dv_g)^{\frac{2}{N}} \leq K_0^2(n,1)\|\nabla_g u\|_2^2 + B\|u\|_2^2. \tag{10}$$

Under the assumption (H), the operator P_g is defined in the weak sense on $H_1^2(M)$, and it is easy to verify that P_g is remained elliptic and self-adjoint. To obtain solutions of (8), we have introduced the functional E:

$$E(u) = \int_M |(\nabla_g u|^2 + C_n S_g u^2) dv_g$$

and we have defined the quantity

$$\mu(M,g) = \inf_{\substack{u \in H \\ u \neq 0}} E(u) \tag{11}$$

where the set

$$H = \{u \in H_1^2(M) \quad \text{such that} \quad \int_M f|u|^N dv_g = 2^{\frac{N}{2}}\}$$

Now, we state our main result :

Theorem 2.1.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Let f be a positive $C^\infty(M)$ function on M. If $n \geq 5$ and $f(P) = \sup_{x \in M} f(x)$ and

$$\frac{\Delta_g f(P)}{f(P)} < \frac{S_g(P)}{3}\left(\frac{(n+2)}{2(n-4)} - 1\right) \tag{12}$$

then the inequality (9) :

$$\mu(M,g) < 2(K_0^{-2}(n,1))(f(P))^{-\frac{2}{N}}$$

is satisfied.

3. The geometric condition and test functions

In this section the principle goal is to find the necessary conditions such that the inequality (9) becomes true. We consider a normal geodesic coordinate system centered at some point P. Let $S(r)$ be the geodesic sphere centered at P and of radius r with $r < r_g(M)$ where $r_g(M)$ is the injectivity radius and we also let $d\Omega$ be the volume element of the $(n-1)$ dimensional Euclidean unit sphere $S^{n-1} \subset \mathbb{R}^n$. Put

$$G(r) = \frac{1}{\omega_{n-1}} \int_{S(r)} \sqrt{|g|} d\Omega$$

where ω_{n-1} is the volume of S^{n-1} and $|g|$ the determinant of the Riemannian metric g. The formula of the Taylor's expansion of $G(r)$ in a neighborhood of P is given (we can see [4] for this) by

$$G(r) = 1 - \frac{S_g(P)}{6n} r^2 + o(r^2)$$

where $S_g(P)$ is just the scalar curvature at P. As in the section 2, let η be a smooth function on M such that

$$\eta(x) = \begin{cases} 1 & \text{on} \quad B(P, \delta) \\ 0 & \text{on} \quad M - B(P, 2\delta) \end{cases}$$

For $\epsilon > 0$, we will use the well-known radial functions u_ϵ, they are defined as follows

$$u_\epsilon = \eta(r)(r^2 + \epsilon^2)^{-\frac{n-2}{2}}$$

where $r = d(P, x)$ is the distance from P to x and these functions are called test functions. For further computations the following integrals are needed (see [18]) : For any real positive numbers p, q such that $p - q > 1$, we put

$$I_p^q = \int_0^{+\infty} (1+t)^{-p} t^q dt.$$

It is easy to check that these integrals satisfy the following identities :

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q \tag{13}$$

Now we are going to prove our main theorem

Proof.
To do that, it suffices to show when the inequality (9)

$$E(u_\varepsilon) < 2(K_0^{-2}(n, 1))(f(P))^{-\frac{2}{N}}$$

will be satisfied where u_ε are defined as above and the point P is chosen such that g is smooth in the ball $B(P, \delta)$. More precisely, we are going to estimate :

$$E(u_\varepsilon) = (\int_M |\nabla_g u_\varepsilon|^2 + C_n S_g u_\varepsilon^2 dv_g)(\int_M f|u_\varepsilon|^N dv_g)^{-\frac{2}{N}}.$$

To do that, the general idea is to compute expansions of each integral in $E(u_\varepsilon)$. In order we set :

$$J_1 = \int_M f|u_\varepsilon|^N dv_g, \quad J_2 = \int_M C_n S_g u_\varepsilon^2 dv_g \quad \text{and } J_3 = \int_M |\nabla_g u_\varepsilon|^2 dv_g.$$

Firstly, we remind the reader that it is easy to show that for : $\varphi = |\nabla_g u_\varepsilon|^2$ or $\varphi = f|u_\varepsilon|^N$, or $\varphi = C_n S_g u_\varepsilon^2$, the corresponding integral satisfies

$$\int_{B(P,2\delta)-B(P,\delta)} \varphi dv_g \underset{\epsilon \longrightarrow 0}{\longrightarrow} 0.$$

Therefore we should only compute expansions of J_1, J_2 and J_3 on the geodesic ball $B(P,\delta)$. To compute the first term, we need the following limited development of f at P,

$$f(x) = f(P) + \frac{1}{2}\nabla_{i,j} f(P) y^i y^j + o(r^2).$$

We have

$$J_1 = \int_M f|u_\varepsilon|^N dv_g = \int_0^\delta |u_\varepsilon|^N (\int_{S(r)} f\sqrt{|g|} d\Omega) r^{n-1} dr$$

where

$$\begin{aligned}
\int_{S(r)} f\sqrt{|g|}d\Omega &= \int_{S(r)} (f(P) + \frac{1}{2}\nabla_{i,j} f(P) y^i y^j)(1 - \frac{1}{6}R_{i,j}) y^i y^j) d\Omega + o(r^2) \\
&= \omega_{n-1}(f(P) - (\frac{\Delta_g f(P)}{2n} + \frac{f(P)S_g(P)}{6n})r^2 + o(r^2)),
\end{aligned}$$

that is to say

$$\begin{aligned}
J_1 &= \omega_{n-1}\int_0^\delta \frac{r^{n-1}}{(r^2+\epsilon^2)^n}(f(P) - (\frac{\Delta_g f(P)}{2n} + \frac{f(P)S_g(P)}{6n})r^2) dr + o(r^2) \\
&= \omega_{n-1}(f(P)\int_0^\delta \frac{r^{n-1}}{(r^2+\epsilon^2)^n} dr - (\frac{\Delta_g f(P)}{2n} + \frac{f(P)S_g(P)}{6n})\int_0^\delta \frac{r^{n+1}}{(r^2+\epsilon^2)^n} dr) + o(r^{n+1}).
\end{aligned}$$

Now, we set

$$t = \frac{r^2}{\epsilon^2}, \quad dr = \frac{\epsilon dt}{2dt} \quad \text{and} \quad r = \epsilon\sqrt{t}.$$

By changing the variable as above, one gets

$$\begin{aligned}
J_1 &= \omega_{n-1}(f(P)\int_0^{(\frac{\delta}{\epsilon})^2} \frac{t^{\frac{n}{2}-1}}{2\epsilon^n(1+t)^n} dt - (\frac{\Delta_g f(P)}{2n} + \frac{f(P)S_g(P)}{6n})\int_0^{(\frac{\delta}{\epsilon})^2} \frac{t^{\frac{n}{2}}}{2\epsilon^{n-2}(1+t)^n} dt) \\
&+ o(\epsilon^{n+1}) \\
&= \frac{\omega_{n-1}}{2\epsilon^n}(f(P) I_n^{\frac{n}{2}-1} - (\frac{\Delta_g f(P)}{2n} + \frac{f(P)S_g(P)}{6n})\epsilon^2 I_n^{\frac{n}{2}}) + o(\epsilon^2).
\end{aligned}$$

Independently by applying (13), one gets

$$I_n^{\frac{n}{2}} = \frac{n}{n-2} I_n^{\frac{n}{2}-1} \quad \text{and} \quad \omega_n = 2^{n-1}\omega_{n-1} I_n^{\frac{n}{2}-1},$$

and by plugging these expressions in J_1, one has

$$\begin{aligned} J_1 &= \frac{\omega_{n-1}}{2\epsilon^n} I_n^{\frac{n}{2}-1}(f(P) - (\frac{\Delta_g f(P)}{2(n-2)} + \frac{f(P)S_g(P)}{6(n-2)})\epsilon^2) + o(\epsilon^2) \\ &= \frac{\omega_{n-1}}{2\epsilon^n} I_n^{\frac{n}{2}-1} f(P)(1 - (\frac{\Delta_g f(P)}{2(n-2)f(P)} + \frac{S_g(P)}{6(n-2)})\epsilon^2) + o(\epsilon^2). \end{aligned}$$

Therefore,

$$\begin{aligned} J_1^{-\frac{2}{N}} &= J_1^{-\frac{n-2}{n}} \\ &= (\frac{\omega_{n-1}}{2\epsilon^n} I_n^{\frac{n}{2}-1} f(P))^{-\frac{n-2}{n}}(1 + \frac{n-2}{n}(\frac{\Delta_g f(P)}{2(n-2)f(P)} + \frac{S_g(P)}{6(n-2)})\epsilon^2) + o(\epsilon^2) \\ &= \frac{2^{\frac{n-2}{n}}\epsilon^{n-2}}{(\omega_{n-1} I_n^{\frac{n}{2}-1} f(P))^{\frac{n-2}{n}}}(1 + (\frac{\Delta_g f(P)}{2nf(P)} + \frac{S_g(P)}{6n})\epsilon^2) + o(\epsilon^2). \end{aligned}$$

Let us now compute the second integral J_2. By using H$\ddot{o}$lder's inequality, we get,

$$\begin{aligned} J_2 &= \int_M C_n S_g u_\varepsilon^2 dv_g \\ &\leq (\int_M (C_n S_g)^p dv_g)^{\frac{1}{p}} (\int_M u^{\frac{2p}{p-1}} dv_g)^{\frac{p-1}{p}} \\ &\leq \|C_n S_g\|_p \|u_\varepsilon\|^2_{\frac{2p}{p-1}}. \end{aligned}$$

Then straightforward computation shows that

$$\begin{aligned} \|u_\varepsilon\|^2_{\frac{2p}{p-1}} &= (\int_M u_\varepsilon^{\frac{2p}{p-1}} dv_g)^{\frac{p-1}{p}} \\ &= (\omega_{n-1})^{\frac{p-1}{p}} (\int_0^\delta (\frac{r^{n-1}}{(r^2+\epsilon^2)^{\frac{(n-2)p}{p-1}}} - \frac{S_g(P)}{6n} \frac{r^{n+1}}{(r^2+\epsilon^2)^{\frac{(n-2)p}{p-1}}} + o(r^{n+1}))dr)^{\frac{p-1}{p}} \\ &= (\frac{1}{2})^{\frac{p-1}{p}} (\omega_{n-1})^{\frac{p-1}{p}} \epsilon^{-n+2+2-\frac{n}{p}} (I^{\frac{n}{2}-1}_{\frac{(n-2)p}{p-1}} - \frac{S_g(P)}{3n}\epsilon^2 I^{\frac{n}{2}}_{\frac{(n-2)p}{p-1}} + o(\epsilon^2))^{\frac{p-1}{p}} \\ &= (\frac{1}{2})^{\frac{p-1}{p}} (\omega_{n-1})^{\frac{p-1}{p}} \epsilon^{-n+2+2-\frac{n}{p}} (I^{\frac{n}{2}-1}_{\frac{(n-2)p}{p-1}} - \frac{S_g(P)(n+2)(p-1)}{3n(pn-8p+4-n)}\epsilon^2 I^{\frac{n}{2}-1}_{\frac{(n-2)p}{p-1}} + o(\epsilon^2))^{\frac{p-1}{p}} \\ &= (\frac{1}{2})^{\frac{p-1}{p}} (\omega_{n-1})^{\frac{p-1}{p}} \epsilon^{-n+2+2-\frac{n}{p}} (I^{\frac{n}{2}-1}_{\frac{(n-2)p}{p-1}})^{\frac{p-1}{p}} (1 - \frac{S_g(P)(n+2)(p-1)}{3n(pn-8p+4-n)}\epsilon^2 + o(\epsilon^2))^{\frac{p-1}{p}} \\ &= (\frac{1}{2})^{\frac{p-1}{p}} (\omega_{n-1})^{\frac{p-1}{p}} \epsilon^{-n+2+2-\frac{n}{p}} (I^{\frac{n}{2}-1}_{\frac{(n-2)p}{p-1}})^{\frac{p-1}{p}} (1 - \beta\epsilon^2 + o(\epsilon^2)) \end{aligned}$$

where we have set

$$\beta = \frac{S_g(P)(n+2)(p-1)^2}{3np(pn-8p+4-n)}.$$

Here it is easy to check that the denominator does not vanish and this gives

$$\|u_\varepsilon\|^2_{\frac{2p}{p-1}} = (\frac{1}{2})^{\frac{p-1}{p}}(\omega_{n-1})^{\frac{p-1}{p}}\epsilon^{-n+2+2-\frac{n}{p}}(I^{\frac{n}{2}-1}_{\frac{(n-2)p}{p-1}})^{\frac{p-1}{p}}(1-\beta\epsilon^2+o(\epsilon^2))$$

we deduce that :

$$J_2 \leq (\frac{1}{2})^{\frac{p-1}{p}}(\omega_{n-1})^{\frac{p-1}{p}}\epsilon^{-n+2+2-\frac{n}{p}}\|C_nS_g\|_p(I^{\frac{n}{2}-1}_{\frac{(n-2)p}{p-1}})^{\frac{p-1}{p}}(1-\beta\epsilon^2+o(\epsilon^2)). \tag{14}$$

Now we compute the last integral. First, we have

$$|\nabla_g u_\varepsilon| = |\frac{\partial u_\varepsilon}{\partial r}| = (n-2)\frac{r}{(r^2+\epsilon^2)^{\frac{n}{2}}},$$

then in similar way one gets

$$\begin{aligned}
J_3 &= \omega_{n-1}\int_0^\delta \frac{(n-2)^2r^2}{(r^2+\epsilon^2)^n}(1-\frac{S_g(P)}{6n}r^2+o(r^2))r^{n-1}dr \\
&= \frac{(n-2)^2\omega_{n-1}}{\epsilon^{n-2}}\int_0^{(\frac{\delta}{\epsilon})^2}\frac{t^{\frac{n}{2}}dt}{2(1+t)^n} - \int_0^{(\frac{\delta}{\epsilon})^2}\frac{S_g(P)\epsilon^2t^{\frac{n}{2}+1}dt}{12n(1+t)^n}+o(\epsilon^2) \\
&= \frac{(n-2)^2\omega_{n-1}}{\epsilon^{n-2}}(\frac{n}{2(n-2)}I_n^{\frac{n}{2}-1} - \frac{S_g(P)\epsilon^2n(n+2)}{12n(n-4)(n-2)}I_n^{\frac{n}{2}-1}+o(\epsilon^2)) \\
&= \frac{(n-2)^2}{\epsilon^{n-2}}\omega_{n-1}I_n^{\frac{n}{2}-1}(\frac{n}{2(n-2)} - \frac{S_g(P)\epsilon^2n(n+2)}{12n(n-4)(n-2)}+o(\epsilon^2)) \\
&= \frac{(n-2)}{\epsilon^{n-2}}\omega_{n-1}I_n^{\frac{n}{2}-1}(\frac{n}{2} - \frac{S_g(P)\epsilon^2n(n+2)}{12n(n-4)}+o(\epsilon^2)).
\end{aligned}$$

which means

$$J_3 = \frac{(n-2)}{\epsilon^{n-2}}\omega_{n-1}I_n^{\frac{n}{2}-1}\frac{n}{2}(1-\frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)}+o(\epsilon^2)). \tag{15}$$

Now plugging all expansions of J_1, J_2 and J_3 together in $E(u_\varepsilon)$, therefore :

$$\begin{aligned}
E(u_\varepsilon) &\leq [(\frac{1}{2})^{\frac{p-1}{p}}(\omega_{n-1})^{\frac{p-1}{p}}\epsilon^{-n+2+2-\frac{n}{p}}\|C_nS_g\|_p(I^{\frac{n}{2}-1}_{\frac{(n-2)p}{p-1}})^{\frac{p-1}{p}}(1-\beta\epsilon^2+o(\epsilon^2)) \\
&+ \frac{(n-2)}{\epsilon^{n-2}}\omega_{n-1}I_n^{\frac{n}{2}-1}\frac{n}{2}(1-\frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)}+o(\epsilon^2))] \\
&\times [\frac{2^{\frac{n-2}{n}}\epsilon^{n-2}}{(\omega_{n-1}I_n^{\frac{n}{2}-1}f(P))^{\frac{n-2}{n}}}(1+(\frac{\Delta_gf(P)}{2nf(P)}+\frac{S_g(P)}{6n})\epsilon^2)+o(\epsilon^2)]
\end{aligned}$$

Next,

$$\begin{aligned} E(u_\varepsilon) &\leq [(\frac{1}{2})^{\frac{p-1}{p}}(\omega_{n-1})^{\frac{p-1}{p}}\epsilon^{2-\frac{n}{p}}\|C_nS_g\|_p(I_{\frac{(n-2)p}{p-1}}^{\frac{n}{2}-1})^{\frac{p-1}{p}}(1-\beta\epsilon^2+o(\epsilon^2)) \\ &+ (n-2)\omega_{n-1}I_n^{\frac{n}{2}-1}\frac{n}{2}(1-\frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)}+o(\epsilon^2))] \\ &\times [\frac{2^{\frac{n-2}{n}}}{(\omega_{n-1}I_n^{\frac{n}{2}-1}f(P))^{\frac{n-2}{n}}}(1+(\frac{\Delta_g f(P)}{2nf(P)}+\frac{S_g(P)}{6n})\epsilon^2)+o(\epsilon^2)]. \end{aligned}$$

Let ϵ sufficiently small such that

$$(\frac{1}{2})^{\frac{p-1}{p}}(\omega_{n-1})^{\frac{p-1}{p}}\epsilon^{2-\frac{n}{p}}\|C_nS_g\|_p(I_{\frac{(n-2)p}{p-1}}^{\frac{n}{2}-1})^{\frac{p-1}{p}})(\frac{2^{\frac{n-2}{n}}}{(\omega_{n-1}I_n^{\frac{n}{2}-1}f(P))^{\frac{n-2}{n}}})\leq A$$

where we have put $A=(n-2)n(\frac{\omega_{n-1}I_n^{\frac{n}{2}-1}}{2})^{\frac{2}{n}}(f(P))^{-\frac{2}{N}}$, and as $2-\frac{n}{p}>0$,

$$\epsilon^{2-\frac{n}{p}}\beta\epsilon^2=o(\epsilon^2).$$

Then it follows that,

$$\begin{aligned} E(u_\varepsilon) &\leq [A+o(\epsilon^2) \\ &+ (n-2)\omega_{n-1}I_n^{\frac{n}{2}-1}\frac{n}{2}(\frac{2^{\frac{n-2}{n}}}{(\omega_{n-1}I_n^{\frac{n}{2}-1}f(P))^{\frac{n-2}{n}}})(1-\frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)}+o(\epsilon^2))] \\ &\times [(1+(\frac{\Delta_g f(P)}{2nf(P)}+\frac{S_g(P)}{6n})\epsilon^2)+o(\epsilon^2)] \end{aligned}$$

again

$$\begin{aligned} E(u_\varepsilon) &\leq [A+(n-2)n(\frac{\omega_{n-1}I_n^{\frac{n}{2}-1}}{2})^{\frac{2}{n}}(f(P))^{-\frac{2}{N}}(1-\frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)}+o(\epsilon^2))] \\ &\times [(1+(\frac{\Delta_g f(P)}{2nf(P)}+\frac{S_g(P)}{6n})\epsilon^2)+o(\epsilon^2)]. \end{aligned}$$

If we put

$$C=\frac{S_g(P)(n+2)}{6n(n-4)} \quad \text{and} \quad D=\frac{\Delta_g f(P)}{2nf(P)}+\frac{S_g(P)}{6n}.$$

the latter equality will be written as follows:

$$\begin{aligned} E(u_\varepsilon) &\leq [A+A-AC\epsilon^2+o(\epsilon^2)]\times[1+D\epsilon^2+o(\epsilon^2)] \\ &\leq A[2-C\epsilon^2+o(\epsilon^2)]\times[1+D\epsilon^2+o(\epsilon^2)]. \end{aligned}$$

and direct calculation gives,

$$E(u_\varepsilon) \leq 2A[1+(D-\frac{C}{2})\epsilon^2]+o(\epsilon^2).$$

Knowing that

$$(n-2)n(\frac{\omega_{n-1}I_n^{\frac{n}{2}-1}}{2})^{\frac{2}{n}} = K_0^{-2}(n,1)$$

it follows that

$$A = K_0^{-2}(n,1)(f(P))^{-\frac{2}{N}}$$

then one can deduce that,

$$E(u_\varepsilon) \leq 2K_0^{-2}(n,1)(f(P))^{-\frac{2}{N}}[1+(D-\frac{C}{2})\epsilon^2]+o(\epsilon^2).$$

Now if

$$D - \frac{C}{2} < 0 \tag{16}$$

we will get the desired inequality

$$E(u_\varepsilon) < 2K_0^{-2}(n,1)(f(P))^{-\frac{2}{N}}$$

which implies that

$$\mu(M,g) < 2(K_0^{-2}(n,1))(f(P))^{-\frac{2}{N}}.$$

The condition (16) means that :

$$\frac{\Delta_g f(P)}{f(P)} < \frac{S_g(P)}{3}\left(\frac{(n+2)}{2(n-4)} - 1\right).$$

□

4. Application

Corollary 4.1.
Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that f is a positive $C^\infty(M)$ function on M and $P \in M$ such that $f(P) = \sup_{x\in M} f(x)$. If

$$\frac{\Delta_g f(P)}{f(P)} < \frac{S_g(P)}{3}\left(\frac{(n+2)}{2(n-4)} - 1\right).$$

Then there exists a metric $\overline{g} = u^{N-2}g$ conformal to g such that the scalar curvature $S_{\overline{g}} = f$.

Proof. Since the inequality (9) is satisfied, The theorem (1.3) asserts that there is $u \in C^1(M)$, $u > 0$ solution of the following equation

$$\Delta_g u + C_n S_g u = f|u|^{N-2}u$$

and we know that the singular Yamabe operator $P_g = \Delta_g + C_n S_g$ is weakly comformally invariant therefore if $\overline{g} = u^{N-2}g$ is conformal to g, one has

$$\Delta_g u + C_n S_g u = C_n S_{\overline{g}}|u|^{N-2}u.$$

then, we deduce that the metric $\overline{g} = u^{N-2}g$ is such that its scalar curvature $S_{\overline{g}} = \frac{f}{C_n}$. □

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